Closed and bounded sets, Heine-Borel Theorem, Bolzano-Weierstrass Theorem, Uniform Continuity and Riemann Integrability

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The aim of this note is to establish that any function that is continuously defined on a closed and bounded interval is also uniformly continuous. This is actually a consequence of the notion of compactness. We shall give explanation of some of the less familiar concepts involved.

Definition 1. A metric space \((M, d)\) is a set \(M\) together with a metric function \(d : M \times M \to \mathbb{R}\) satisfying the following: For all \(x, y \) and \(z \) in \(M\),
1. \(d (x, y) \geq 0\),
2. \(d (x, y) = 0\) if and only if \(x = y\),
3. \(d (x, y) = d (y, x)\) and
4. \(d (x, y) \leq d (x, z) + d (z, y)\).

Then for each \(r > 0\), and each \(x\) in \(M\), the open balls \(B(x, r) = \{ y \in M : d(y, x) < r \}\) are crucial in defining a new object. Any subset of \(M\) is said to be open if and only if it is a union of a family of open balls or if it is empty. We can easily show that this collection of all open sets form a topology on \(M\), called the metric topology in the following sense.

Definition 2. A topology on a set \(X\) is a family \(\mathcal{T}\) of subsets of \(X\) satisfying
1. \(\emptyset, X \in \mathcal{T}\),
2. If \(\mathcal{S}\) is any subfamily of \(\mathcal{T}\), then the union \(\bigcup \mathcal{S} = \bigcup \{ U : U \in \mathcal{S} \} \in \mathcal{T}\),
3. If \(U_1, U_2, \ldots, U_n \in \mathcal{T}\), then the finite intersection \(U_1 \cap U_2 \cap \ldots \cap U_n \in \mathcal{T}\).

Example. 1. \((\mathbb{R}, d)\) with \(d(x, y) = |x - y|\).
2. For integer \(n > 1\), \((\mathbb{R}^n, d)\) with the Euclidean metric
\[d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\]

Definition 3. An open cover of a set \(A\) in \(\mathbb{R}\) (topological space), is a family \(\mathcal{U}\) of open intervals (open sets) such that the union \(\bigcup \mathcal{U} = \bigcup \{ U : U \in \mathcal{U} \} \supseteq A\).

Example. For each \(x\) in the closed interval \([a, b]\) and for each natural number \(n\), let \(B(x, 1/n) = (x - 1/n, x + 1/n)\). Then \(B(x, 1/n)\) is open. Then the family or collection of open sets \(\mathcal{U} = \{ B(x, 1/2) : x \in [a, b] \}\) is an open cover for \([a, b]\). This collection is most effective when we can select a finite subset of \(\mathcal{U}\) which also covers \([a, b]\). It is indeed the case that we can do this but not for any other subsets of \(\mathbb{R}\) and for any open cover. Hence the following definition.

Definition 4. A subspace \(A\) of a topological space \(X\) is compact, if and only if, any open cover \(\mathcal{C}\) of \(A\) have a finite subcover, that is, a finite subfamily (subset) \(\mathcal{B}\) of \(\mathcal{C}\) such that \(A \subseteq \bigcup \{ U : U \in \mathcal{B} \}\).
A subset \( A \) of \( \mathbb{R} \) is compact if and only if any open cover \( \mathcal{C} \) of \( A \) by open intervals has a finite subcover, that is a finite subfamily (subset) \( \mathcal{E} \) of \( \mathcal{C} \) such that \( A \subseteq \bigcup \{ U : U \in \mathcal{E} \} \).

Example.

1. \( \mathbb{R} \) (with usual metric topology) is not compact. Take for example \( \mathcal{C} = \{(n, n+2) : n \in \mathbb{Z} \} \). Then \( \mathcal{C} \) covers \( \mathbb{R} \) but does not have a finite subcover.

2. \( A = \{1, 1/2, 1/3, 1/n, \ldots \} \subseteq \mathbb{R} \) is not compact. Take \( \mathcal{C} = \{(1/(n+1), 1/(n-1)) : n \in \mathbb{Z} \} \cup \{(1/2,3/2)\}. \mathcal{C} \) covers \( A \) but does not have a finite subcover.

3. \( A = \{0, 1, 1/2, 1/3, 1/n, \ldots \} \subseteq \mathbb{R} \) is compact.

Proof. Suppose \( \mathcal{C} \) is an open cover covering \( A \). Then 0 \( \in \) \( U \) for some \( U \) in \( \mathcal{C} \).

Then since \( 1/n \) converges to 0 as \( n \) tends to infinity, there exists an integer \( N \) such that for all \( n > N, 1/n \in U \). Now for \( n = 1, \ldots, N, 1/n \in U \). Hence \( \{U_1, \ldots, U_N, U\} \) is a finite subfamily that covers \( A \) too.

The next notion is the notion of boundedness. A subset \( A \) of a metric space \((M, d)\) is said to be bounded, if and only if, there exists a real positive number \( k \) such that \( d(x, y) < k \) for all \( x, y \) in \( A \).

Theorem 5 (Heine-Borel). A subset \( A \) of \( \mathbb{R} \) is compact if and only if \( A \) is closed and bounded.

Before we proceed with the proof. The following results will contribute to it and are important and useful on their own merits

Theorem 6. A compact subset \( A \) of a metric space \((M, d)\) is bounded.

Proof. We are going to use an open cover of \( A \) by open balls. A typical open ball centred at \( x \) in \( A \) and of radius \( \delta > 0 \) is the set \( B(x, \delta) = \{ y \in M : d(y, x) < \delta \} \). For each \( a \) in \( A \), let \( U(a) = B(a, 1) \) be the unit ball centred at \( a \). Then \( \mathcal{C} = \{U(a) : a \in A\} \) is an open cover for \( A \). Since \( A \) is compact, \( \mathcal{C} \) has a finite subcover, say \( \{U(a_i) : i = 1, \ldots, n\} \). Let \( k = \max \{d(a_i, a_j) : 1 \leq i, j \leq n\} \). Therefore, for any \( x, y \) in \( A \), \( x \in a_i \) and \( y \in a_j \) for some \( 1 \leq i, j \leq n \), \( d(x, y) \leq d(x, a_i) + d(a_i, a_j) + d(a_j, y) < 2 + k \) and so \( A \) is bounded.

Theorem 7. Any compact subset \( A \) of a metric (Hausdorff) space is closed.

Proof. The proof uses the fact that any two distinct points \( x, y \) in a metric space can be separated in the sense that there are two disjoint open sets \( U \) and \( V \) with \( x \in U \) and \( y \in V \). We can take for instance, \( U = B(x, d(x, y)/2) \) and \( V = B(y, d(x, y)/2) \). This is the concept of a Hausdorff space. Let us fix an element \( y \) not in \( A \). Then for each \( a \) in \( A \), we have an open set \( U(a) \) and an open set \( V(a) \) such that \( a \in U(a), y \in U(a) \) and \( U(a) \cap V(a) = \emptyset \). Then \( \mathcal{C} = \{U(a) : a \in A\} \) is an open cover for \( A \). Since \( A \) is compact \( \mathcal{C} \) has a finite subcover, say \( \{U(a_i) : i = 1, \ldots, n\} \). Then if we let \( U = \bigcup \{U(a_i) : i = 1, \ldots, n\} \) and \( V = \bigcap \{V(a) : i = 1, \ldots, n\} \). Then \( U \) is a finite union of open sets and is therefore open and \( V \) is a finite intersection of open sets and is also open. Also \( A \subseteq U \) and \( U \cap V = \emptyset \). This is because \( U \cap V \subseteq \bigcup \{U(a_i) \cap V : i = 1, \ldots, n\} \subseteq \bigcup \)
Proof.

\{U(a) \cap V(a) : i = 1, \ldots, n\} = \emptyset. Hence \(V\) is an open set containing \(y\) and \(V \subseteq\) complement of \(A\) since \(V \cap A \subseteq U \cap V = \emptyset\). Hence each point \(y\) in the complement of \(A\) has an open set contained entirely in the complement of \(A\), therefore the complement of \(A\) is a union of open sets and so is open. Therefore, \(A\) is closed. This completes the proof.

**Proof of Theorem 5.**

(⇒) Suppose \(A\) is a compact subset of \(\mathbb{R}\). Then by Theorem 6, \(A\) is bounded and is closed by Theorem 7.

(⇐) Suppose \(A\) is a closed and bounded subset of \(\mathbb{R}\). Then \(A \subseteq [a, b]\) for some closed and bounded interval \([a, b]\). If we can show that \([a, b]\) is compact, then \(A\) being a closed subspace of a compact space is therefore compact. (This is because any open cover for \(A\) together with the complement of \(A\) constitute an open cover for \([a, b]\) and if \([a, b]\) is compact there will be a finite subcover for \(A\).) Now let \(C\) be open cover for \([a, b]\). Define \(c = \sup \{ x \in [a, b] : \text{a finite subfamily of } C \text{ covers } [a, x]\}\). Obviously the set \(\{ x \in [a, b] : \text{a finite subfamily of } C \text{ covers } [a, x]\}\) is not empty since \(a\) belongs to it and is clearly bounded above by \(b\). Therefore, by the completeness property of \(\mathbb{R}\), \(c\) exists. Then \(c > a\). Why? \(a\) open set \(U\) in \(C\) since \(C\) is an open cover for \([a, b]\). Therefore, there exists a \(\delta > 0\) such that \((a - \delta, a + \delta) \subseteq U\). Thus for any \(a < y < a + \delta\), \([a, y] \subseteq U\) and so \(y \in \{ x \in [a, b] : \text{a finite subfamily of } C \text{ covers } [a, x]\}\). Therefore, by the definition of supremum \(c \geq y > a\).

We shall show next that \(c = b\). Now we have \(a < c \leq b\). Thus there exists an open set \(U\) in \(C\) such that \(c \in \text{open set } U\). Then there exists \(\delta > 0\) such that \((c - \delta, c + \delta) \subseteq U\). Take any \(d\) such that \(c - \delta < d < c\). Then \([d, c] \subseteq U\). Now since \(d < c\), by the definition of supremum, there exists a point \(z\) in \(\{ x \in [a, b] : \text{a finite subfamily of } C \text{ covers } [a, x]\}\) such that \(d < z \leq c\). Hence there is a finite subfamily of \(C\) covering \([a, z]\) and since \([a, z] \cup [d, c] = [a, c]\) and \([d, c] \subseteq U\), this subfamily together with \(U\) constitute a finite subfamily covering \([a, c]\). Hence, \(c \in \{ x \in [a, b] : \text{a finite subfamily of } C \text{ covers } [a, x]\}\) and \(U \subseteq C\). This subfamily and \(U\) constitute a finite subfamily covering \([a, e]\). Thus \(e \in \{ x \in [a, b] : \text{a finite subfamily of } C \text{ covers } [a, x]\}\). Therefore, \(c = \sup \{ x \in [a, b] : \text{a finite subfamily of } C \text{ covers } [a, x]\}\). This completes the proof.

**Theorem 8 (Bolzano-Weierstrass).** Any bounded sequence in \(\mathbb{R}\) has a convergent subsequence.

We shall give a proof of this theorem that can be adapted to a proof for a bounded sequence in \(\mathbb{R}^n\).

**Proof.** By the Heine-Borel Theorem (Theorem 5), a bounded sequence \(\{a_n\}\) in \(\mathbb{R}\) lies inside a compact set, a large closed interval \([c, d]\). Let us use the following
notation for the sequence. Consider \( \{a_n\} \) as the image of a function \( a : \mathbb{N} \rightarrow \mathbb{R} \), where \( a(n) = a_n \).

If the image \( A = a(\mathbb{N}) \) is finite, then there must exist an element \( y \) in \( a(\mathbb{N}) \) such that \( a^{-1}(y) \) is infinite. Therefore \( \{a_j : j \in a^{-1}(y)\} \) is a convergent constant subsequence. We now consider the case \( A \) is infinite. Then of course \( A \) is contained in \([c, d]\).

Consider now the set of accumulation point \( A' \) of \( A \) in \( \mathbb{R} \). A point \( x \) in \( \mathbb{R} \), is an accumulation point of \( A \) if any open set containing \( x \) contains a point of \( A \) distinct from \( x \). Claim that \( A' \neq \emptyset \). Suppose \( A' = \emptyset \). That means each point \( x \) in \([c, d]\) has an open set \( U_x \) such that \( U_x \cap A \) is finite. Then the family of open sets \( \{U_x : x \in [c, d]\} \) covers \([c, d]\). Since \([c, d]\) is compact by the Heine-Borel Theorem, this family has a finite sub family \( \{U_i, i = 1, \ldots, n\} \) such that \([c, d]\) is a union of finite set and so is finite. Hence \( A \) being a subset of a finite set must be finite. We have thus arrived at a contradiction since we have started with an infinite \( A \). Take a point \( x \) in \( A' \). Then we shall construct a sequence \( \{x_i\} \) in \( A \) such that \( x_i \neq x_j \) for \( i \neq j \) and \( \{x_j\} \) converges to \( x \) as \( j \) tends to infinity. A consequence of this is that \( x \) is in \([c, d]\). Take \( x_1 \) in \( B(x, 1) \) such that \( x_1 \neq x \) and so \( d(x_1, x) > 0 \). This point \( x_1 \) exists by definition of accumulation point. As we shrink the Ball \( B(x, 1/n) \), we shall exclude the point \( x_1 \). For instance there exists an integer \( n_2 \) such that \( 1/n_2 < d(x_1, x) \), then by virtue of \( x \) being an accumulation point of \( A \), there exists \( x_2 \) in \( B(x, 1/n_2) \) such that \( x_2 \neq x \) and so \( d(x_2, x) > 0 \). Obviously \( x_2 \neq x_1 \) for otherwise if \( x_2 = x_1 \), then \( d(x_2, x_1) = 0 \) and we have \( d(x_1, x) \leq d(x_2, x_1) + d(x_2, x) < 0 + 1/n_2 = 1/n_2 \) contradicting \( 1/n_2 < d(x_1, x) \). In this way, there exists \( n_3 \) such that \( 1/n_3 < d(x_2, x) \), \( x_2, x_1 \in B(x, 1/n_3) \) and there exists \( x_3 \) in \( B(x, 1/n_3) \) such that \( x_3 \neq x \). So inductively, we find integers \( l < n_2 < n_3 \) ... and points \( x_1, x_2, x_3, \ldots \) such that \( x_j \in B(x, 1/n_j), x_i \neq x_j \) for \( i \neq j \). Then obviously \( \{x_j\} \) converges to \( x \) as \( j \) tends to infinity since for any open set \( U \) containing \( x \) there exists an integer \( J \) such that \( x \in B(x, 1/n_J) \subseteq U \). Therefore, for all \( j > J \), \( x_j \in B(x, 1/n_J) \subseteq B(x, 1/n_J) \subseteq U \).

Now based on this sequence we are going to construct a subsequence of \( \{a_n\} \) converging to \( x \). Start with \( x_1 \), consider \( a^{-1}(x_1) \). Choose \( i_1 \) in \( a^{-1}(x_1) \). Then \( a(i_1) = x_1 \). Next observe that since not all \( a^{-1}(x_j) \) for \( j > 1 \) can be bounded above by \( i_1 \) because otherwise \( a^{-1}(\{x_j : j > 1\}) \) would be finite which implies that \( \{x_j : j > 1\} \) is finite contradicting that \( \{x_j : j > 1\} \) is infinite since the sequence \( \{x_j\} \) is a sequence of distinct terms. Thus there is a \( j > 1 \) such that \( a^{-1}(x_{j_2}) \) is not bounded by \( i_1 \). There exists \( i_2 \) in \( a^{-1}(x_{j_2}) \) such that \( i_2 > i_1 \) and \( a(i_2) = x_{j_2} \). Next not all \( a^{-1}(x_{j}) \) for \( j > j_2 \) can be bounded above by \( i_2 \). So there exists \( j_3 > j_2 \) such that \( a^{-1}(x_{j_3}) \) is not bounded by \( i_2 \). So there exists \( i_3 \) in \( a^{-1}(x_{j_3}) \) such that \( i_3 > i_2 \) and \( a(i_3) = x_{j_3} \). In this way we obtain a subsequence \( \{x_{j_n} : n = 1, \ldots, \infty\} \) of \( \{x_j\} \) and this subsequence is equal to the subsequence \( \{a_{i_n} : n = 1, \ldots, \infty\} \) of \( \{a_n\} \). That means \( a_{i_n} = x_{j_n} \) for \( n = 1, 2, \ldots \). Since \( \{x_j\} \) converges to \( x \), any subsequence of it also converges to \( x \). Hence, \( \{x_{j_n}\} \) converges to \( x \). Therefore, \( \{a_{i_n}\} \) also converges to \( x \). This completes the proof.

Remark. The Bolzano-Weierstrass Theorem for bounded sequence in \( \mathbb{R}^n \) follows the same proof above by replacing \( \mathbb{R} \) by \( \mathbb{R}^n \), \([c, d]\) by a large closed disk or ball and using the Heine-Borel Theorem for \( \mathbb{R}^n \).

2. We can use the Bolzano-Weierstrass Theorem to prove the Extreme Value Theorem.

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A consequence of the compactness of the domain on continuity.

**Uniform Continuity**

We shall stick to the one variable case. Let $D$ be a subset of $\mathbb{R}$.

**Definition 9.** A function $f : D \to \mathbb{R}$ is said to be uniformly continuous if given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $x, y \in D$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

The next result is a consequence of the closed and bounded interval being a compact set of $\mathbb{R}$.

Notice that uniform continuity implies continuity.

**Theorem 9.** If the function $f : [a, b] \to \mathbb{R}$ is continuous, then it is also uniformly continuous.

**Proof.** The most important result we use here is the compactness of $[a, b]$. That means we are going to produce a family of open cover of $[a, b]$. Since $f$ is continuous at each $x$ in $[a, b]$, given $\varepsilon > 0$, there exists a $\delta(x) > 0$ ($\delta$ here may depend on $x$) such that for any $y$ in $[a, b]$, $|y - x| < \delta(x) \Rightarrow |f(y) - f(x)| < \varepsilon/2$. This means whenever $y$ is in the open set $B(x, \delta(x)) = \{z: |z - x| < \delta(x)\} \cap [a, b]$ then $|f(y) - f(x)| < \varepsilon/2$. Therefore the collection $\mathcal{C} = \{B(x, \delta(x)/2): x \in [a, b]\}$ is an open cover for $[a, b]$. Since $[a, b]$ is compact by the Heine-Borel Theorem (Theorem 5), $\mathcal{C}$ has a finite subcover say $\mathcal{E} = \{B(x_1, \delta(x_1)/2), B(x_2, \delta(x_2)/2), \ldots, B(x_n, \delta(x_n)/2),\}$, where $n$ is some positive integer. Now let $\delta = \min \{ \delta(x_1)/2, \delta(x_2)/2, \ldots, \delta(x_n)/2 \}$. Take any $x, y$ in $[a, b]$ such that $|y - x| < \delta$. Since $\mathcal{E}$ covers $[a, b]$, $x \in B(x_k, \delta(x_k)/2)$ for some $1 \leq k \leq n$.

Therefore, $|f(x_k) - f(x)| < \varepsilon/2$  \hspace{1cm} (1)

Now, let us see how far away from $x_k$ is $y$.

$|y - x_k| = |y - x + x - x_k| \leq |y - x| + |x - x_k| < \delta + \delta(x_k)/2 \leq \delta(x_k)/2 + \delta(x_k)/2 = \delta(x_k)$.

Hence $y \in B(x_k, \delta(x_k))$ and we have $|f(y) - f(x_k)| < \varepsilon/2$. \hspace{1cm} (2)

Therefore,

$|f(y) - f(x)| = |f(y) - f(x_k) + f(x_k) - f(x)|$

$\leq |f(y) - f(x_k)| + |f(x_k) - f(x)|$ by the triangle inequality

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$ by (1) and (2) above.

Hence, $f$ is uniformly continuous.

This notion of uniform continuity proves useful to tell us that any continuous function on a closed and bounded interval is Riemann integrable.

**Theorem 10.** If the function $f : [a, b] \to \mathbb{R}$ is continuous, then it is Riemann integrable on $[a, b]$.

**Proof.** If $f : [a, b] \to \mathbb{R}$ is continuous, then it is also uniformly continuous. Therefore given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y$ in $[a, b]$,

$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon(b - a)$. \hspace{1cm} (3)
Let $P: a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a partition with norm $||P|| < \delta$ that is, $||P|| = \max \{ |x_i - x_{i+1}| : i = 1, \ldots, n \} < \delta$. For $i = 1, \ldots, n$, let $M_i = \sup \{ f(x) : x \in [x_{i-1}, x_i] \}$. Then since $f$ is continuous on $[x_{i-1}, x_i]$, for each $i$, by the Extreme Value Theorem, $M_i = f(c_i)$ for some $c_i \in [x_{i-1}, x_i]$. Similarly, for each $i = 1, \ldots, n$, let $m_i = \inf \{ f(x) : x \in [x_{i-1}, x_i] \}$. Then again by the Extreme Value Theorem, for each $i = 1, \ldots, n$, there exists $d_i \in [x_{i-1}, x_i]$ such that $m_i = f(d_i)$. Then the upper Riemann sum with respect to $P$ is

$$U(P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

and the lower Riemann sum with respect to $P$ is

$$L(P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \sum_{i=1}^{n} f(d_i)(x_i - x_{i-1})$$.

Then the difference,

$$U(P) - L(P) = \sum_{i=1}^{n} (f(c_i) - f(d_i))(x_i - x_{i-1}) = \sum_{i=1}^{n} |f(c_i) - f(d_i)|(x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} \frac{\varepsilon}{b-a} (x_i - x_{i-1}) \cdot \text{by (3) since } |c_i - d_i| \leq ||P|| < \delta, 1 \leq i \leq n .$$

Therefore, $U(P) - L(P) < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\varepsilon}{b-a} (x_n - x_0) = \varepsilon$.

Hence, Riemann's condition holds and so by Theorem 1 in Riemann Integral and Bounded function, $f$ is Riemann integrable. This completes the proof.