The function \( f \) is defined by

\[
f(x) = \begin{cases} 
  \frac{2}{3}x^3 + \frac{1}{3}, & x > 1 \\
  x^2 \sin\left(\frac{\pi}{2x}\right), & -1 \leq x \leq 1 \text{ and } x \neq 0 \\
  x^3 - 1, & x < -1 \\
  0, & x = 0
\end{cases}
\]
(a) For $x < -1$, $f(x) = x^3 - 1 < -2$.
    Also, for $x < -1$, $x^3 - 1 < -2 \Leftrightarrow x < -1$.
    Thus $f$ maps $(-\infty, -1)$ onto $(-\infty, -2)$.
    (Because for any $y < -2$, we can take $x = \sqrt[3]{y + 1}$ so that $f(x) = y$)

    Also, for $-1 \leq x \leq 1$, $-1 \leq f(x) \leq 1$.
    This is seen as follows.

For $-1 \leq x \leq 1$ and $x \neq 0$, $|f(x)| = \left| x^2 \sin \left( \frac{\pi}{2x} \right) \right| \leq x^2 \leq 1$.
Also we know $f(0) = 0$.
Thus $-1 \leq f(x) \leq 1$. Therefore, $f(1) = 1$ is the absolute maximum of $f$ on $[-1, 1]$ and $f(-1) = -1$ is the absolute minimum of $f$ on $[-1, 1]$.
Assuming that $f$ is continuous on $[-1, 1]$ (as we shall show in part (d) below),
by the Intermediate Value Theorem)
$f$ maps the interval $[-1,1]$ onto $[-1,1]$.

Finally for $x > 1$, $f(x) = \frac{2}{3}x^3 + \frac{1}{3} > 1$.
And for any $y > 1$, we can take
\[ x = \sqrt[3]{\frac{3y - 1}{2}} > 1 \] so that $f(x) = y$.
Hence $f$ maps $(1, \infty)$ onto $(1, \infty)$.

Hence the range of $f$ is
$(-\infty, -2) \cup [-1, 1] \cup (1, \infty) = (-\infty, -2) \cup [-1, \infty)$.
(b) By part (a) \( \text{Range}(f) = (-\infty, -2) \cup [-1, \infty) \neq \mathbb{R} = \text{codomain}(f) \), therefore \( f \) is not surjective.

(c) (i) By part (a)

1 is in the image of \([-1, 1]\) under \( f \).

Thus, to find the preimage we need to solve the equation

\[
x^2 \sin\left(\frac{\pi}{2x}\right) = 1 \quad \text{for } x \text{ in } [-1, 1]-\{0\}.
\]

For \( x \neq 0 \) and \(-1 < x < 1\), \( \left|x^2 \sin\left(\frac{\pi}{2x}\right)\right| \leq x^2 < 1 \).

Since we know \( f(1) = 1 \), and \( f(-1) < 0 \), \( x = 1 \).
(ii) From part (a)
- 2 is not in the range of \( f \).
Thus, the solution of \( f(x) = -2 \) does not exist.
Therefore, there is no value of \( x \) such that \( f(x) = -2 \)

(d) When \( x < -1 \), \( f(x) = x^3 - 1 \), \( f \) is continuous on \( (-\infty, -1) \).
When \( -1 < x < 1 \) and \( x \neq 0 \), \( f(x) = x^2 \sin\left(\frac{\pi}{2x}\right) \).
Since \( x^2 \sin\left(\frac{\pi}{2x}\right) \) is continuous on \( (-1, 0) \) and on \( (0, 1) \), \( f \) is continuous on the union of these two intervals.
When \( x > 1 \), \( f(x) \) is a polynomial function and so it is continuous for \( x > 1 \).
Thus it remains to check if \( f \) is continuous at \( x = -1, 0 \) or 1.
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 \sin\left(\frac{\pi}{2x}\right) = 1^2 \sin\left(\frac{\pi}{2}\right) = 1
\]
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{2}{3}x^3 + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = 1 = f(1)
\]
Therefore, \( \lim_{x \to 1} f(x) = f(1) \) and so \( f \) is continuous at \( x = 1 \).

Now \( \lim_{x \to (-1)^-} f(x) = \lim_{x \to (-1)^-} x^3 - 1 = -2 \)
\[
\lim_{x \to (-1)^+} f(x) = \lim_{x \to (-1)^+} x^2 \sin\left(\frac{\pi}{2x}\right) = 1^2 \sin\left(-\frac{\pi}{2}\right) = -1
\]

Thus the left and the right limits of \( f \) at \( x = -2 \) are not the same and so \( f \) is not continuous at \( x = -1 \).
Now \( \lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{\pi}{2x}\right) = 0 \) by the Squeeze Theorem.

Since \( f(0) = 0 \), \( f \) is continuous at \( x = 0 \). Hence \( f \) is continuous at \( x \) for all \( x \neq -1 \).

(e) \( f \) is differentiable at \( x = 1 \). This is seen as follows.

\[
\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x^2 \sin\left(\frac{\pi}{2x}\right) - 1}{x - 1} = \lim_{x \to 1^-} \frac{2x \sin\left(\frac{\pi}{2x}\right) - \frac{\pi}{2} \cos\left(\frac{\pi}{2x}\right)}{1} \quad \text{by L’ Hôpital’s Rule}
\]

\[= 2\]
\[
\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{2}{3}x^3 + \frac{1}{3} - 1
\]

\[
= \frac{2}{3} \lim_{x \to 1^+} \frac{x^3 - 1}{x - 1} = 2.
\]

Thus \( f \) is differentiable at \( x = 1 \) and \( f \ ' (1) = 2 \).

(f) Note that \( f \) is an odd function on the interval \([-1, 1]\), since \( f(-x) = x^2 \sin(-\pi/(2x)) = -x^2 \sin(\pi/(2x)) = -f(x) \).

\[
\int_{-1}^{0} f(x)dx = -\int_{1}^{0} f(-t)dt \quad \text{where} \quad t = -x
\]

\[
= \int_{1}^{0} f(t)dt = -\int_{0}^{1} f(t)dt
\]

Therefore,

\[
\int_{-1}^{1} f(x)dx = \int_{0}^{1} f(x)dx + \int_{-1}^{0} f(x)dx = 0.
\]
Question 2

(a) \( \lim_{x \to \infty} \frac{61x^7 + 2x^3 + 1}{907x^7 + 7x^3 + 5x^2 + 7} \)

\[
= \lim_{x \to \infty} \frac{61 + \frac{2}{x^4} + \frac{1}{x^7}}{907 + \frac{7}{x^4} + \frac{5}{x^5} + \frac{7}{x^7}} = \frac{61 + 0 + 0}{907 + 0 + 0 + 0} = \frac{61}{907}. 
\]

(b) \( \lim_{x \to 0} \frac{\sqrt{7x^2 + 121} - 11}{14x^2} \)

\[
= \lim_{x \to 0} \frac{7x^2}{14x^2(\sqrt{7x^2 + 121} + 11)} \\
= \lim_{x \to 0} \frac{1}{2(\sqrt{7x^2 + 121} + 11)} = \frac{1}{44}.
\]
(c) \( \lim_{x \to \infty} \frac{x^5}{e^{x^2}} = \lim_{x \to \infty} \frac{5x^4}{2xe^{x^2}} \)

\[ = \frac{5}{2} \lim_{x \to \infty} \frac{x^3}{e^{x^2}} = \frac{5}{2} \lim_{x \to \infty} \frac{3x^2}{2xe^{x^2}} \]

\[ = \frac{15}{4} \lim_{x \to \infty} \frac{x}{e^{x^2}} = \frac{15}{4} \lim_{x \to \infty} \frac{1}{2xe^{x^2}} = 0 \]

by repeated use of L’Hôpital’s rule and \( \lim_{x \to \infty} 2xe^{x^2} = \infty \) so that the limit of the reciprocal function \( \lim_{x \to \infty} \frac{1}{2xe^{x^2}} \) is 0.
(d) \[ \lim_{{x \to 0}} \frac{\sin(\tan(x))}{\tan(\sin(x))} \]

\[ = \lim_{{x \to 0}} \frac{\cos(\tan(x)) \sec^2(x)}{\sec^2(\sin(x)) \cos(x)} \quad \text{by L’ Hôpital’s rule} \]

\[ = \frac{\cos(\tan(0)) \sec^2(0)}{\sec^2(\sin(0)) \cos(0)} = 1 \]
(e) Let \( y = (e^{x^3} + 3x^2)^{(1/x^2)} \).

Then \( \ln(y) = \frac{1}{x^2} \ln(e^{x^3} + 3x^2) \).

Now \( \lim_{x \to 0} \ln(y) = \lim_{x \to 0} \frac{\ln(e^{x^3} + 3x^2)}{x^2} \)

\[ = \lim_{x \to 0} \frac{3x^2 e^{x^3} + 6x}{2x(e^{x^3} + 3x^2)} \]

\[ = \frac{3}{2} \frac{0 + 2}{1 + 0} = 3 \]

Therefore, \( \lim_{x \to 0} y = \lim_{x \to 0} e^{\ln(y)} = e^{\lim_{x \to 0} \ln(y)} = e^3 \)
Question 3

(a) \[ \int \frac{dx}{(x^2 + 2)(x^2 + 3)} = \int \left( \frac{1}{x^2 + 2} - \frac{1}{x^2 + 3} \right) dx \]

\[= \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{x}{\sqrt{2}} \right) - \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x}{\sqrt{3}} \right) + C \]
(b) \[ \int \sin^{-1}(4x) \, dx = x \sin^{-1}(4x) - \int 4x \frac{1}{\sqrt{1 - 16x^2}} \, dx \]
by integration by parts

\[ = x \sin^{-1}(4x) + \frac{1}{8} \int \frac{-32x}{\sqrt{1 - 16x^2}} \, dx \]

\[ = x \sin^{-1}(4x) + \frac{1}{4} \sqrt{1 - 16x^2} + C \]
by change of variable or substitution
using e.g. \( u = \sqrt{(1 - 16x^2)} \)
(c) \[\int e^{2x} \sin(5x)\,dx = \frac{1}{2} e^{2x} \sin(5x) - \frac{1}{2} \int e^{2x} 5 \cos(5x)\,dx\]

\[= \frac{1}{2} e^{2x} \sin(5x) - \frac{5}{2} \left[ \frac{1}{2} e^{2x} \cos(5x) + \frac{1}{2} \int e^{2x} 5 \sin(5x)\,dx \right]\]

\[= \frac{1}{2} e^{2x} \sin(5x) - \frac{5}{4} e^{2x} \cos(5x) - \frac{25}{4} \int e^{2x} \sin(5x)\,dx\]

by integration by parts.

Therefore,

\[\frac{29}{4} \int e^{2x} \sin(5x)\,dx = \frac{1}{2} e^{2x} \sin(5x) - \frac{5}{4} e^{2x} \cos(5x) + C\]

Thus

\[\int e^{2x} \sin(5x)\,dx = \frac{2}{29} e^{2x} \sin(5x) - \frac{5}{29} e^{2x} \cos(5x) + C'.\]
Therefore,

\[
\int_{0}^{\frac{\pi}{5}} e^{2x} \sin(5x) \, dx = \frac{1}{29} \left[ 2e^{2x} \sin(5x) - 5e^{2x} \cos(5x) \right]_{0}^{\frac{\pi}{5}}
\]

\[
= \frac{1}{29} \left[ 5e^{0} \cos(0) - 5e^{\frac{2\pi}{5}} \cos(\pi) \right]
\]

\[
= \frac{5}{29} \left( 1 + e^{\frac{2\pi}{5}} \right).
\]
\[(d) \quad \int \frac{x + 3}{x^2 + 2x + 4} \, dx = \int \left( \frac{1}{2} \frac{2x + 2}{x^2 + 2x + 4} + \frac{2}{x^2 + 2x + 4} \right) \, dx \]

\[= \frac{1}{2} \ln(x^2 + 2x + 4) + 2 \int \frac{1}{(x + 1)^2 + 3} \, dx \]

\[= \frac{1}{2} \ln(x^2 + 2x + 4) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C \]

Therefore,

\[\int_0^2 \frac{x + 3}{x^2 + 2x + 4} \, dx = \left[ \frac{1}{2} \ln(x^2 + 2x + 4) + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) \right]_0^2 \]

\[= \frac{1}{2}((\ln(12) - \ln(4)) + \frac{2}{\sqrt{3}}(\tan^{-1}(\sqrt{3}) - \tan^{-1}(\frac{1}{\sqrt{3}})) \]

\[= \frac{1}{2} \ln(3) + \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{1}{2} \ln(3) + \frac{\pi}{3\sqrt{3}} \]
(a) Let \( f(x) = 3x - 2 \). First note that
\[
|f(x) - 1| = |3x - 2 - 1| = 3|x - 1|.
\]
Therefore,
\[
given \text{ any } \varepsilon > 0, \text{ take } \delta = \varepsilon / 3
\]
Thus,
\[
0 < |x - 1| < \delta \Rightarrow |f(x) - 1| = 3|x - 1| < 3\delta = \varepsilon.
\]
Therefore, by the definition of limit,
\[
\lim_{x \to 1} f(x) = 1.
\]
(b) First note that 
\( f \) is differentiable at \( x \) for \( x \) in \((-\infty, -\pi)\) or \((\pi, \infty)\) 
since on these intervals the function is the same 
as \( \sin(x) \) and \( \sin(x) \) is differentiable on these 
intervals.

Now for \( x \) such that \(-\pi < x < \pi\), \( f(x) \) is given 
by a polynomial and any polynomial is 
differentiable on the interval \((-\pi, \pi)\).

Then a necessary condition for \( f \) to be 
differentiable at \( \pi \) is that \( f \) be continuous at \( \pi \).

That is, \( \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi) \).
Now \( \lim_{x \to \pi} f(x) = \lim_{x \to \pi} ax^3 + bx = a\pi^3 + b\pi = f(\pi) \) and \( \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} \sin(x) = \sin(\pi) = 0. \)

And so our first condition is

\[ a\pi^2 + b = 0 \quad \text{------------------ (1)} \]

Since we know the derivative of \( \sin(x) \) is \( \cos(x) \), that is

\[
\lim_{y \to x} \frac{\sin(y) - \sin(x)}{y - x} = \cos(x),
\]

\[
\lim_{x \to \pi^+} \frac{f(x) - f(\pi)}{x - \pi} = \lim_{x \to \pi^+} \frac{\sin(x) - 0}{x - \pi}
\]

\[
= \lim_{x \to \pi} \frac{\sin(x) - \sin(\pi)}{x - \pi} = \cos(\pi) = -1.
\]
Similarly,
\[
\lim_{x \to \pi^-} \frac{f(x) - f(\pi)}{x - \pi} = \lim_{x \to \pi^-} \frac{ax^3 + bx - (a\pi^3 + b\pi)}{x - \pi} = 3a\pi^2 + b.
\]
Therefore, in addition to equation (1), for differentiability at \(\pi\), we must have
\[
3a\pi^2 + b = -1 \quad \text{------------------ (2)}
\]
Solving (1) and (2) gives \(b = \frac{1}{2}\) and \(a = -\frac{1}{2\pi^2}\).

For differentiability at \(-\pi\), we get the same equations (1) and (2) above. Thus the same values for \(a\) and \(b\) above will guarantee differentiability at \(-\pi\) too.
(c) Let \( f(x) = 2x^3 + 3x + 1 - 3 \sin(x) \cos(x) \)
\[ = 2x^3 + 3x + 1 - \frac{3}{2} \sin(2x). \]
Then \( f'(x) = 6x^2 + 3 - 3 \cos(2x) \)
\[ = 6x^2 + 3(1 - \cos^2(x) + \sin^2(x)) \]
\[ = 6(x^2 + \sin^2(x)). \]
Therefore, \( f'(x) > 0 \) for \( x \neq 0. \)

Since \( f \) is continuous on \( \mathbb{R}, \) \( f \) is continuous at \( x = 0. \)
Thus \( f \) is increasing on \((-\infty, 0] \) and on \([0, \infty) \) and so it is increasing on \( \mathbb{R}. \) Therefore, \( f \) is injective.
• Now $f(0) = 1 > 0$ and $f(-\pi) = -2\pi^2 - 3\pi + 1 < 0$.
• Therefore, by the Intermediate Value Theorem, there exists a point $c$ in $\mathbb{R}$ such that $f(c) = 0$.
• That is, $f$ has a root in $\mathbb{R}$.
• Since $f$ is injective, it has exactly one real root.
Question 5

Observe that

\[
f(x) = \begin{cases} 
\frac{2x|x|}{1+x^2}, & x < 1 \\
\frac{1}{x}, & x \geq 1
\end{cases}
\]

We note that

1. \( f \) is continuous on \((1, \infty)\) because \( f \) is a rational function on \((1, \infty)\).
2. \( f \) is continuous on \((-\infty, 1)\) because \( f \) is a product of a rational function and \( |x| \) and \( |x| \) is a continuous function.
Now \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x} = 1 = f(1) \) and
\( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{2x|x|}{1 + x^2} = 1. \)

Therefore, \( \lim_{x \to 1} f(x) = f(1) \) and so \( f \) is continuous at \( x = 1. \)
Thus \( f \) is continuous on \( \mathbb{R}. \)

Then
\[
f'(x) = \begin{cases} 
-\frac{4x}{(1 + x^2)^2}, & x < 0 \\
\frac{4x}{(1 + x^2)^2}, & 0 < x < 1 \\
-\frac{1}{x^2}, & x > 1 
\end{cases}
\] \( \quad \text{------- (1)} \)
\[ f''(x) = \begin{cases} 
4 \frac{(3x^2 - 1)}{(1 + x^2)^3}, & x < 0 \\
-4 \frac{(3x^2 - 1)}{(1 + x^2)^3}, & 0 < x < 1 \\
\frac{2}{x^3}, & x > 1 
\end{cases} \]
(a) For \( x < 0 \), \(-4x > 0\) and so from (1) \[
\frac{-4x}{(1 + x^2)^2} > 0 \quad \text{for } x \in (-\infty, 0)
\]
since \((1 + x^2) > 0\).

Thus \(f\) is increasing on the interval \((-\infty, 0]\) since \(f\) is continuous at \(x = 0\).

Now for \( x \) in \((0, 1)\) \(f'(x) = \frac{4x}{(1 + x^2)^2} > 0\).

Therefore, \(f\) is increasing on \([0, 1]\) since \(f\) is continuous at \(x = 0\) and at \(x = 1\).
Thus \(f\) is increasing on the interval \((-\infty, 1]\).
For \( x > 1 \), \( f'(x) = -\frac{1}{x^2} < 0 \).

Thus \( f \) is decreasing on the interval \( [1, \infty) \) since \( f \) is continuous at \( x = 1 \).

(b) Now \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x} = 0 \)

and so the line \( y = 0 \) is a horizontal asymptote of the graph of \( f \).

Next we check the following limit.

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -\frac{2x^2}{1 + x^2} = \lim_{x \to -\infty} -\frac{2}{1 + \frac{1}{x^2}} = -2
\]

Therefore, the line \( y = -2 \) is another horizontal asymptote of the graph of \( f \).
(c) When \( x < -\frac{1}{\sqrt{3}} \), from (2),

\[
  f''(x) = 12 \frac{(x^2 - \frac{1}{3})}{(1 + x^2)^3} > 0 \text{ since } (x^2 - \frac{1}{3}) > 0.
\]

Hence the graph of \( f \) is concave upward on the interval \((-\infty, -\frac{1}{\sqrt{3}})\).

Also from (2), when \( -\frac{1}{\sqrt{3}} < x < 0 \),

\[
  f''(x) = 12 \frac{(x^2 - \frac{1}{3})}{(1 + x^2)^3} < 0.
\]

Therefore, the graph of \( f \) is concave downward on the interval \((-\frac{1}{\sqrt{3}}, 0)\).
Again from (2), for \( 0 < x < \frac{1}{\sqrt{3}} (< 1) \),

\[
f''(x) = -12 \frac{(x^2 - \frac{1}{3})}{(1 + x^2)^3} > 0 \text{ since } (x^2 - \frac{1}{3}) < 0.
\]

Therefore, the graph of \( f \) is concave upward on \((0, \frac{1}{\sqrt{3}})\).

For \( \frac{1}{\sqrt{3}} < x < 1 \), \( f''(x) = -12 \frac{(x^2 - \frac{1}{3})}{(1 + x^2)^3} < 0 \)

and therefore the graph of \( f \) is concave downward on \((\frac{1}{\sqrt{3}}, 1)\).

Finally for \( x > 1 \), \( f''(x) = \frac{2}{x^3} > 0 \) and so the graph of \( f \) is concave upward on \((1, \infty)\).
(d) Since from part (a) \( f \) is increasing on \((-\infty, 1]\) and decreasing on \([1, \infty)\), \( f \) has a relative maximum value at \( x = 1 \).

- Indeed the relative maximum value is \( f(1) = 1 \).

Since \( f \) is increasing on \((-\infty, 1]\), \( f \) has no relative minimum in \((-\infty, 1]\).

- Likewise since \( f \) is decreasing on \([1, \infty)\), \( f \) has no relative minimum value in \([1, \infty)\).

Therefore \( f \) has no relative minimum value.
(e) 
From part (c), there are changes of concavity before and after the following points in the graph:

\((-\frac{1}{\sqrt{3}}, f(-\frac{1}{\sqrt{3}})) = (-\frac{1}{\sqrt{3}}, -\frac{1}{2}))\),

\((0, f(0)) = (0, 0),\)

\((\frac{1}{\sqrt{3}}, f(\frac{1}{\sqrt{3}})) = (\frac{1}{\sqrt{3}}, \frac{1}{2}))\)

and \((1, f(1)) = (1, 1)\).

Therefore, these are the points of inflection.
(f) The graph of \( f \) (not drawn to scale)
(a) \( g(x) = \int_{-x}^{x^3} \frac{1}{1 + \sin^2(2t) + t^2} \, dt \)

\[= \int_{0}^{x^3} \frac{1}{1 + \sin^2(2t) + t^2} \, dt + \int_{-x}^{0} \frac{1}{1 + \sin^2(2t) + t^2} \, dt\]

\[= \int_{0}^{x^3} \frac{1}{1 + \sin^2(2t) + t^2} \, dt - \int_{0}^{x} \frac{1}{1 + \sin^2(2t) + t^2} \, dt\]

\[= F(x^3) - F(-x)\]

where \( F(x) = \int_{0}^{x} \frac{1}{1 + \sin^2(2t) + t^2} \, dt \)
Therefore, 
\[ g'(x) = F'(x^3) \cdot 3x^2 - F'(-x) \cdot (-1) \]
by the Chain Rule

\[ = \frac{3x^2}{1 + \sin^2(2x^3) + x^6} + \frac{1}{1 + \sin^2(-2x) + x^2} \]
by the FTC
(b) (i)

Since \( k(x) = \int_{1}^{x} \frac{1}{\sqrt{1 + t^4}} \, dt \), by the FTC,

\[
k'(x) = \frac{1}{\sqrt{1 + x^4}} > 0 \quad \text{since} \quad 1 + x^4 > 0.
\]

Therefore, \( k \) is increasing on the whole of \( \mathbb{R} \).
Thus \( k \) is injective.
(ii) \(( k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))}\).

- Need to know the value of \(k^{-1}(0)\).

\[k^{-1}(0) = x \iff k(x) = 0 \iff \int_1^x \frac{1}{\sqrt{1 + t^4}} dt = 0\]

Since \(k(1) = \int_1^1 \frac{1}{\sqrt{1 + t^4}} dt = 0\) and \(k\) is injective, \(x = 1\).

Therefore,

\[(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))} = \frac{1}{k'(1)} = \frac{1}{\sqrt{2}} = \sqrt{2}\]
Let \( f(x) = \int_{a}^{x} h(t) dt + \int_{h(a)}^{h(x)} h^{-1}(s) ds - xh(x) \)

- We want to show that this is a constant function.
- At this point, it is reasonable to make some assumption that allows us to proceed to show that this is true under this assumption. We then assume that \( h \) is differentiable. This is to make sure that the function \( f \) is differentiable.
- Notice that \( f \) is continuous on \([a, b]\).
With this assumption, by the Fundamental Theorem of Calculus, $f$ is indeed differentiable and

$$f'(x) = h(x) + h^{-1}(h(x))h'(x) - (h(x) + xh'(x))$$

$$= h(x) + x h'(x) - h(x) - x h'(x) = 0$$

Therefore,

$$f(x) = C$$ for some constant $C$.

Thus $C = f(a) = -ah(a)$.

Hence

$$\int_{a}^{x} h(t)dt = xh(x) - ah(a) - \int_{h(a)}^{h(x)} h^{-1}(s)ds$$

In particular

$$\int_{a}^{b} h(t)dt = bh(b) - ah(a) - \int_{h(a)}^{h(b)} h^{-1}(s)ds$$
The solution to this part without assuming the differentiability of $h$ is given at the end of the page
(ii) Let \( h(x) = \sqrt{1 + (x - 1)^{\frac{1}{3}}} \) for \( x \) in \([0,1]\).

\[ h(x) = y \quad \text{if and only if} \]

\[ 1 + (x - 1)^{\frac{1}{3}} = y^2 \iff (x - 1)^{\frac{1}{3}} = y^2 - 1 \quad \text{so that} \]

\[ x = (y^2 - 1)^3 + 1 = y^6 - 3y^4 + 3y^2 \]

Therefore \( h^{-1}(y) = y^6 - 3y^4 + 3y^2 \)

Now \( h(0) = 0 \) and \( h(1) = 1 \).

• Before we use part (i), note that in part (i) we only require that \( h \) be differentiable on \((a, b)\).
Hence by part (i),

\[ \int_0^1 \sqrt{1 + (x - 1)^{\frac{1}{3}}} \, dx = h(1) - \int_0^1 (y^6 - 3y^4 + 3y^2) \, dy \]

\[ = 1 - \left[ \frac{1}{7}y^7 - \frac{3}{5}y^5 + y^3 \right]_0^1 = 1 - (1 + \frac{1}{7} - \frac{3}{5}) = \frac{16}{35}. \]
Or use substitution $u = 1 + (x - 1)^{\frac{1}{3}}$.

Then $x = u^3 - 3u^2 + 3u$.

$$\int_0^1 \sqrt{1 + (x - 1)^{\frac{1}{3}}} \, dx = \int_0^1 (3u^\frac{5}{2} - 6u^\frac{3}{2} + 3u^\frac{1}{2}) \, du$$

$$= 3 \left[ \frac{2}{7}u^\frac{7}{2} - \frac{4}{5}u^\frac{5}{2} + \frac{2}{3}u^\frac{3}{2} \right]_0^1$$

$$= \frac{6}{7} - \frac{12}{5} + 2 = \frac{16}{35}$$