National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 7

1. Find all those x for which the following series converge.

   (i) $\sum_{n=1}^{\infty} \frac{n^2(n+2)}{(n+5)3^n} x^n$ (ii) $\sum_{n=1}^{\infty} \frac{3\sqrt{n}}{n} x^n$ (iii) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{n^2} (2x+1)^n$

   (Hint: Use ratio test.)

2. Use trigonometric formula to prove that $4\sin^3(x) = 3\sin(x) - \sin(3x)$. Use this and the power series expansion for $\sin(x)$ to show that

   (i) $\sin^3(x) = \frac{3}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{2n}-1}{(2n+1)!}x^{2n+1}$ for all real x**.**

   (ii) Use partial fraction and obvious series expansion of the resulting rational function**s,** or otherwise, show that $\frac{x}{1+x-2x^2} = \frac{1}{3}\sum_{n=1}^{\infty}[1-(-2)^n]x^n$ for $|x| < ½$.

3. Assuming that $y'' + y = 0$, $y(0) = 0$, $y'(0)=1$ has a solution given by a power series. Find the power series and determine its radius of convergence.
   (Hint: Use the three conditions to obtain relation among the coefficients of the power series and solving the relation.)

4. Find the radius of convergence of $.y(x) = a_0(1-x^2) - a_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)(2n-1)}$, where a_0 and a_1 are arbitrary real numbers.
   Show that $y(x)$ satisfies the differential equation $(1-x^2)y'' = -2y$ on its interval of convergence.

5. Show that $f_n(x) = (1-x^2)x^n$ converges uniformly on $[-1, 1]$ and find its limiting function g. Hence conclude that $\int_0^1 f_n(x)dx \to 0$.

6. Explain what results you would use to show that $\sum_{n=1}^{\infty}\frac{e^{-nx^2}}{n^2}$ is continuous on $\mathbf{R}$.