National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 5

1. Test the following series for convergence.

   (i) $\sum_{1}^{\infty} \frac{2^n+n}{3^n-n}$ (ii) $\sum_{1}^{\infty} \frac{e^{-n}}{\sqrt{n+1}}$ (iii) $\sum_{1}^{\infty} \sin\left(\frac{1}{n^p}\right)$, p real $p > 0$ (iv) $\sum_{1}^{\infty} \frac{\ln(n+1)-\ln(n)}{\tan^{-1}(\frac{2}{n})}$

2. Determine the conditional convergence, absolute convergence or divergence of the following series.

   (i) $\sum_{1}^{\infty}(-1)^n e^{-n^2}$ (ii) $\sum_{1}^{\infty} \frac{(-1)^n}{\ln(\cosh(n))}$ (iii) $\sum_{1}^{\infty}(-1)^n \frac{n^2}{2+n^2}$

3. (i) Prove that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \frac{\pi}{4}$.

   (ii) Let S_n (n ≥ 1) be the n-th partial sum of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$. Show by induction that $S_{2n} = \sum_{k=1}^{n} \frac{1}{n+k}$. Use integral calculus to deduce that $\sum_{1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln(2)$.

4. Prove that if $\sum a_n^2$ and $\sum b_n^2$ are absolutely convergent , then $\sum a_n b_n$ is also absolutely convergent. [Hint: $|a_n b_n| \le (a_n^2 + b_n^2)/2$]. Hence deduce that if $\sum a_n^2$ is absolutely convergent, then so is $\sum_{n=1}^{\infty} \frac{a_n}{n}$.

5. Given that $\sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, show that (i) $\sum_{1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$ and $\sum_{1}^{\infty}(-1)^n \frac{1}{n^4} = -\frac{7\pi^4}{720}$

6. Find the radius of convergence of the following power series $\sum_{n=1}^{\infty} a_n x^n$, where a_n =

   (i) n^2 (ii) $1/n$ (iii) $1/n^2$ (iv) 2^n (v) $2^n/n$ (vi) $1/3^n$ (vii) $\frac{n+1}{2^n+n}$ (viii) $\frac{(2n)!}{(n!)^2}$

7. Use the Cauchy Hadamard formula to show that the three series

   $$\sum_{n=1}^{\infty} a_n x^n \quad , \quad \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } \sum_{n=1}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

   have the same radius of convergence.

8. Compare the regions of convergence of the three (real) power series (say if they are the same and state the precise region of convergence).

   $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$
   $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$
   $$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots$$

9. (i) Give an example of a power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence 1, which is divergent at each point on the circle of convergence (i.e., the boundary of the disk of convergence)..

   (ii) Give an example of a power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence 1, which is divergent at some points on the circle of convergence and divergent at other points.

   (iii) Give an example of a power series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence 1, which is convergent at each point on the circle of convergence.