Closed and bounded sets, Heine-Borel Theorem, Bolzano-Weierstrass Theorem, Uniform Continuity and Riemann Integrability

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The aim of this note is to establish that any function that is continuously defined on a closed and bounded interval is also uniformly continuous. This is actually a consequence of the notion of compactness. We shall give explanation of some of the less familiar concepts involved.

Definition 1. A *metric space* (M, d) is a set M together with a metric function $d : M \times M \to \mathbf{R}$ satisfying the following: For all x, y and z in M,

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$ and
4. $d(x, y) \leq d(x, z) + d(z, y)$.

Then for each $r > 0$, and each x in M, the open balls $B(x, r) = \{y \in M: d(y, x) < r\}$ are crucial in defining a new object. Any subset of M is said to be open if and only if it is a union of a family of open balls or if it is empty. We can easily show that this collection of all open sets form a topology on M, called the metric topology in the following sense.

Definition 2. A *topology* on a set X is a family $\mathcal{T}$ of subsets of X satisfying

1. $\emptyset$, $X \in \mathcal{T}$,
2. If $\mathcal{S}$ is any subfamily of $\mathcal{T}$, then the union $\cup \mathcal{S} = \cup\{ U: U \in \mathcal{S} \} \in \mathcal{T}$,
3. If $U_1, U_2, \ldots, U_n \in \mathcal{T}$, then the finite intersection $U_1 \cap U_2 \cap \ldots \cap U_n \in \mathcal{T}$.

Example. 1. $(\mathbf{R}, d)$ with $d(x, y) = |x - y|$.
2. For integer $n > 1$, $(\mathbf{R}^n, d)$ with *the Euclidean metric* $d(x, y) = \sqrt{(\sum_{i=1,\ldots,n} (x_i - y_i)^2)}$

Definition 3. An *open cover* of a set A in $\mathbf{R}$ (topological space), is a family $\mathcal{U}$ of open intervals (open sets) such that the union $\cup \mathcal{U} = \cup\{ U: U \in \mathcal{U} \} \supseteq A$.

Example. For each x in the closed interval $[a, b]$ and for each natural number n, let $B(x, 1/n) = (x-1/n, x+1/n)$. Then $B(x, 1/n)$ is open. Then the family or collection of open sets $\mathcal{U} = \{ B(x, 1/2): x \in [a, b]\}$ is an open cover for $[a, b]$. This collection is most effective when we can select a finite subset of $\mathcal{U}$ which also covers $[a, b]$. It is indeed the case that we can do this but not for any other subsets of $\mathbf{R}$ and for any open cover. Hence the following definition.

Definition 4. A subspace A of a topological space X is *compact*, if and only if, any open cover $\mathcal{C}$ of A have a finite *subcover*, that is, a finite subfamily (subset) $\mathcal{B}$ of $\mathcal{C}$ such that $A \subseteq \cup\{ U: U \in \mathcal{B} \}$.

A subset A of **R** is compact if and only if any open cover $\mathcal{C}$ of A by open intervals has a finite subcover, that is a finite subfamily (subset) $\mathcal{B}$ of $\mathcal{C}$ such that $A \subseteq \cup\{ U: U \in \mathcal{B} \}$.

Example.

1. **R** (with usual metric topology) is not compact. Take for example $\mathcal{C} = \{(n, n+2): n \in \mathbf{Z} \}$. Then $\mathcal{C}$ covers **R** but does not have a finite subcover.
2. $A = \{1, 1/2, 1/3, 1/n, \ldots\} \subseteq \mathbf{R}$ is not compact. Take $\mathcal{C} = \{(1/(n+1), 1/(n-1)): n \in \mathbf{Z} \} \cup (1/2,3/2)$. $\mathcal{C}$ covers A but does not have a finite subcover.
3. $A = \{0, 1, 1/2, 1/3, 1/n, \ldots\} \subseteq \mathbf{R}$ is compact.
   Proof. Suppose $\mathcal{C}$ is an open cover covering A . Then $0 \in U$ for some U in $\mathcal{C}$. Then since $1/n$ converges to 0 as n tends to infinity, there exists an integer N such that for all $n > N$, $1/n \in U$. Now for $n = 1, \ldots, N$, $1/n \in Un$. Hence $\{U_1, \ldots, U_N, U\}$ is a finite subfamily that covers A too.

   The next notion is the notion of boundedness. A subset A of a metric space (M, d) is said to be *bounded*, if and only if, there exists a real positive number k such that $d(x, y) < k$ for all x, y in A.

Theorem 5 (Heine-Borel). A subset A of **R** is compact if and only if A is closed and bounded.

Before we proceed with the proof. The following results will contribute to it and are important and useful on their own merits

Theorem 6. A compact subset A of a metric space (M, d) is bounded.

Proof. We are going to use an open cover of A by open balls. A typical open ball centred at x in A and of radious $\delta > 0$ is the set $B(x, \delta) = \{y \in M: d(y, x) < \delta\}$. For each a in A, let $U(a) = B(a, 1)$ be the unit ball centred at a. Then $\mathcal{C} = \{U(a) : a \in A\}$ is an open cover for A. Since A is compact, $\mathcal{C}$ has a finite subcover, say $\{ U(a_i) : i = 1, \ldots, n\}$. Let $k = \max \{d(a_i, a_j): 1 \leq i, j \leq n \}$. Therefore, for any x, y in A, $x \in a_i$ and $y \in a_j$ for some $1 \leq i, j \leq n$, $d(x, y) \leq d(x, a_i) + d(a_i, a_j) + d(a_j, y) < 2 + k$ and so A is bounded.

Theorem 7. Any compact subset A of a metric (*Hausdorff*) space is closed.
Proof. The proof uses the fact that any two distinct points x, y in a metric space can be separated in the sense that there are two disjoint open sets U and V with $x \in U$ and $y \in V$. We can take for instance, $U = B(x, d(x, y)/2)$ and $V = B(y, d(x, y)/2)$. This is the concept of a *Hausdorff space*. Let us fix an element y not in A. Then for each a in A, we have an open set $U(a)$ and and an open set $V(a)$ such that $a \in U(a)$, $y \in U(a)$ *and* $U(a) \cap V(a) = \varnothing$. Then $\mathcal{C} = \{U(a) : a \in A\}$ is an open cover for A. Since A is compact $\mathcal{C}$ has a finite subcover, say $\{U(a_i) : i = 1, \ldots, n\}$. Then if we let $U = \cup \{ U(a_i) : i = 1, \ldots, n\}$ and $V = \cap \{V(a_i) : i = 1, \ldots, n\}$. Then U is a finite union of open sets and is therefore open and V is a finite intersection of open sets and is also open. Also $A \subseteq U$ and $U \cap V = \varnothing$. This is because $U \cap V \subseteq \cup \{U(a_i) \cap V : i = 1, \ldots, n\} \subseteq \cup$

$\{U(a_i) \cap V(a_i) : i = 1, \ldots, n\} = \emptyset$. Hence V is an open set containing y and $V \subseteq$ complement of A since $V \cap A \subseteq U \cap V = \emptyset$. Hence each point y in the complement of A has an open set contained entirely in the complement of A, therefore the complement of A is a union of open sets and so is open. Therefore, A is closed. This completes the proof.

Proof of Theorem 5.

($\Rightarrow$) Suppose A is a compact subset of **R**. Then by Theorem 6, A is bounded and is closed by Theorem 7.

($\Leftarrow$) Suppose A is a closed and bounded subset of **R**. Then $A \subseteq [a, b]$ for some closed and bounded interval $[a, b]$. If we can show that $[a, b]$ is compact, then A being a closed subspace of a compact space is therefore compact. (This is because any open cover for A together with the complement of A constitute an open cover for $[a, b]$ and if $[a, b]$ is compact there will be a finite subcover for A.) Now let $\mathcal{C}$ be open cover for $[a, b]$. Define $c = sup\ \{ x \in [a, b] :$ a finite subfamily of $\mathcal{C}$ covers $[a, x]\}$. Obviously the set $\{ x \in [a, b] :$ a finite subfamily of $\mathcal{C}$ covers $[a, x]\}$ is not empty since a belongs to it and is clearly bounded above by b. Therefore, by the completeness property of **R**, c exists. Then $c > a$. Why? $a \in$ open set U in $\mathcal{C}$ since $\mathcal{C}$ is an open cover for $[a, b]$. Therefore, there exists a $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U$. Thus for any $a < y < a + \delta$, $[a, y] \subseteq U$ and so $y \in \{ x \in [a, b] :$ a finite subfamily of $\mathcal{C}$ covers $[a, x]\}$. Therefore, by the definition of supremum $c \geq y > a$. We shall show next that $c = b$. Now we have $a < c \leq b$. Thus there exists an open set U in $\mathcal{C}$ such that $c \in$ open set U. Then there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq U$. Take any d such that $c - \delta < d < c$. Then $[d, c] \subseteq U$. Now since $d < c$, by the definition of supremum, there exists a point z in $\{ x \in [a, b] :$ a finite subfamily of $\mathcal{C}$ covers $[a, x]\}$ such that $d < z \leq c$. Hence there is a finite subfamily of $\mathcal{C}$ covering $[a, z]$ and since $[a, z] \cup [d, c] = [a, c]$ and $[d, c] \subseteq U$, this subfamily together with U constitute a finite subfamily covering $[a, c]$. Hence, $c \in \{ x \in [a, b] :$ a finite subfamily of $\mathcal{C}$ covers $[a, x]\}$. Hence, $c = b$. This is because if $c < b$, then as above we can take a point e this time in $(c, b) \cap (c - \delta, c + \delta) \subseteq U$. Thus $c < e < b$ and $[c, e] \subseteq U$, and so since there is a finite subfamily of $\mathcal{C}$ covering $[a, c]$ and $U \in \mathcal{C}$, this subfamily and U constitute a finite subfamily covering $[a, e]$. Thus $e \in \{ x \in [a, b] :$ a finite subfamily of $\mathcal{C}$ covers $[a, x]\}$. Therefore, $c = sup\{ x \in [a, b] :$ a finite subfamily of $\mathcal{C}$ covers $[a, x]\} \geq e$ contradicting $c < e$. Hence $c = b$ and so there is a finite subfamily covering $[a, b]$ (Why? Reason as above.) and so $[a, b]$ is compact. This completes the proof.

Theorem 8 (Bolzano-Weierstrass). Any bounded sequence in **R** has a convergent subsequence.

We shall give a proof of this theorem that can be adapted to a proof for a bounded sequence in $\mathbf{R}^n$.

Proof. By the Heine -Borel Theorem (Theorem 5), A bounded sequence $\{a_n\}$ in **R** lies inside a compact set, a large closed interval $[c, d]$ Let us use the following

notation for the sequence. Consider $\{a_n\}$ as the image of a function $a : \mathbf{N} \to \mathbf{R}$, where $a(n) = a_n$.
If the image $A = a(\mathbf{N})$ is finite, then there must exist an element y in $a(\mathbf{N})$ such that $a^{-1}(y)$ is infinite. Therefore $\{a_j : j \in a^{-1}(y)\}$ is a convergent constant subsequence.
We now consider the case A is infinite. Then of course A is contained in $[c, d]$.
Consider now the set of *accumulation* point A' of A in $\mathbf{R}$. A point x in $\mathbf{R}$, is an *accumulation* point of A, if any open set containing x contains a point of A distinct from x. Claim that $A' \neq \varnothing$. Suppose $A' = \varnothing$. That means each point x in $[c, d]$ has an open set U_x such that $U_x \cap A$ is finite. Then the family of open sets $\{U_x : x \in [c, d]\}$ covers $[c, d]$. Since $[c, d]$ is compact by the Heine-Borel Theorem, this family has a finite sub family $\{U_n, i = 1, \ldots, n\}$ such that $[c, d] \subseteq U_1 \cup U_2 \cup \ldots \cup U_n$.
Therefore, $A \subseteq A \cap [c, d] \subseteq (U_1 \cap A) \cup (U_2 \cap A) \cup \ldots \cup (U_n \cap A)$. *But* $(U_1 \cap A) \cup (U_2 \cap A) \cup \ldots \cup (U_n \cap A)$ is a union of finite set and so is finite. Hence A being a subset of a finite set must be finite. We have thus arrived at a contradiction since we have started with an infinite A. Take a point x in A'. Then we shall construct a sequence $\{x_j\}$ in A such that $x_i \neq x_j$ for $i \neq j$ *and* $\{x_j\}$ converges to x as j tends to infinity. A consequence of this is that x is in $[c, d]$. Take x_1 in $B(x, 1)$ such that $x_1 \neq x$ and so $d(x_1, x) > 0$. This point x_1 exists by definition of accumulation point. As we shrink the Ball $B(x, 1/n)$, we shall exclude the point x_1. For instance there exists an integer n_2 such that $1/n_2 < d(x_1, x)$, then by virtue of x being an accumulation point of A, there exists x_2 in $B(x, 1/n_2)$ such that $x_2 \neq x$ and so $d(x_2, x) > 0$. Obviously $x_2 \neq x_1$ for otherwise if $x_2 = x_1$ then $d(x_2, x_1) = 0$ *and we have* $d(x_1, x) \leq d(x_1, x_2) + d(x_2, x) < 0 + 1/n_2 = 1/n_2$ contradicting $1/n_2 < d(x_1, x)$. In this way, there exists n_3 such that $1/n_3 < d(x_2, x)$, $x_2, x_1 \notin B(x, 1/n_3)$ and there exists x_3 in $B(x, 1/n_3)$ such that $x_3 \neq x$. *So* inductively, we find integers $1 < n_2 < n_3 \ldots$ and points $x_1, x_2, x_3, \ldots$ such that $x_j \in B(x, 1/n_j)$, $x_i \neq x_j$ for $i \neq j$. Then obviously $\{x_j\}$ converges to x as j tends to infinity since for any open set U containing x there exists an integer J such that $x \in B(x, 1/n_J) \subseteq U$. Therefore, for all $j > J$, $x_j \in B(x, 1/n_j) \subseteq B(x, 1/n_J) \subseteq U$.
Now based on this sequence we are going to construct a subsequence of $\{a_n\}$ converging to x. Start with x_1, consider $a^{-1}(x_1)$. Choose i_1 in $a^{-1}(x_1)$. Then $a(i_1) = x_1$. Next observe that since not all $a^{-1}(x_j)$ for $j > 1$ can be bounded above by i_1 because otherwise $a^{-1}(\{x_j : j > 1\})$ would be finite which implies that $\{x_j : j > 1\}$ is finite contradicting that $\{x_j : j > 1\}$ is infinite since the sequence $\{x_j\}$ is a sequence of distinct terms. Thus there is a $j_2 > 1$ such that $a^{-1}(x_{j_2})$ is not bounded by i_1. There exists i_2 in $a^{-1}(x_{j_2})$ such that $i_2 > i_1$ and $a(i_2) = x_{j_2}$. Next not all $a^{-1}(x_j)$ for $j > j_2$ can be bounded above by i_2. So there exists $j_3 > j_2$ such that $a^{-1}(x_{j_3})$ is not bounded by i_2. So there exists i_3 in $a^{-1}(x_{j_3})$ such that $i_3 > i_2$ and $a(i_3) = x_{j_3}$. In this way we obtain a subsequence $\{x_{j_n} : n = 1, \cdots, \infty\}$ of $\{x_j\}$ and this subsequence is equal to the subsequence $\{a_{i_n} : n = 1, \cdots, \infty\}$ of $\{a_n\}$. That means $a_{i_n} = x_{j_n}$ for $n = 1, 2, \ldots$. Since $\{x_j\}$ converges to x, *any* subsequence of it also converges to x. Hence, $\{x_{j_n}\}$ converges to x. Therefore, $\{a_{i_n}\}$ also converges to x. This completes the proof.

Remark. The Bolzano-Weierstrass Theorem for bounded sequence in $\mathbf{R}^n$ follows the same proof above by replacing $\mathbf{R}$ by $\mathbf{R}^n$, $[c, d]$ by a large closed disk or ball and using the Heine-Borel Theorem for $\mathbf{R}^n$.
2. We can use the Bolzano-Weierstrass Theorem to prove the Extreme Value Theorem.

A consequence of the compactness of the domain on continuity.

Uniform Continuity

We shall stick to the one variable case. Let D be a subset of **R**.

Definition 9. A function $f: D \to \mathbf{R}$ is said to be *uniformly continuous* if given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any x, y in D, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

The next result is a consequence of the closed and bounded interval being a compact set of **R**.

Notice that uniform continuity implies continuity.

Theorem 9. If the function $f: [a, b] \to \mathbf{R}$ is continuous, then it is also uniformly continuous.

Proof. The most important result we use here is the compactness of $[a, b]$. That means we are going to produce a family of open cover of $[a, b]$. Since f is continuous at each x in $[a, b]$, given $\varepsilon > 0$, there exists a $\delta(x) > 0$ (δ here may depend on x) such that for any y in $[a, b]$, $|y - x| < \delta(x) \Rightarrow |f(y) - f(x)| < \varepsilon/2$. This means whenever y is in the open set $B(x, \delta(x)) = \{z: |z - x| < \delta(x)\} \cap [a, b]$ then $|f(y) - f(x)| < \varepsilon/2$. Therefore the collection $\mathcal{C} = \{B(x, \delta(x)/2): x \in [a, b]\}$ is an open cover for $[a, b]$. Since $[a, b]$ is compact by the Heine-Borel Theorem (Theorem 5), $\mathcal{C}$ has a finite subcover say $\mathcal{B} = \{B(x_1, \delta(x_1)/2), B(x_2, \delta(x_2)/2), \ldots, B(x_n, \delta(x_n)/2),\}$, where n is some positive integer. Now let $\delta = \min\{\delta(x_1)/2, \delta(x_2)/2), \ldots, \delta(x_n)/2)\}$. Take any x, y in $[a, b]$ such that $|y - x| < \delta$. Since $\mathcal{B}$ covers $[a, b]$, $x \in B(x_k, \delta(x_k)/2)$ for some $1 \leq k \leq n$.

Therefore, $|f(x_k) - f(x)| < \varepsilon/2$ ——————————— (1)

Now, let us see how far away from x_k is y.

$|y - x_k| = |y - x + x - x_k| \leq |y - x| + |x - x_k| < \delta + \delta(x_k)/2 \leq \delta(x_k)/2 + \delta(x_k)/2 = \delta(x_k)$.

Hence $y \in B(x_k, \delta(x_k))$ and we have

$$|f(y) - f(x_k)| < \varepsilon/2 . \qquad \text{———————————} \quad (2)$$

Therefore,

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f(x_k) + f(x_k) - f(x)| \\ &\leq |f(y) - f(x_k)| + |f(x_k) - f(x)| \text{ by the triangle inequality} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ by (1) and (2) above.} \end{aligned}$$

Hence, f is uniformly continuous.

This notion of uniform continuity proves useful to tell us that any continuous function on a closed and bounded interval is Riemann integrable.

Theorem 10. If the function $f: [a, b] \to \mathbf{R}$ is continuous, then it is Riemann integrable on $[a, b]$.

Proof. If $f: [a, b] \to \mathbf{R}$ is continuous, then it is also uniformly continuous. Therefore given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, y in $[a, b]$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/(b-a). \qquad \text{—————————} \quad (3)$$

Let $P : a = x_0 < x_1 < x_2 < \dots < x_n = b$ be a partition with norm $||P|| < \delta$ that is, $||P|| = \max\{ |x_i - x_{i-1}| : i = 1, \dots, n \} < \delta$. For $i = 1, \dots, n$, let $M_i = \sup\{ f(x) : x \in [x_{i-1}, x_i]\}$. Then since f is continuous on $[x_{i-1}, x_i]$, for each i, by the Extreme Value Theorem, $M_i = f(c_i)$ for some c_i *in* $[x_{i-1}, x_i]$. Similarly, for each $i = 1, \dots, n$, let $m_i = \inf\{ f(x) : x \in [x_{i-1}, x_i]\}$. Then again by the Extreme Value Theorem, for each $i = 1, \dots, n$, there exists d_i *in* $[x_{i-1}, x_i]$ such that $m_i = f(d_i)$. Then the upper Riemann sum with respect to P is

$$U(P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

and the lower Riemann sum with respect to P is

$$L(P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} f(d_i)(x_i - x_{i-1}).$$

Then the difference,

$$U(P) - L(P) = \sum_{i=1}^{n} (f(c_i) - f(d_i))(x_i - x_{i-1}) = \sum_{i=1}^{n} |f(c_i) - f(d_i)|(x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}). \text{ by (3) since } |c_i - d_i| \le ||P|| < \delta,\ 1 \le i \le n .$$

Therefore, $U(P) - L(P) < \dfrac{\varepsilon}{b-a} \displaystyle\sum_{i=1}^{n}(x_i - x_{i-1}) = \dfrac{\varepsilon}{b-a}(x_n - x_0) = \varepsilon.$

Hence, Riemann's condition holds and so by Theorem 1 in *Riemann Integral and Bounded function,* f is Riemann integrable. This completes the proof.