National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 6

1. Determine whether each of the following sequences (of functions) converge uniformly on the given domain.

   (i) $\dfrac{\sin(nx)}{n}$ on [0, 1] (ii) $\dfrac{1}{3n-x}$ on [0, 1] (iii) $\dfrac{1}{nx+2}$ on [0, 1]

   (iv) $(x-\frac{1}{n})^2$ on [0, 1] (v) $x - x^n$ on [0, 1] (vi) $\dfrac{2n+x}{n+3}$ on $[a, b]$, $a < b$.

2. Use the Weierstrass M-Test to prove that each of the following series is uniformly convergent on the given domain...

   (i) $\sum\limits_{n=1}^{\infty} \dfrac{\sin(nx)}{2^n}$ on **R** (ii) $\sum\limits_{n=1}^{\infty} \dfrac{x^2+n}{x^2+n^4}$ on $[-a, a]$, $a > 0$ (iii) $\sum\limits_{n=1}^{\infty}(n+1)x^n$ on $[-a, a]$, $0 < a < 1$

   (iv) $\sum\limits_{n=1}^{\infty} \dfrac{x^n(1-x)}{n}$ on [0, 1] (Hint: Find maximum value of $x^n(1-x)$ in [0, 1].)

3. Let $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{x^{n/2}}{n(n!)^2}$. Discuss how you might prove that f is continuous on [0, 1].

4. (Realising function as a power series.)

   (i) Prove that $\dfrac{1}{1+x} = 1 - x + x^2 + \cdots = \sum\limits_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$.

   Discuss how you might prove that (ii) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n+1} x^{n+1}$ for $|x| < 1$.

   and (iii) $\dfrac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + \cdots = \sum\limits_{n=1}^{\infty}(-1)^n n x^{n-1}$ for $|x| < 1$.

5. Use the power series expansion of $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots = \sum\limits_{n=0}^{\infty} x^n$ for $|x| < 1$ to prove that

   (a) $\sum\limits_{n=1}^{\infty} n x^n = \dfrac{x}{(1-x)^2}$ if $|x| < 1$;

   (b) $\sum\limits_{n=0}^{\infty} \dfrac{n}{n+1} x^n = \begin{cases} \dfrac{x + (1-x)\ln(1-x)}{(1-x)x} & \text{if } 0 < |x| < 1 \\ 0, & \text{if } x = 0 \end{cases}$

   (c) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n(n+1)} = \begin{cases} (\frac{1}{x} - 1)\ln(1-x) + 1 & \text{if } 0 < |x| < 1 \\ 0, & \text{if } x = 0 \end{cases}$

   Give reasons for the steps you take.

   (This question is an example of power series manipulation.)

6. (Optional) Determine the radius of convergence of the Bessel function of the first kind of order zero $J_0(x)$ given by $J_0(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(n!)^2}\left(\dfrac{x}{2}\right)^{2n}$. Write out the first 4 terms of $J_0(x)$.

   Show that $J_0(x)$ satisfies the differential equation $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$

   (Bessel's differential equation of order zero).