**Composition and Riemann Integrability**
**by Ng Tze Beng**

It is natural to ask whether composition of Riemann integrable functions is again Riemann integrable. Unlike the case for differentiable functions, where composition of differentiable functions is again differentiable, composition of Riemann integrable functions need not be Riemann integrable. The starting point is of course to establish that non-Riemann integrable functions exist. The following example will be used in subsequent proceeding.

**Example 1.** The function $h : [0, 1] \to \mathbf{R}$, defined by

$$h(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases},$$

is not Riemann integrable.

We shall prove that $h$ is not Riemann integrable, using Theorem 1 of *Riemann Integral and Bounded function* or more precisely part 1 of this theorem.

Let $P : 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ be any partition for the interval [0, 1]. Then by the density of the rational numbers and irrational numbers, in each of the subinterval $[x_{i-1}, x_i]$, ($i = 1,\ldots, n$) we can always find a rational number and an irrational number. Hence for $i = 1,\ldots, n$., $\{h(x): x \in [x_{i-1}, x_i]\} = \{0, 1\}$. Therefore, for each $i = 1,\ldots, n$,

$$M_i(P, h) = \sup\{h(x): x \in [x_{i-1}, x_i]\} = \sup\{0, 1\} = \max\{0, 1\} = 1 \text{ and}$$
$$m_i(P, h) = \inf\{h(x): x \in [x_{i-1}, x_i]\} = \inf\{0, 1\} = \min\{0, 1\} = 0.$$

Therefore, the upper Riemann sum with respect to the Partition $P$ is

$$U(P,h) = \sum_{i=1}^{n} M_i(P,h)\Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1$$

and the lower Riemann sum with respect to the Partition $P$ is

$$L(P,h) = \sum_{i=1}^{n} m_i(P,h)\Delta x_i = 0.$$

The above is true for any partition $P$ for [0, 1]. Hence, the lower Riemann integral of $h$,

$$\underline{\int} h = \sup\{L(P,h) : P \text{ is a partition for } [0,1]\} = \max\{0\} = 0$$

and the upper Riemann integral of $h$,

$$\overline{\int} h = \inf\{U(P,h) : P \text{ is a partition for } [0,1]\} = \min\{1\} = 1.$$

Therefore, the lower Riemann integral $\underline{\int} h$ is not equal to the upper Riemann integral $\overline{\int} h$ and so by Theorem 1 of *Riemann Integral and Bounded Function*, $h$ is not Riemann integral over [0, 1].

There is a deeper theorem that we can invoke, namely the Theorem of Lebesgue.

**Theorem 1 (Lebesgue).** A bounded real valued function $f : [a, b] \to \mathbf{R}$ from the closed and bounded interval $[a, b]$ into $\mathbf{R}$ is Riemann integrable if and only if $f$ is continuous except perhaps on a set of *measure* zero in $[a, b]$.

Here we have a new concept, more precisely, the concept of Lebesgue measure. This is a kind of generalisation of length. How can we use this theorem?. It is useful to know that the set of rational numbers is a set of *measure* zero and that the set of irrational numbers in a non-trivial interval is not zero. We shall also use other exotic sets of non-zero measure. For now what we need to examine is the set of discontinuities of $h$.

**Assertion 1**. $h$ is nowhere continuous on [0, 1].

**Proof.** Let $x$ be an irrational number in [0, 1]. Then $h(x) = 0$. Take any $\varepsilon > 0$ with $\varepsilon < 1$, for any $\delta > 0$, there exists, by the density of rational numbers, a rational number $p_\delta$ in $(x - \delta, x + \delta)\cap[0,1]$ such that $|h(p_\delta) - h(x)| = |1 - 0| = 1 > \varepsilon$. Hence, $\lim_{y \to x} h(y) \neq h(x)$. Therefore, $h$ is not continuous at $x$. Similarly let $x$ be a rational number in [0, 1]. Then $h(x) = 0$. Take any $\varepsilon > 0$ with $\varepsilon < 1$, for any $\delta > 0$, there exists, by the density of the irrational numbers, an irrational number $q_\delta$ in $(x - \delta, x + \delta)\cap[0,1]$ such that $|h(q_\delta) - h(x)| = |0 - 1| = 1 > \varepsilon$. Hence $\lim_{y \to x} h(y) \neq h(x)$. Therefore, $h$ is not continuous at $x$.

For our purpose we only need to know that the function $h$ is discontinuous on the set of irrational numbers in [0, 1] and that the measure of the irrational numbers in [0, 1] is not zero (in fact, the measure is equal to 1). We cannot easily establish the fact that the measure of the irrational numbers in [0, 1] is not zero without going into the theory of Lebesgue measure. Also the measure of [0, 1] is 1. For now we accept these facts.

**Another proof of the non Riemnann integrability of *h*.**
By assertion 1, $h$ is discontinuous on a set of measure bigger than zero. By Theorem 1, $h$ is not Riemann integrable on [0, 1].

How can we produce a counter example to the assertion that composite of Riemann integrable functions is Riemann integrable? If we factor $h$ into a composite of Riemann integrable functions, then we are done. Indeed, we can do so. Let us first describe the factors.

**Example 2.** The function $f: [0, 1] \to \mathbf{R}$ defined by $f(x) = 0$ if $x = 0$ and $f(x) = 1$, for $0 < x \leq 1$ is Riemann integrable.

**Proof.** We shall use the method of Example 9.2.6 of *Calculus, An Introduction* to show this. Given any $\varepsilon > 0$, there exists a positive integer $m$ such that $1/m < \varepsilon$. Take any partition $P : 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$ such that the norm of $P$, $\|P\| = \max\{x_i - x_{i-1} : i = 1, \ldots, n\} < 1/m$. Then the difference of the upper and lower Riemann sum,

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} M_i(P, f)\Delta x_i - \sum_{i=1}^{n} m_i(P, f)\Delta x_i ,$$

where $M_i(P, f) = \sup\{f(x): x \in [x_{i-1}, x_i]\}$ and
$m_i(P, f) = \inf\{f(x): x \in [x_{i-1}, x_i]\}$,

$$= (M_1(P, f) - m_1(P, f))(x_1 - x_0),$$

since $M_i(P, f) = m_i(P, f) = 1$ for $i > 1$ ,

$$= (x_1 - x_0), \quad \text{since } M_1(P, f) = 1 \text{ and } m_1(P, f) = 0,$$

$\leq \|P\| < 1/m < \varepsilon.$

Hence, by the Riemann's condition (Theorem 1, *Riemann Integral and Bounded Function*), $f$ is Riemann integrable.

We consider now the next example.

**Example 3.** The real valued function g : [0, 1] → [0, 1] defined by

$$g(x) = \begin{cases} 0, & x \text{ is irrational} \\ \dfrac{1}{q}, & 0 < x < 1 \text{ and } x = \dfrac{p}{q} \text{ in its lowest term} \\ 1, & x = 0 \text{ or } 1 \end{cases}.$$

is Riemann integrable.

Before we show that g is Riemann integrable we observe that $f$ and $g$ are the required factors for our counter example.

**Example 4.** The functions $f$, $g$ and $h$ as defined in example 1, 2 and 3 satisfy $h = f \circ g$. The functions $f$ and $g$ are Riemann integrable but $h$ is not Riemann integrable.

**Proof.** If $x$ is an irrational number in [0, 1], then $f \circ g\,(x) = f(g(x)) = f(0) = 0 = h(x)$. Suppose now that $x$ is a rational number in [0, 1]. Then by the definition of g, $0 < g(x) \leq 1$. Hence by the definition of $f$, $f \circ g\,(x) = f(g(x)) = 1 = h(x)$ for any rational number in [0, 1]. Thus, for all $x$ in [0, 1], $f \circ g\,(x) = h(x)$. Therefore, $h = f \circ g$.

**Assertion 2. The function g: [0, 1] → [0, 1] as defined in Example 3 is Riemann integrable.**

We shall use Lebesgue theorem. This requires us to show that g is continuous at every irrational point in [0, 1]. Thus, since the rational numbers in [0, 1] is of measure zero, by the Lebesgue Theorem, $g$ is Riemann integrable. Since the proof of the continuity at irrational points is of some interest, especially the simple logic involved, we shall present the proof along this line. We shall do more, we shall prove that g is discontinuous at every rational point in [0,1]. Given any $\varepsilon > 0$, by the Archimedean property of **R**, there exists a positive integer $m > 1$ such that $1/m < \varepsilon$. Next we observe that there can only be a finite number of reciprocals of integers that are greater than or equal to $1/m$: For any rational number $p/q$ with $p/q$ in its lowest terms and $0 < p/q \leq 1$, $g(p/q) \geq 1/m$ if and only if $1/q \geq 1/m$ if and only if $1 \leq q \leq m$. This means that $1 \leq p \leq q$ and the greatest common divisor of $p$ and $q$ is 1. Thus, the number of rational numbers in [0, 1] that have values greater than or equal to $1/m$ is finite and this set includes the point 0 since g(0) = 1. That is, the finite set $S_m = \{ p/q: q = 1, \ldots, m; p = 1, \ldots, q\} \cup \{0\}$ is precisely the set on which the values of g are greater than or equal to $1/m$. Take an irrational number $x$ in [0, 1]. Then let $\delta = \min\{|x - y| : y \in S_m\}$. Then $\delta > 0$ since $x$ is irrational. Obviously, the open interval $(x - \delta, x + \delta)$ do not meet $S_m$. This is because if $(x - \delta, x + \delta) \cap S_m \neq \emptyset$, then there exists $p/q$ in $S_m$ such that $|x - p/q| < \delta$ contradicting that $|x - p/q| \geq \delta$. That means for all $y$ in

$(x-\delta, x+\delta) \cap [0, 1]$, $y \notin S_m$ and so consequently, $g(y) < 1/m < \varepsilon$. Therefore, $|g(y) - g(x)| = |g(y) - 0| = g(y) < \varepsilon$. Hence, for all $y$ in [0, 1] such that $|y - x| < \delta$ we have $|g(y) - g(x)| < \varepsilon$. Thus g is continuous at $x$. Now we shall show that g is discontinuous at rational point $x$. For any rational point $x$ in [0, 1], $g(x) > 0$. Let $\varepsilon = g(x)/2 > 0$. Then for any $\delta > 0$, by the density of the irrational numbers in any interval, there exists an irrational number $y_\delta$ in $(x-\delta, x+\delta) \cap [0, 1]$. Then $|g(y_\delta) - g(x)| = |0 - g(x)| = g(x) > g(x)/2 = \varepsilon$. That means g is not continuous at $x$. Hence g is not continuous at any rational number in [0,1]. This completes the proof of the assertion.

**Another proof of assertion 2.**

We shall show that g satisfies the Riemann's condition. For any $\varepsilon > 0$, take any positive integer $m > 1$ such that $1/m < \varepsilon/2$. Recall that the finite set $S_m$ is precisely the set $\{y \in [0, 1]: g(y) \geq 1/m\}$. Note that $0, 1 \in S_m$. Let the number of points of $S_m$ be $k+1$. Order the elements $y_0, y_1, y_2, \ldots, y_k$ of $S_m$ as follows:

$$0 = y_0 < y_1 < y_2 < \ldots y_{k-1} < y_k = 1.$$

Choose $k-1$ pair of points, each pair constitute an interval containing each $y_i$ in its interior and of length $< \varepsilon/2$ for $i = 1, 2, \ldots, k-1$ and such that they are all mutually disjoint. That is, we choose $x_0 < x_1 < x_2 < \ldots < x_{2k-2}$ such that

$0 = y_0 < x_1 < y_1 < x_2 < x_3 < y_2 < x_4 < x_5 < \ldots x_{2k-4} < x_{2k-3} < y_{k-1} < x_{2k-2} < y_k = 1$

We choose further two more points, $x_0$ and $x_{2k-1}$ and name $y_0$ as $x_{-1}$, $y_k$ as $x_{2k}$ such that $0 = x_{-1} = y_0 < x_0 < x_1$ and $x_{2k-2} < x_{2k-1} < x_{2k} = y_k = 1$. We further require that

$$\sum_{j=0}^{k}(x_{2j} - x_{2j-1}) < \frac{\varepsilon}{2} \text{ . ---------------------------- (1)}$$

Obviously,

$P$: $0 = x_{-1} < x_0 < x_1 < x_2 < x_3 < x_4 < x_5 < \ldots x_{2k-4} < x_{2k-3} < x_{2k-2} < x_{2k-1} < x_{2k} = 1$

forms a partition for [0, 1].

Now, by the density of the irrational numbers in any interval, for $i = 0, 1, \ldots, 2k$,

$$m_i(P, g) = \inf\{g(x): x \in [x_{i-1}, x_i]\} = 0. \text{ ---------------- (2)}$$

For each $j = 0, 1, 2, \ldots, k$, $y_j \in [x_{2j-1}, x_{2j}]$ and so by the definition of $S_m$,

$$M_{2j}(P, g) = \sup\{g(x): x \in [x_{2j-1}, x_{2j}]\} = g(y_j) \text{ ------------------- (3).}$$

for $j = 0, 1, 2, \ldots, k$.

Now because for $j = 1, 2, \ldots, k$, $[x_{2j-2}, x_{2j-1}] \cap S_m = \varnothing$,

$$M_{2j-1}(P, g) = \sup\{g(x): x \in [x_{2j-2}, x_{2j-1}]\} < 1/m, \text{ ---------------- (4).}$$

for $j = 1, 2, \ldots, k$.

Hence,
$$\begin{aligned}
U(P, g) - L(P, g) &= \sum_{i=0}^{2k} M_i(P, g)\Delta x_i - \sum_{i=0}^{2k} m_i(P, g)\Delta x_i, \\
&= \sum_{i=0}^{2k} M_i(P, g)\Delta x_i \text{ , by (2),} \\
&= \sum_{j=0}^{k} M_{2j}(P, g)\Delta x_{2j} + \sum_{j=0}^{k} M_{2j-1}(P, g)\Delta x_{2j-1} \\
&\leq \sum_{j=0}^{k} g(y_j)\Delta x_{2j} + \sum_{j=0}^{k} \frac{1}{m}\Delta x_{2j-1} \text{ , by (3) and (4),} \\
&\leq \sum_{j=0}^{k} \Delta x_{2j} + \frac{1}{m}\sum_{j=0}^{k} \Delta x_{2j-1} \text{ , since } g(y_j) \leq 1, \\
&< \varepsilon/2 + \frac{1}{m}\sum_{j=0}^{2k} \Delta x_j \text{ , by (1),}
\end{aligned}$$

$< \varepsilon/2 + 1/m$ , since $\sum_{j=0}^{2k} \Delta x_j = x_{2k} - x_{-1} = 1$,

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Thus g satisfies the Riemann's condition and so by Theorem 1 of *Riemann Integral and Bounded Function ,* g is Riemann integrable. This completes the proof.

This wraps up our demonstration of a counter example. It is then natural to ask: Under what condition can we deduce that the composite of two Riemann integrable functions is Riemann integrable? In view of Lebesgue's Theorem, it is necessary to examine the set of discontinuities of the composite function. If the set of discontinuities of the composite function is contained within the set of discontinuities of the first function, then it becomes an easy matter to make a deduction. Note that it is the set of discontinuities that determines the Riemann integrability of a function. We have the following theorem.

**Theorem 2.** Suppose g: $[a, b] \to \mathbf{R}$ is a Riemann integrable function and $f : [c, d] \to \mathbf{R}$ is a continuous function such that the range of g, g( $[a, b]$ ) is contained in $[c, d]$. Then the composite function $f \circ g : [a, b] \to \mathbf{R}$ is Riemann integrable on $[a, b]$.

**Proof.** We shall prove this using Lebesgue's theorem. Note that if g is continuous at a point $x$ in $[a, b]$, then since $f$ is continuous at $g(x)$ (because $f$ is a continuous function), $f \circ g$ is continuous at $x$. This means that whenever $g$ is continuous at $x$ the composite $f \circ g$ is also continuous at $x$. Hence the set of discontinuities of the composite function $f \circ g$, $D_{f \circ g}$ is contained in the set of discontinuities of g, $D_g$. Since g is Riemann integrable, $D_g$ is of measure zero and since $D_{f \circ g} \subseteq D_g$ , $D_{f \circ g}$ too is of measure zero. Therefore, by Lebesgue's theorem $f \circ g$ is Riemann integrable.

This proof demonstrates the power of Lebesgue's theorem. But one need not use Lebesgue's theorem. One can make use of the uniform continuity of g too. It will involve a clever manipulation of the Riemann sums.

**Another Proof of Theorem 2.**

As $f : [c, d] \to \mathbf{R}$ is a continuous function, by Theorem 1 of *The Boundedness Theorem, Extreme Value theorem and Intermediate Value Theorem,* $f$ is bounded. Therefore, there exists a real number $M > 0$, such that $|f(x)| < M$ for all $x$ in $[c, d]$. Also by Theorem 9 of *Closed and bounded sets, Heine Borel Theorem, etc*, $f$ is uniformly continuous. Therefore, given any $\varepsilon > 0$, there exists $0 < \delta < \varepsilon$ such that for all $x, y$ in $[c, d]$,

$$|x - y| < \delta \text{ implies that } |f(x) - f(y)| < \frac{\varepsilon}{4(b-a)} \text{ ------------------ (5).}$$

Next, since g is Riemann integrable by Theorem 1 of *Riemann Integral and Bounded Function*, we can find a partition $P : a = x_0 < x_1 < x_2 < \ldots < x_n = b$ for $[a, b]$ such that

$$U(P, g) - L(P, g) < \delta^2/(4M),$$

where $U(P,g) = \sum_{i=1}^{n} M_i(P, g)\Delta x_i$, $L(P,g) = \sum_{i=1}^{n} m_i(P, g)\Delta x_i$ , for $i = 1, \ldots, n$,

$M_i(P, g) = \sup\{ g(x) : x \in [x_{i-1}, x_i] \}$ and $m_i(P, g) = \inf\{ g(x) : x \in [x_{i-1}, x_i] \}$.

Note that

$$U(P, g) - L(P, g) = \sum_{i=1}^{n}(M_i(P, g) - m_i(P, g))\Delta x_i < \delta^2/(4M) \text{ ---- (6)}$$

Now,

$$M_i(P, g) - m_i(P, g) = \sup\{g(x): x \in [x_{i-1}, x_i]\} - \inf\{g(x): x \in [x_{i-1}, x_i]\}$$
$$= \sup\{g(x) - g(y): x, y \in [x_{i-1}, x_i]\}.$$

Similarly,

$$M_i(P, f\circ g) - m_i(P, f\circ g) = \sup\{f\circ g(x) - f\circ g(y): x, y \in [x_{i-1}, x_i]\}.$$

Therefore, using the same partition $P$ for the composite $f\circ g$, the difference of the upper and lower Riemann sum with respect to $P$ for $f\circ g$ is,

$$U(P, f\circ g) - L(P, f\circ g) = \sum_{i=1}^{n} \sup\{f\circ g(x) - f\circ g(y) : x, y \in [x_{i-1}, x_i]\}\Delta x_i .$$

Let $J = \{i : 1 \le i \le n, \sup\{g(x) - g(y): x, y \in [x_{i-1}, x_i]\} < \delta\}$. So we can rewrite the above difference as

$$U(P, f\circ g) - L(P, f\circ g) = \sum_{i \in J} \sup\{f\circ g(x) - f\circ g(y) : x, y \in [x_{i-1}, x_i]\}\Delta x_i$$
$$+ \sum_{i \notin J} \sup\{f\circ g(x) - f\circ g(y) : x, y \in [x_{i-1}, x_i]\}\Delta x_i . \text{ ----- (7)}$$

If $J \neq \emptyset$, then for $i \in J$,

$\sup\{f\circ g(x) - f\circ g(y): x, y \in [x_{i-1}, x_i]\}$

$= \sup\{|f(g(x)) - f(g(y))|: x, y \in [x_{i-1}, x_i]\} \le \dfrac{\varepsilon}{4(b-a)}$, by (5),

because for any $x, y \in [x_{i-1}, x_i]$, $|g(x) - g(y)| \le \sup\{g(x) - g(y): x, y \in [x_{i-1}, x_i]\} < \delta$.

Hence,

$$\sum_{i \in J} \sup\{f\circ g(x) - f\circ g(y) : x, y \in [x_{i-1}, x_i]\}\Delta x_i$$
$$\le \sum_{i \in J} \frac{\varepsilon}{4(b-a)}\Delta x_i \le \frac{\varepsilon}{4(b-a)} \sum_{i \in J} \Delta x_i \le \frac{\varepsilon}{4(b-a)} \sum_{i=1}^{n} \Delta x_i < \frac{\varepsilon}{2}.$$

------------------------ (8)

If $\{1, \ldots, n\} - J \neq \emptyset$, then $i \in \{1, \ldots, n\} - J$ implies that $M_i(P, g) - m_i(P, g) \ge \delta$. Therefore,

$$\sum_{i \notin J} (M_i(P, g) - m_i(P, g))\Delta x_i \ge \delta \sum_{i \notin J} \Delta x_i .$$

Hence, by (6),

$$\delta \sum_{i \notin J} \Delta x_i \le \sum_{i \notin J} (M_i(P, g) - m_i(P, g))\Delta x_i \le \sum_{i=1}^{n} (M_i(P, g) - m_i(P, g))\Delta x_i \le \frac{\delta^2}{4M}$$

and so,

$$\sum_{i \notin J} \Delta x_i \le \frac{\delta}{4M}. \text{ -------------------------------- (9)}$$

Therefore, using (9),

$$\sum_{i \notin J} \sup\{f\circ g(x) - f\circ g(y) : x, y \in [x_{i-1}, x_i]\}\Delta x_i \le 2M \sum_{i \notin J} \Delta x_i \le 2M\frac{\delta}{4M} = \frac{\delta}{2} < \frac{\varepsilon}{2}.$$

---------------------------- (10)

Therefore, by (7), if $J = \emptyset$ using (10), $U(P, f\circ g) - L(P, f\circ g) < \varepsilon/2 < \varepsilon$. If $J = \{1, \ldots, n\}$, by (8) $U(P, f\circ g) - L(P, f\circ g) < \varepsilon/2 < \varepsilon$. If $J \neq \emptyset$ and $J \neq \{1, \ldots, n\}$, by (7), (8) and (10), $U(P, f\circ g) - L(P, f\circ g) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus, by the Riemann's conndition (Theorem 1 of *Riemann Integral and Bounded Function* ), $f\circ g$ is Riemann integrable on $[a, b]$. This completes the proof.

Now we ask the question: If g is Riemann integrable and $f$ is a continuous function, is it true that the composite $g\circ f$ is Riemann integrable? In fact it need not be the case. More specifically, if $f : [a, b] \to [c, d]$ is continuous and $g : [c, d] \to \mathbf{R}$ is Riemann integrable, the composite function $g\circ f : [a, b] \to \mathbf{R}$ need not necessarily be

Riemann integrable, even if $f$ is a monotonically increasing bijective continuous function. Our counter example will involve two types of Cantor set, one of measure zero and another of positive measure. We shall not go into the construction of these Cantor sets. But we shall state the properties these Cantor sets enjoy.

**Example 5 Cantor sets.** There are Cantor sets in [0, 1] with measure $k$ with $0 \le k < 1$. Let $C_k$ denote the Cantor set of measure $k$. Thus $C_0$ is the Cantor set of measure zero. This is the usual cantor set we meet. The construction of these Cantor sets is similar. They satisfy the following properties:
1. Any Cantor set $C$ is closed in [0,1].
2. Any Cantor set $C$ is nowhere dense, that is to say, $C$ does not contain any interval, meaning $C$ has empty interior.
3. Any Cantor set is uncountable, more precisely, it has the cardinality of the real numbers **R**.
4. Any Cantor set is perfect, that is, it is its own accumulation points.
5. At the $n$-th stage of the construction of $C_k$ for $1 > k > 0$ there is left a disjoint union of $2^n$ closed intervals. The ($n$+1)-stage is obtained by deleting from the centre of each of these disjoint intervals, an open interval of length $2^{-2n-1}(1-k)$.
6. At the $n$-th stage of the construction of $C_0$, we have a disjoint union of $2^n$ closed intervals. The ($n$+1)-stage is obtained by deleting from the centre of each of these disjoint intervals, the middle open interval.
7. The complement of any of these Cantor sets is a disjoint union of a countable set of open intervals.

Our example for the continuous function $f$ mentioned above will be given by the following lemma.

**Lemma 1.** For each $k$ with $0 < k < 1$, there exists a function $f: [0, 1] \to [0, 1]$ such that
1. $f$ is monotonically increasing and onto and so $f$ is continuous. (Refer to Theorem 2 of *Monotonicity and Continuity of Inverse Function.*)
2. $f$ maps the Cantor set $C_k$ *onto* the Cantor set $C_0$.

**Proof.** We shall look at the complement of the Cantor set $C_k$ and $C_0$. They are disjoint unions of open intervals. Let $\{I_1, I_2, I_3, \ldots, I_n, \ldots\}$ be the open intervals of the complement of $C_k$, listed from left to right following the construction of $C_k$ in the order of deletion and the natural ordering of the open intervals in each stage of the construction. Similarly, let $\{J_1, J_2, J_3, \ldots, J_n, \ldots\}$ be the open intervals of the complement of $C_0$, listed from left to right following the construction of $C_0$. For each $n$ denote $I_n$ by $(a_n, b_n)$ and $J_n$ by $(c_n, d_n)$. Now we shall define our function $f$: $[0, 1] \to [0, 1]$ as follows.

1. $f(0) = 0$.
2. For $x$ in $I_n = (a_n, b_n)$, $f(x) = \dfrac{d_n - c_n}{b_n - a_n}(x - a_n) + c_n$. This maps $I_n$ bijectively onto $J_n$.
3. For $x \neq 0$ and $x$ in $C_k$,

$$f(x) = \sup\{ f(y) : y < x \text{ and } y \in \bigcup_{n=1}^{\infty} I_n\} = \sup\{ f(y) : y < x \text{ and } y \in [0, 1] - C_k\} \ .$$

This is well defined by the completeness property of **R**, since the set $\{ f(y) : y < x \text{ and } y \in \bigcup_{n=1}^{\infty} I_n\}$ is bounded above by 1.

***The function $f$ is increasing on* [0,1]−$C_k$**

Firstly, we shall show that $f$ is increasing on the complement of $C_k$ and maps the complement of $C_k$ bijectively onto the complement of $C_0$ in [0, 1].

By definition $f$ is increasing on each $I_n$ and maps $I_n$ bijectively onto $J_n$. At the $n$-th stage of the construction of the Cantor set we obtained $2^n - 1$ disjoint open intervals, that have been deleted from [0, 1], $I_1, I_2, I_3, \ldots, I_{2^n-1}$. The ordering of these intervals is in the order of the deletion starting from the left to the right. The natural ordering defined as follows follows a very simple rule. $I_k < I_l$, if and only if, there exists some $x$ in $I_k$ such that $x < y$ for some $y$ in $I_l$, if and only if, for any $x$ in $I_k$, $x < y$ for all $y$ in $I_l$. At the $n$-th stage we can map $\{I_1, I_2, I_3, \ldots, I_n, \ldots I_{2^n-1}\}$ onto the set $\{ j / 2^n : j = 1, 2, \ldots, 2^n - 1\}$ according to the order of dissection of [0, 1] into $2^n$ parts: $I_1$ corresponds to $2^{n-1}/2^n = 1/2$, $I_2$ corresponds to $2^{n-2}/2^n = 1/4$, $I_3$ corresponds to $3\times 2^{n-2}/2^n = 3/4$, $I_4$ corresponds to $2^{n-3}/2^n = 1/8$, $I_5$ corresponds to $3\times 2^{n-3}/2^n = 3/8$ and so on. This map g($n$) : $\{I_1, I_2, I_3, \ldots, I_n, \ldots I_{2^n-1}\} \to \{ j / 2^n : j = 1, 2, \ldots, 2^n - 1\}$ is canonical, meaning that it is exactly the same for any of the Cantor set. Hence, the natural ordering follows the simple rule $I_k < I_l$, if and only if, the corresponding image g($n$)($I_k$) < g($n$) ($I_l$). Thus, we can conclude that $\{J_1, J_2, J_3, \ldots, J_n, \ldots\}$ is ordered in exactly the same way and so $I_j < I_k$ if and only if $J_j < J_k$. We now claim that $f$ is increasing on the complement of $C_k$ in [0, 1] = $\bigcup_{n=1}^{\infty} I_n$. Let $x$, $y$ be in [0, 1] − $C_k$ be such that $x < y$. If $x$ and $y$ are in some $I_k$, then since $f$ is by definition increasing on $I_k$, $f(x) < f(y)$. Suppose now $x$ is in $I_j$ and $y$ is in $I_l$. Then $x < y$ implies that $I_j < I_l$. This is easily seen by taking a positive integer $m$ such that $\max(j, l) \leq 2^m - 1$ and consider the ordering map g($m$) at the $m$-th stage of the construction of the Cantor set. Hence $J_j < J_l$. Therefore, $f(x) < f(y)$ since $f(x) \in J_j$ and $f(y) \in J_l$. We have thus shown that $f$ is increasing on the complement of $C_k$ in [0, 1]. Now for any $y$ in [0, 1] − $C_0$ = $\bigcup_{n=1}^{\infty} J_n$, $y \in J_k$ for some $k$. Since $f$ maps $I_k$ onto $J_k$, there exists $x$ in $I_k$ such that $f(x) = y$. Hence $f$ maps the the complement of $C_k$ in [0, 1] onto the complement of $C_0$ in [0, 1].

***The function $f$ maps $C_k$ into $C_0$.***

We shall next show that $f$ maps $C_k$ into $C_0$. For $x = 0$, $f(x) = 0$ by definition. We now assume $x \neq 0$ and $x \in C_k$. Recall then that

$$f(x) = \sup\{ f(y) : y < x \text{ and } y \in \bigcup_{n=1}^{\infty} I_n\} = \sup\{ f(y) : y < x \text{ and } y \in [0, 1] - C_k\} \ .$$

Suppose that $f(x) \notin C_0$. Then for some integer $l$, $f(x) \in J_l$ and since $J_l = f(I_l)$, there exists $x_0$ in $I_l$ such that $f(x_0) = f(x)$. Then since $I_l$ is open there exists $y_0$ in

$I_l$ with $y_0 < x_0$ such that $f(y_0) < f(x_0) = f(x)$. Thus, by the definition of supremum, there exists $y'$ in $[0, 1] - C_k$ with $y' < x$ and $f(y_0) < f(y') \le f(x) = f(x_0)$. Since $f$ is increasing on $[0, 1] - C_k$, $y_0 < y' < x$. Then since $y_0 \in I_l$ and so for all $y$ in $I_l$, $y < x$ for otherwise if there exists $z$ in $I_l$ with $z > x$, then $x$ would belong to $(y_0, z) \subseteq I_l \subseteq [0, 1] - C_k$, contradicting $x \in C_k$. Now since $I_l$ is open, there exists $x'$ in $I_l$ such that $x' > x_0$. Thus, $f(x') > f(x_0) = f(x)$. Also since $x' < x$, $f(x') \le \sup\{ f(y): y < x \text{ and } y \in [0, 1] - C_k\} = f(x)$, contradicting $f(x') > f(x)$. We can thus deduce that $f(x)$ is in $C_0$. Therefore, this shows that $f$ maps $C_k$ into $C_0$.

***The function f is strictly increasing on* [0, 1].**

Next we shall show that $f$ is (strictly) increasing on [0, 1]. We have already shown that $f$ is increasing on $[0, 1] - C_k$. Thus if $x, y$ are in $[0, 1] - C_k$. and $x < y$, then $f(x) < f(y)$. Suppose now $x \in C_k$ and $y \notin C_k$ and $x < y$. Then for any $z \in \bigcup_{n=1}^{\infty} I_n$, $z < x$ implies that $z < y$. Therefore, since $c$ and $z$ are in $[0, 1] - C_k$, $f(z) < f(y)$. Hence, $f(x) = \sup\{ f(z): z < x \text{ and } z \in \bigcup_{n=1}^{\infty} I_n\} \le f(y)$. Now since $f(x) \in C_0 = [0, 1] - \bigcup_{n=1}^{\infty} J_n$, $f(x) \ne f(y)$ and so $f(x) < f(y)$.

Suppose now $x \in C_k$ and $y \notin C_k$ and $x > y$. Then $f(y) \le \sup\{ f(z): z < x \text{ and } z \in [0, 1] - C_k\} = f(x)$. Again since $f(x) \ne f(y)$, we must have $f(x) > f(y)$.

Suppose both $x$ and $y$ are in $C_k$ and $x < y$. This time we shall use the property of the Cantor set here. Because $C_k$ is nowhere dense, the intersection $(x, y) \cap ([0, 1] - C_k) \ne \varnothing$. Therefore, there exists $z \in [0, 1] - C_k$ such that $x < z < y$. By what we have just proved $f(x) < f(z)$ and $f(z) < f(y)$. Therefore, we can conclude that $f(x) < f(y)$. Hence, we have shown that $f$ is strictly increasing on [0, 1].

***The function f is onto and maps $C_k$ onto $C_0$.***

Now we shall show that $f$ is onto. Since $f$ maps the complement of $C_k$ in [0, 1] onto the complement of $C_0$ in [0, 1], it is sufficient to show that $f$ maps $C_k$ onto $C_0$. By examining the definition of $f$ we can consider a similar function mapping $C_0$ into $C_k$ which is the inverse of $f$. We are going to use this inverse function to construct a pre image of $y$ in $C_0$ under $f$. For $y = 0$ in $C_0$, by definition of $f$, $f(0) = 0$ and 0 is also in $C_k$. For a fixed $y \ne 0$ in $C_0$, define the following

$$x = \sup\{ f^{-1}(z) : z < y \text{ and } z \in \bigcup_{n=1}^{\infty} J_n\} .$$

Note that this is well defined because $\bigcup_{n=1}^{\infty} J_n$ is in the range of $f$, the set $\{ f^{-1}(z) : z < y \text{ and } z \in \bigcup_{n=1}^{\infty} J_n\}$ is non-empty and bounded above by 1 and so the supremum exists by the completeness property of **R**. Note that we also have

$$x = \sup\{ f^{-1}(z) : z < y \text{ and } z \in \bigcup_{n=1}^{\infty} f(I_n)\} .$$

Essentially the same argument for showing that for any $l \ne 0$ in $C_k$, $f(l)$ is in $C_0$, applies here to conclude that $x \in C_k$. Now we claim that $f(x) = y$.

*Note that* $\{f^{-1}(z) : z < y \text{ and } z \in \bigcup_{n=1}^{\infty} f(I_n)\}$

$$= \{x' : f(x') < y \text{ and } f(x') \in \bigcup_{n=1}^{\infty} f(I_n)\}$$

$$= \{x' : f(x') < y \text{ and } x' \in \bigcup_{n=1}^{\infty} I_n\} .$$

Therefore, $x = \sup\{x' : f(x') < y \text{ and } x' \in \bigcup_{n=1}^{\infty} I_n\}$ . We now claim that for any $z'$ in $\bigcup_{n=1}^{\infty} I_n$ ,

$$z' < x \Leftrightarrow f(z') < y. \quad \text{-------------------------- (*)}$$

This is deduced as follows. Let $z'$ be in $\bigcup_{n=1}^{\infty} I_n$.

$z' < x = \sup\{x' : f(x') < y \text{ and } x' \in \bigcup_{n=1}^{\infty} I_n\}$

$\Leftrightarrow$ there exists $z_0$ *in* $\{x' : f(x') < y \text{ and } x' \in \bigcup_{n=1}^{\infty} I_n\}$ such that $z' < z_0 \leq x$

$\Rightarrow$ there exists $z_0$ *in* $\{x' : f(x') < y \text{ and } x' \in \bigcup_{n=1}^{\infty} I_n\}$ such that $f(z') < f(z_0) < y$

$\Rightarrow f(z') < y.$

Conversely, if $z'$ in $\bigcup_{n=1}^{\infty} I_n$ and $f(z') < y$ *, then by definition of* $x$*,* $z' \leq x$ *and so since* $z' \in [0, 1] - C_k$ *,* $z' < x$*. This proves our claim.*

Therefore,

$$\{f(z') : z' < x \text{ and } z' \in \bigcup_{n=1}^{\infty} I_n\} = \{f(z') : f(z') < y \text{ and } z' \in \bigcup_{n=1}^{\infty} I_n\}$$

$$= \{y' : y' < y \text{ and } y' \in \bigcup_{n=1}^{\infty} J_n\}.$$

Thus,

$$f(x) = \sup\{f(z') : z' < x \text{ and } z \in \bigcup_{n=1}^{\infty} I_n\} = \sup\{y' : y' < y \text{ and } y' \in \bigcup_{n=1}^{\infty} J_n\} = y .$$

This is seen as follows. Obviously, $f(x) \leq y$.

If $f(x) < y$, then there exists $y_0$ in $\bigcup_{n=1}^{\infty} J_n$ such that $f(x) < y_0 < y$ since both $f(x)$ and $y$ are in $C_0$ and $C_0$ is nowhere dense. Therefore, there exists $x_0$ in $\bigcup_{n=1}^{\infty} I_n$ with $f(x) < y_0 = f(x_0) < y$. Since $f$ is increasing , $x < x_0$. But by (*), $f(x_0) < y$ implies that $x_0 < x$ contradicting $x < x_0$. Therefore, $f(x) = y$. *Thus we have shown that* $f$ maps $C_k$ onto $C_0$ and consequently $f$ is onto..

We have thus shown that $f$ is a strictly increasing function mapping the closed and bounded interval [0, 1] onto itself and so by Theorem 3 of *Inverse Function and Continuity*, $f$ is continuous on [0, 1]. This completes the proof of Lemma 1.

**Example 5. Riemann integrable function of a continuous function need not necessarily be Riemann Integrable.**

Let g: $[0, 1] \to \mathbf{R}$ be defined by $g(x) = 0$ if $x \notin C_0$ and $g(x) = 1$ if $x \in C_0$ . Let $f$: $[0, 1] \to [0,1]$ be the continuous strictly increasing bijective function mapping a Cantor set of positive measure $C_k$ onto the Cantor set of measure zero $C_0$ given by

Lemma 1. Then $g$ is Riemann integrable, $f$ is continuous but $g \circ f$ is not Riemann integrable.

**Proof.** The function g is obviously bounded. Since $[0, 1] - C_0 = \bigcup_{n=1}^{\infty} J_n$ is a countable disjoint union of open interval and g is zero on each of these open intervals, g is continuous on $[0, 1] - C_0$. The function g is discontinuous at every point in $C_0$. This is seen as follows. Take $\varepsilon = 1/2$. For any $\delta > 0$ and any $x$ in $C_0$, $(x - \delta, x + \delta) \cap [0,1] \not\subseteq C_0$ because $C_0$ is nowhere dense and so there exists $x_\delta \in (x - \delta, x + \delta) \cap [0,1] - C_0$ . Hence, $|f(x_\delta) - f(x)| = |0 - 1| = 1 > \varepsilon = 1/2$. This implies that $f$ is discontinuous at $x$ in $C_0$ . Since $C_0$ is of measure zero, by Lebesgue theorem, g is Riemann integrable. Take any $k$ such that $0 < k < 1$, for instance, $k = 1/2$ . Then $C_k$ is of positive measure. The function $f$ defined by Lemma 1 is a continuous bijection of [0, 1] onto [0, 1] and maps the Cantor set of measure $k$ , $C_k$ , bijectively onto $C_0$. Then

$$g \circ f(x) = \begin{cases} 0, & \text{if } x \notin C_k \\ 1, & \text{if } x \in C_k \end{cases} .$$

Thus $g \circ f$ is constant on the complement of $C_k$ *in* [0, 1] which is a disjoint union $\bigcup_{n=1}^{\infty} I_n$ of open intervals and so $g \circ f$ is continuous on each of these intervals and so $g \circ f$ is continuous on $\bigcup_{n=1}^{\infty} I_n = [0, 1] - C_k$ . As before we can check that $g \circ f$ is discontinuous at any point $x$ in $C_k$ . Take again $\varepsilon = 1/2$. Then for any $\delta > 0$ and any $x$ in $C_k$ , $(x - \delta, x + \delta) \cap [0,1]$ contains a point $x_\delta$ not in $C_k$ because $C_k$ is nowhere dense. Thus $| g \circ f(x_\delta) - g \circ f(x)| = |0 - 1| = 1 > \varepsilon = 1/2$. This means $g \circ f$ is discontinuous at any point in $C_k$ . Therefore, by Lebesgue's Theorem, since $C_k$ is of positive measure , $g \circ f$ is not Riemann integrable.

There are other examples: one example would be to take g to be the function defined on [0, 1] such that g($x$) = 0 for all $x$, such that $0 \le x < 1$ and g($x$) =1 when $x = 1$ and $f$ to be a function on [0, 1] such that on each of the disjoint interval $I_n$ the graph of $f$ is a *'U'* - shape graph with limit tending to 1 at both end points. Then the composite $g \circ f$ would be the same as the composite in the above example.

Example 5 inspires the next example.

**Example 6.** Let g: $[0, 1] \to \mathbf{R}$ be defined by $g(x) = 0$ if $x \notin C_k$ for $k = 1/2$ and $g(x) = 1$ if $x \in C_k$. Let $f^{-1}$: $[0, 1] \to [0,1]$ be the inverse of the continuous strictly increasing bijective function $f$, given by Lemma 1, which maps a Cantor set of positive measure $C_k$ onto the Cantor set of measure zero $C_0$. Then $g$ is not Riemann integrable, $f^{-1}$ is continuous but $g \circ f^{-1}$ is Riemann integrable.

The next question that we would ask is: What about composite of Lebesgue integrable functions? This will be discussed next.