Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation

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The aim of this article is to discuss absolute continuity in a wider sense. That is, we shall study absolutely continuous function with domain an arbitrary subset of $\mathbb{R}$, not necessary an interval. We examine the properties that an absolutely continuous function must satisfy. It must be a Lusin function, that is, it must satisfy the Lusin condition that it maps null set to null set. It must be a function of bounded variation on bounded subset of its domain. Hence, the property of bounded variation for a function is included in the study. We explore when these properties, when satisfied by a continuous function, are also sufficient to guarantee that the function is absolutely continuous.

Absolute Continuity and Bounded Variation

We recall the definition of an absolutely continuous function.

Definition 1. Let *A* be an arbitrary subset of $\mathbb{R}$. Let $f : A \to \mathbb{R}$ be a finite-valued function. The function *f* is said to be *absolutely continuous* on *A*, if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with end points, $a_i, b_i \in A$,

$$\sum_i |b_i - a_i| < \delta \Rightarrow \sum_i |f(b_i) - f(a_i)| < \varepsilon . \quad \text{---------------------} \quad (1)$$

Plainly, if *f* is absolutely continuous on *A*, *f* is uniformly continuous on *A*.

Note that absolutely continuous function need not be of bounded variation on its domain. For instance, the function $f(x) = x$ is absolutely continuous on $\mathbb{R}$ but is certainly not of bounded variation on $\mathbb{R}$ since it is not bounded on $\mathbb{R}$.

Recall that a function $f: A \to \mathbb{R}$ is said to be of *bounded variation* (with bound *M*), if for any sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with end points, $a_i, b_i \in A$, $\sum_i |f(b_i) - f(a_i)| < M$.

Many results and facts concerning function of bounded variation may be found in the two articles, "*Functions of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*" and "*Functions of Bounded Variation and de La Vallée Poussin's Theorem*".

Lemma 2. Let *A* be an arbitrary subset of $\mathbb{R}$. Suppose $f: A \to \mathbb{R}$ is absolutely continuous on *A*. Let *E* be a bounded subset of *A*. Then *f* is of bounded variation on *E*.

Proof.

Since *f* is absolutely continuous on *A*, it is absolutely continuous on *E*. Therefore, there exists $\delta > 0$ such that for any sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with end points, $a_i, b_i \in E$,

$$\sum_i |b_i - a_i| < \delta \Rightarrow \sum_i |f(b_i) - f(a_i)| < 1. \quad \text{-------------} \quad (2)$$

Let $a = \inf E$ and $b = \sup E$. Partition $[a, b]$ into a finite number of subintervals $\{[x_{i-1}, x_i]; i = 1, \cdots, n\}$ with $a = x_0 < x_1 < x_2 < \cdots < x_n$ and $|x_i - x_{i-1}| < \delta$ for $i = 1, \cdots, n$.

If $[x_{i-1}, x_i] \cap E \neq \varnothing$, pick an $a_i \in [x_{i-1}, x_i] \cap E$. Then for all $x \in [x_{i-1}, x_i] \cap E$, $|f(x) - f(a_i)| < 1$. Hence, $|f(x)| < |f(a_i)| + 1$. It follows that for all *x* in *E*,

$$|f(x)| < \sum_{i, [x_{i-1}, x_i] \cap E \neq \varnothing} |f(a_i)| + n .$$

Thus, *f* is bounded on *E*.

Now we shall show that *f* is of bounded variation on *E*.

Note that the total variation of *f* on each $[x_{i-1}, x_i] \cap E$ is less than or equal to 1.

Now let $M = \sup\{|f(x)| : x \in E\}$.

Take any non-overlapping sequence of closed intervals, $\{[a_i, b_i]\}$ with end points, $a_i, b_i \in E$. Observe that if $[a_i, b_i] \subseteq [x_{k-1}, x_k]$ for some *k*, then $|f(b_i) - f(a_i)| \leq$ total variation of *f* on $[x_{k-1}, x_k] \leq 1$.

If $x_{j-1} < a_i < x_j < b_i$ for some *j* and $[a_i, b_i] \cap \{x_1, x_3, \cdots, x_n\} = x_j$, then $|f(b_i) - f(a_i)| \leq 2M$.

If for some *j*< *k* < *n* and $x_{j-1} \leq a_i < x_j < x_{j+1} < \cdots < x_{k-1} < x_k \leq b_i < x_{k+1}$, then

$$|f(b_i) - f(a_i)| \leq \sum_{m=j}^{k} \left(\text{total variation of } f \text{ on } [x_{m-1}, x_m] \cap E\right) + \sum_{m=j}^{k} 2M .$$

Note that it is possible that $[x_i, x_{i+1}] \cap E = \varnothing$ or a singleton set, in which case we define the total variation of *f* on $[x_i, x_{i+1}] \cap E$ to be zero.

If for some *k*, $x_{n-2} < a_k < x_{n-1} < b_k = x_n$, then

$$|f(b_k) - f(a_k)| \leq \text{total variation of } f \text{ on } [x_{n-1}, x_n] \cap E + 2M .$$

Similarly, if for some *k*, $x_{j-1} \leq a_i < x_j < x_{j+1} < \cdots x_{k-1} < x_k < b_i \leq x_{k+1}$, then

$$|f(b_i) - f(a_i)| \leq \sum_{m=j}^{k} \left(\text{total variation of } f \text{ on } [x_{m-1}, x_m] \cap E\right) + \sum_{m=j}^{k} 2M$$

Since $\{[a_i, b_i]\}$ is a sequence of non-overlapping closed intervals with end points, $a_i, b_i \in E$, it follows that

$$\sum_i |f(b_i) - f(a_i)| \leq \sum_{m=1}^{n} \left(\text{total variation of } f \text{ on } [x_{m-1}, x_m] \cap E\right) + \sum_{m=1}^{n-1} 2M \leq n + (2n-2)M .$$

This proves that *f* is of bounded variation on *E*.

Lusin Condition

A function $f : A \to \mathbb{R}$ is said to satisfy the *Lusin condition*, if *f* maps sets of measure zero to sets of measure zero. The function with this condition is said to be a *Lusin function*. We also call a set of measure zero, a *null set*.

Lemma 3. Let *A* be an arbitrary subset of $\mathbb{R}$. Suppose $f : A \to \mathbb{R}$ is absolutely continuous on *A*. Then *f* is a Lusin function on *A*, i.e., *f* maps sets of measure zero to sets of measure zero.

Proof.

Let $H \subseteq A$ be a set of measure zero, i.e., $m(H) = 0$.

We claim that there exists a $\delta > 0$ and a sequence of non-overlapping intervals $\{I_k\}$ with $\sum_k m(I_k) < \delta$ and covering *H* such that

$$\sum_k \left(\sup\left(f(H \cap I_k)\right) - \inf\left(f(H \cap I_k)\right)\right) < \varepsilon . \text{ ----------------------------- } (1)$$

Since *f* is absolutely continuous, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any sequence of non-overlapping closed intervals, $\{J_k\}$ with end points in *A*,

$$\sum_k m(J_k) < \delta \Rightarrow \sum_k \left| f(d_k) - f(c_k) \right| < \frac{\varepsilon}{2} , \text{ --------------------------- } (2)$$

where $J_k = [c_k, d_k]$.

Since $m(H) = 0$, for any $\varepsilon_1 > 0$, there exists an open set $U_{\varepsilon_1} \supseteq H$ such that $m(U_{\varepsilon_1}) < \varepsilon_1$. Since U_{ε_1} is open, it is a disjoint union of at most a countable number of bounded open intervals, $\{I_i\}$. That is, $U_{\varepsilon_1} = \bigcup_i I_i \supseteq H$ and $\sum_{i=1}^{\infty} m(I_i) < \varepsilon_1$.
Take $\varepsilon_1 = \delta$ as given in (2). Let $I_k = (a_k, b_k)$. Let
$m_k = \inf\left(f(H \cap (a_k, b_k))\right)$ and $M_k = \sup\left(f(H \cap (a_k, b_k))\right)$. By the definition of infimum and supremum, there exists a sequence $\{\ell_{n,k} : n = 1, \cdots\}$ in $H \cap (a_k, b_k)$ such that $\lim_{n\to\infty} f(\ell_{n,k}) = m_k$ and a sequence $\{L_{n,k} : n = 1, \cdots\}$ in $H \cap (a_k, b_k)$ such that $\lim_{n\to\infty} f(L_{n,k}) = M_k$. It follows that

$\left| f(L_{n,k}) - f(\ell_{n,k}) \right| \to M_k - m_k$ as $n \to \infty$. Since $\sum_{i=1}^{\infty} m(I_i) < \varepsilon_1$, it follows that

$\sum_k \left| L_{n,k} - \ell_{n,k} \right| \le \sum_k m(I_k) < \delta$. Therefore, by (2), for any positive integer *N*,

$$\sum_{k=1}^{N} \left| f(L_{n,k}) - f(\ell_{n,k}) \right| < \frac{\varepsilon}{2} .$$

Letting n tends to infinity we get,

$$\sum_{k=1}^{N}\left(M_k - m_k\right) \le \frac{\varepsilon}{2}.$$

Now, letting N tends to infinity we get, $\sum_{k=1}^{\infty}\left(M_k - m_k\right) \le \frac{\varepsilon}{2} < \varepsilon$.

This proves our claim.

Note that $m^*(f(H\cap I_k)) \le \sup\left(f(H\cap I_k)\right) - \inf\left(f(H\cap I_k)\right) = M_k - m_k$.

We have,

$$m^*(f(H)) = m^*(f(H\cap \bigcup_k I_k)) \le \sum_k m^*\left(f(H\cap I_k)\right) \le \sum_k \left(M_k - m_k\right) < \varepsilon \ .$$

Since ε is arbitrary, we conclude that $m^*(f(H)) = 0$. Thus, f is a Lusin function.

This completes the proof of Lemma 3.

We have shown that if A is bounded and $f : A \to \mathbb{R}$ is absolutely continuous on A, then f is continuous of bounded variation and is a Lusin function. Thus, these two properties are indispensable for a function on a bounded set to be absolutely continuous.

Note that not all continuous function of bounded variation, satisfying the Lusin condition, is absolutely continuous, even on bounded domain. Consider the function $f : \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \to \mathbb{R}$ defined by $f(x) = \begin{cases} x, & 0 \le x < \frac{1}{2}, \\ x+1, & \frac{1}{2} < x \le 1 \end{cases}$. This function is continuous of bounded variation on $\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$. Plainly, it is a Lusin function. But it is obviously not an absolutely continuous function.

Banach Zarecki Theorem

The next theorem is usually known as the Banach-Zarecki Theorem.

Theorem 4. Suppose *A* is a closed and bounded subset of $\mathbb{R}$. Suppose $f : A \to \mathbb{R}$ is a finite-valued function, continuous of bounded variation on *A* and is a Lusin function. Then *f* is absolutely continuous on *A*.

Proof.

The function *f* is continuous of bounded variation implies that its total variation function, v_f is continuous and bounded on the domain *A*. (See Theorem 13, *Functions of Bounded Variation and de La Valléee Poussin's Theorem*.) Since *f* is a Lusin function, by Theorem 10 of *Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*, v_f is also a Lusin function. It is sufficient to prove that v_f is an absolutely continuous function on *A*, since v_f is absolutely continuous implies that *f* is absolutely continuous.

Since *A* is bounded and closed, inf *A* and sup *A* belong to *A*. Let *a* = inf *A* and *b* =sup *A*. Then $A \subseteq [a,b]$. The idea is to extend v_f to a continuous function of bounded variation satisfying the Lusin condition on the whole interval [*a*, *b*]. We can then use the result known on such function on a closed interval.

Since *A* is closed and $a,b \in A$, $[a,b]-A=(a,b)-A$ is open and so is at most a countable disjoint union of open intervals. Moreover, its boundary is at most denumerable and is in *A*, as $\partial([a,b]-A)=\partial((a,b)-A)=\partial A \in A$. This implies that each of the open interval in $[a,b]-A$ has its end points in *A*. Suppose (*c*, *d*) is one of these intervals, and since $c,d \in A$, we define a function on (*c*, *d*) linearly by taking the value at *c* to be $v_f(c)$ and the value of *d* to be $v_f(d)$. In this way we extend v_f to all of $[a,b]-A$. We thus obtain a continuous and increasing function $F:[a,b]\to\mathbb{R}$, such that $F(x)=v_f(x)$ for *x* in *A* and *F* is differentiable with finite derivative on $[a,b]-A$. Since v_f is a Lusin function on *A* and *F* is a Lusin function on $[a,b]-A$ because *F* is differentiable with finite derivative on $[a,b]-A$ (see *Theorem 17 of Functions of Bounded Variation and de La Vallee Poussin's Theorem*), *F* is a Lusin function on [*a*, *b*].

Now we appeal to the well-known results about increasing function on a closed interval. Since *F* is increasing on [*a*, *b*], its derivative, $F'(x)$ exists almost everywhere on [*a*, *b*] and is Lebesgue integrable on [*a*, *b*]. Therefore, by Theorem 7 of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, *F* is absolutely continuous on [*a*, *b*]. Hence, *F* is absolutely continuous on *A* and so ν_f is absolutely continuous on *A*.

Hence, Lemma 2, Lemma3 and Theorem 4 give the necessary and sufficient condition for a continuous function on a closed and bounded domain to be absolutely continuous:

Suppose *A* is a closed and bounded subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a finite-valued continuous function. Then *f* is absolutely continuous on *A* if, and only if, *f* is of bounded variation on *A* and is a Lusin function.

Integral of the Derivative and Bound for the Measure of the Image

The next result gives the bound for the measure of a function on the subset, where the derivative exists and is finite for each point in the subset. It is a useful result as we shall use it quite often.

Theorem 5. Let *A* be an arbitrary measurable subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a finite-valued measurable function. Suppose *f* has finite derivative at every point of a measurable set *D* in *A*. Then ${}_A Df(x)$ is measurable on *D* and

$$m^*(f(D)) \le \int_D \left| {}_A Df(x) \right| dx .$$

Proof.

We may assume that every point of *D* is a 2-sided limit point of *D* for non-limit points or only one-sided limit points of *D* constitute at most a denumerable set.

Since ${}_A\underline{D}f(x) \le {}_D\underline{D}f(x) \le {}_D\overline{D}f(x) \le {}_A\overline{D}f(x)$ for every *x* in *D*, and as ${}_A\underline{D}f(x) = {}_A\overline{D}f(x)$ for *x* in *D*, ${}_D Df(x)$ exists and is finite. Hence, by Theorem 11 of *Functions of Bounded Variation and de LA Vallée Poussin's Theorem*, since the restriction of *f* to *D* is measurable, ${}_D Df : D \to \mathbb{R}$ is measurable. Note that ${}_D Df(x) = {}_A Df(x)$ for *x* in *D*.

Suppose *D* is bounded. Take an arbitrary $\varepsilon > 0$.

Let $E_n = \left\{ x \in D : (n-1)\varepsilon \le \left| {}_A Df(x) \right| < n\varepsilon \right\}$. Then $D = \bigcup_{n=1}^{\infty} E_n$. Therefore, $m^*(f(D)) \le \sum_{n=1}^{\infty} m^*\left(f(E_n) \right)$. Now, since for all *x* in E_n, $\left| {}_A Df(x) \right| < n\varepsilon$, by Theorem 12 of *Denjoy Saks Young Theorem for Arbitrary Function*,

$$m^*(f(E_n)) \le n\varepsilon m^*(E_n).$$

It follows that

$$m^*(f(D)) \le \sum_{n=1}^{\infty} n\varepsilon m^*\left(E_n\right) = \sum_{n=1}^{\infty} (n-1)\varepsilon m^*\left(E_n\right) + \varepsilon \sum_{n=1}^{\infty} m^*\left(E_n\right)$$

$$\le \sum_{n=1}^{\infty} \int_{E_n} \left| {}_A Df(x) \right| dx + \varepsilon m^*(D) = \int_D \left| {}_A Df(x) \right| dx + \varepsilon m^*(D).$$

Since ε is arbitrary, it follows that $m^*(f(D)) \le \int_D \left| {}_A Df(x) \right| dx$.

If *D* is not bounded, let $D_n = [-n, n] \cap D$. Then

$$m^*(f(D)) = \lim_{n\to\infty} m^*(f(D_n)) \le \lim_{n\to\infty} \int_{D_n} \left| {}_A Df(x) \right| dx = \int_D \left| {}_A Df(x) \right| dx.$$

The last equality is given by the Lebesgue Monotone Convergence Theorem.

Bound for the Integral of the Derivative and the Total Variation Function

When the function is of bounded variation, we have also the bound for the integral of the derivative of the function over the set of points, where the derivative of the function exists and is finite. The bound is given by the measure of the image of this set under the total variation function of the function.

Theorem 6. Suppose *A* is a measurable subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a finite-valued function of bounded variation on *A*. Then ${}_A Df(x)$ exists and is finite for almost all *x* in *A*. The function ${}_A Df(x)$ is measurable on *A*. Suppose $E \subseteq A$ is a measurable subset of *A* on which ${}_A Df(x)$ exists and is finite. Then ${}_A Df(x)$ is integrable on *E*, i.e., $-\infty < \int_E {}_A Df(x)dx < \infty$. More precisely, $\int_E \left| {}_A Df(x) \right| dx = m^*\left(v_f(E)\right)$.

Let $F = \{x \in A : {}_A Df(x) \text{ exists and is finite}\}$. Then

$$\int_A \left| {}_A Df(x) \right| dx = \int_F \left| {}_A Df(x) \right| dx = m^*\left(v_f(F)\right) \leq m^* v_f(A).$$

Moreover, if *f* is defined and increasing on the interval, [*a*, *b*],

$$\int_a^b {}_A Df(x)dx \leq f(b) - f(a).$$

Proof. Firstly, note that *f* is a measurable function on *A*. By Theorem 5, ${}_A Df(x)$ is measurable on $\{x \in A : {}_A Df(x) \text{ exists and is finite}\}$.

By Theorem 18 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, there is a null set $N \subseteq A$ such that $m(N) = m(f(N)) = m\left(v_f(N)\right) = 0$ and for all $x \in A - N$, ${}_A Df(x)$ and ${}_A Dv_f(x)$ exist finitely or infinitely and that $\left| {}_A Df(x) \right| = {}_A Dv_f(x)$. Moreover, the set $H = \{x \in A : {}_A Df(x) = \pm\infty\}$ has measure zero.

Therefore, $m\left(A - \{x \in A : {}_A Df(x) \text{ exists and is finite}\}\right) = 0$. It follows that ${}_A Df(x)$ is measurable on *A*.

Suppose $E \subseteq A$ is a measurable subset of *A* on which ${}_A Df(x)$ exists and is finite.

We may assume that $E \subseteq A - N$ and further assume that $E \subseteq A - N - H$, since ${}_A Df(x)$ is finite for all *x* in *E*. Thus, $\left| {}_A Df(x) \right| = {}_A Dv_f(x)$ and is finite for all *x* in *E*.

Note that ${}_A Df(x)$ is measurable on *E*.

To prove that ${}_A Df(x)$ is integrable on *E*, it is sufficient to prove that ${}_A Dv_f(x)$ is Lebesgue integrable on *E*. Note that $v_f(x)$ is a bounded function on *E*.

Let $E_n = E \cap [-n, n]$. Then E_n is measurable and bounded. Let $c = \inf E_n$ and $d = \sup E_n$ and so $E_n \subseteq [c, d]$. The total variation function v_f is defined on E_n. We shall extend ν_f to all of [*c*, *d*]. Firstly, we extend ν_f to the boundary of E_n, ∂E_n. Let $x \in \partial E_n$. If $x \in E_n$, define $G(x) = \nu_f(x)$. Suppose $x \in \partial E_n - E_n$. If *x* is a left limit point of E_n, define $G(x) = \sup\{\nu_f(y) : y < x, y \in E_n\}$. If *x* is not a left limit point of E_n, then it must be a right limit point of E_n, define $G(x) = \inf\{\nu_f(y) : y > x, y \in E_n\}$. Thus, *G* so defined is increasing on $E_n \cup \partial E_n$, which is the closure of E_n. The closure of E_n, $\overline{E_n} = E_n \cup \partial E_n \subseteq [c,d]$. Note that both *c* and *d* belong to ∂E_n. Therefore, $[c,d] - \overline{E_n} = (c,d) - \overline{E_n}$ is at most a countable disjoint union of bounded open intervals. The end points of these open intervals are in ∂E_n. Now extend *G* to $[c,d] - \overline{E_n}$ by linear maps joining the values of *G* at the end points of these open intervals. Thus, *G* is defined and increasing on the closed interval, [*c*, *d*], and $G(x) = \nu_f(x)$ for *x* in E_n. Moreover, $G'(x) = {}_A D\nu_f(x)$ for *x* in $E_n = E \cap [-n,n]$.

By Theorem 15 of *Arc Length, Functions of Bounded Variation and Total Variation*, we can decompose *G* = *g* + *h*, where g is absolutely continuous and increasing on [*c*, *d*] and *h* is an increasing singular function on [*c*, *d*].

By Lemma 10 of *Functions of Bounded Variation and Johnson's Indicatrix*, $m^*(G(E_n)) \geq m^*(g(E_n)) = m(g(E_n))$. By Theorem 11 of *Functions of Bounded Variation and Johnson's Indicatrix*, $\int_{E_n} g'(x)dx = m(g(E_n))$ as *g* is increasing and absolutely continuous on [*c*, *d*]. Since $G'(x) = g'(x)$ almost everywhere on [*c*, *d*],

$$\int_{E_n} G'(x)dx = \int_{E_n} g'(x)dx = m(g(E_n)) \leq m^*(G(E_n)) = m^*(\nu_f(E_n)) .$$

It follows that $\int_{E_n} {}_A D\nu_f(x)dx = \int_{E_n} G'(x)dx \leq m^*(\nu_f(E)) \leq m^*(\nu_f(A)) < \infty$.

Therefore, $\int_E {}_A D\nu_f(x)dx = \lim_{n\to\infty} \int_{E_n} {}_A D\nu_f(x)dx \leq m^*(\nu_f(E)) < \infty$.

Hence, ${}_A D\nu_f(x)$ is Lebesgue integrable on *E* and it follows that ${}_A Df(x)$ is Lebesgue integrable on *E*. It follows that

$$\int_E |{}_A Df(x)|\, dx = \int_E {}_A D\nu_f(x)dx \leq m^*(\nu_f(E)). \quad \text{--------------- (1)}$$

Since $|{}_A Df(x)| = {}_A D\nu_f(x)$ and is finite for all *x* in *E*, by Theorem 5,

$$m^*(v_f(E)) \le \int_E \left| {}_A Dv_f(x) \right| dx = \int_E {}_A Dv_f(x) dx \text{ . ------------ (2)}$$

It follows from (1) and (2) that $\int_E \left| {}_A Df(x) \right| dx = m^*\left(v_f(E)\right)$.

Suppose $F = \{x \in A : {}_A Df(x) \text{ exists and is finite}\}$. Then $m(A-F)=0$, ${}_A Df(x)$ is measurable on F and since $m(A-F)=0$, we may extend ${}_A Df(x)$ to a measurable function on A. Hence,

$$\int_A \left| {}_A Df(x) \right| dx = \int_F \left| {}_A Df(x) \right| dx = m^*(v_f(F)) \le m^*(v_f(A)) .$$

If f is defined and increasing on the interval, $[a, b]$, then we have just proved by the above argument that,

$$\int_{[a,b]} {}_A Df(x) dx = \int_{[a,b]} {}_A Dv_f(x) dx \le m^*(v_f([a,b])) \le f(b) - f(a) .$$

This completes the proof of Theorem 6.

Constancy of a Function

Note that in the proof of Theorem 11 of *Functions of Bounded Variation and Johnson's Indicatrix*, we have made use of the special case that if f is an absolutely continuous function on an interval $[a, b]$, then $\int_a^b f'(x) dx = f(b) - f(a)$. This is a standard characterisation of absolutely continuous function on a closed and bounded interval. Theorem 11 cited above is a special case when the absolutely continuous function is also monotone.

We present a proof of this fact.

We shall need the following well-known result.

Theorem 7. Suppose I is an interval and $f : I \to \mathbb{R}$ is an absolutely continuous function on I. Suppose $f'(x) = 0$ almost everywhere on I. Then f is a constant function.

Proof. By Lemma 3, f is a Lusin function. Let E be the set on which $f'(x) = 0$. Then $m(I - E) = 0$. By Theorem 11 of *Arbitrary Function, Limit Superior, Dini*

Derivative and Lebesgue Density Theorem, $m\left(f(E)\right)=0$. Since *f* is a Lusin function, $m\left(f\left(I-E\right)\right)=0$ and so $m\left(f(I)\right)=0$. Since *I* is an interval and *f* is continuous, $f(I)$ is an interval. The only interval with measure zero is the trivial interval consisting of one point, a singleton set. Therefore, *f* is a constant function.

Fundamental Theorem of Calculus for Absolutely Continuous Function on Interval

Theorem 8. Suppose $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function on [*a*, *b*]. Then $\int_a^b f'(x)dx = f(b)-f(a)$.

Proof.

Since *f* is an absolutely continuous function on the bounded interval [*a*, *b*], by Lemma 2, it is of bounded variation on [*a*, *b*]. Hence, by Theorem 6 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, *f* is the difference of two increasing functions, whose derivatives are measurable and Lebesgue integrable on [*a*, *b*]. (See Theorem 3, page 100 of Royden's *Real Analysis, Third Edition*.) It follows that $f'(x)$ is Lebesgue integrable and so $|f'(t)|$ is also Lebesgue integrable over [*a*, *b*].

Let $F(x)=\int_a^x f'(t)dt$. By Lemma 10 of *Convergence of Fourier Series*, almost every point of [*a*, *b*] is a *Lebesgue point* of f'. That is to say, for almost all *x* in [*a*, *b*], $\lim_{t\to 0}\frac{1}{t}\int_0^t \left|f'(x+u)-f'(x)\right|du=0$. Now,

$$\frac{F(x+t)-F(x)}{t}-f'(x)=\frac{1}{t}\int_x^{x+t}\left(f'(u)-f'(x)\right)du=\frac{1}{t}\int_0^t\left(f'(u+x)-f'(x)\right)dt$$

and so $\left|\frac{F(x+t)-F(x)}{t}-f'(x)\right|=\left|\frac{1}{t}\int_0^t\left(f'(u+x)-f'(x)\right)dt\right|\le\frac{1}{|t|}\int_0^t\left|f'(u+x)-f'(x)\right|dt\to 0$.

It follows that if x is a Lebesgue point of f, then $F'(x) = f'(x)$. Hence, $F'(x) = f'(x)$ almost everywhere on [a, b].

To show that F is absolutely continuous on [a, b], it is sufficient to show that $H(x) = \int_a^x |f'(t)|dt$ is absolutely continuous. To show this, we use the following result. Its elegant proof is due to Royden.

Proposition 9. If g is a non-negative function integrable over a set A, then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any set $E \subseteq A$, with $m(E) < \delta$, then $\int_E g(x)dx < \varepsilon$.

Proof. If g is bounded say by M. Then given any ε > 0, take δ to be less than $\frac{\varepsilon}{M}$. Thus, $\int_E g(x)dx \leq \int_E M \leq m(E)M < \varepsilon$. If g is not bounded, let

$g_n(x) = g(x)$, if $g(x) \leq n$ and $g_n(x) = n$, if $g(x) > n$. Then $g_n(x) \to g(x)$ pointwise and $\{g_n(x)\}$ is an increasing sequence of functions. By the Lebesgue Dominated Convergence Theorem, $\int_A g_n(x)dx \to \int_A g(x)dx$ and that, given any $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$,

$$\int_A g(x)dx - \frac{\varepsilon}{2} < \int_A g_n(x)dx \leq \int_A g(x)dx \text{ and } \int_E |g(x) - g_n(x)|dx < \frac{\varepsilon}{2}.$$

(See Theorem 33, *Introduction to Measure Theory*.)

Taking $n = N$, we get

$$\int_E g(x)dx = \int_E (g(x) - g_N(x))dx + \int_E g_N(x)dx < \int_E (g(x) - g_N(x))dx + Nm(E)$$

$$< \frac{\varepsilon}{2} + Nm(E)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$ for any subset E with $m(E) < \delta$ any $\delta < \frac{\varepsilon}{2N}$.

This completes the proof of Proposition 9.

Now, we continue with the proof of Theorem 8.

Thus, applying Proposition 9, for any sequence of non-overlapping closed intervals, $\{I_i\}$, in [a, b], with $\sum_i m(I_i) = m\left(\bigcup_i I_i\right) < \delta$, we have that

$$\int_{\bigcup_i I_i} |f'(x)|\,dx = \sum_i \int_{I_i} |f'(x)|\,dx = \sum_i |H(b_i) - H(a_i)| < \varepsilon ,$$

where $I_i = [a_i, b_i]$ and $H(x) = \int_a^x |f'(t)|\,dt$. Therefore, H is absolutely continuous on [a, b]. Then F is absolutely continuous follows by observing that for the same δ > 0 as above,

$$\sum_i \left| \int_{I_i} f'(x)dx \right| \le \sum_i \int_{I_i} |f'(x)|\,dx < \varepsilon .$$

Let $G(x) = f(x) - F(x)$. Then $G'(x) = 0$ almost everywhere on [a, b]. Thus, by Theorem 7, $G(x) = f(x) - F(x) = c$ for all x in [a, b], since the function, G, is also absolutely continuous because it is the difference of two absolutely continuous functions. Therefore, $c = f(a) - F(a) = f(a) - 0 = f(a)$. It follows that $F(b) = \int_a^b f'(x)dx = f(b) - f(a)$.

Remark.

It follows from Theorem 8 that if $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function on [a, b], then $f(x) = \int_a^x f'(x)dx + f(a)$. That is, f is an indefinite integral of its derivative. We have shown in the proof of Theorem 8, that if $g : [a,b] \to \mathbb{R}$ is Lebesgue integrable, then the indefinite integral, $G(x) = \int_a^x g(t)dt$ is absolutely continuous and that $G'(x) = g(x)$ for almost all x in [a, b]. Thus, f is absolutely continuous if, and only if, it is the indefinite integral of its derivative.

Measurability of Image of Measurable Set

The next result is known for absolutely continuous function on closed and bounded interval.

Theorem 10. Suppose A is a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a finite-valued continuous function and also a Lusin function. Then f maps measurable subsets of A to measurable sets.

Proof.

Take a measurable subset E of A.

Suppose E is bounded. Then by the inner regularity of the Lebesgue measure, there is a F_σ set K in E such that $m^*(E-K)=0$. Now, a F_σ set is a countable union of closed sets and so K is a countable union of closed sets in E. Let $K=\bigcup_n K_n$, where each K_n is a closed subset of E. Since E is bounded, each K_n is closed and bounded and so by the Heine-Borel Theorem, each K_n is compact. Since f is continuous, each $f(K_n)$ is compact and so is measurable. Therefore, $f(K)=\bigcup_n f(K_n)$ is measurable. Since f is a Lusin function, $m^*\left(f(E-K)\right)=0$ and so $f(E-K)$ is measurable. It follows that $f(E)$ is measurable.

Suppose E is not bounded, then let $E_n = E\cap[-n,n]$. Note that E_n is measurable. Hence, $f(E_n)$ is measurable. Therefore, $f(E)=\bigcup_n f(E_n)$ is measurable.

Corollary 11. Suppose A is a non-empty subset of $\mathbb{R}$ and $f:A\to\mathbb{R}$ is an absolutely continuous function. Then f maps measurable subsets of A to measurable sets.

Proof. By Lemma 3, f is a Lusin function. Since f is continuous, by Theorem 10, f maps measurable subsets of A to measurable sets.

Lusin Function

Note that Lusin condition is a necessary condition for absolute continuity.

The next result gives a criterion for a function to be a Lusin function. The most general result may be deduced from Denjoy-Saks-Young Theorem, Theorem 13 of *Denjoy Saks Young Theorem for Arbitrary Function*.

Theorem 12. Let A be an arbitrary subset of $\mathbb{R}$ and $f:A\to\mathbb{R}$ is a finite-valued function. Suppose f has finite derivative at every point of a measurable set D in A, then f is a Lusin function on D.

Proof.

Suppose A is measurable and f is measurable. Suppose $E \subseteq D$ is a set of measure zero in D. Then by Theorem 5, $m^*(f(E)) \le \int_E |{}_A Df(x)|\, dx = 0$ implies that $m(f(E)) = 0$. Hence, f is a Lusin function on D.

In general, since ${}_A Df(x)$ is finite implies that ${}_A D^+ f(x)$ is also finite, by Theorem 5 of *Denjoy Saks Young Theorem for Arbitrary Function*, f is a Lusin function on D.

Remark. In view of Theorem 5 of *Denjoy Saks Young Theorem for Arbitrary Function* and its analogue for the other extreme Dini derivates, Theorem 12 holds true if we replace the condition of f, by "f has any one of its Dini derivates finite at every point of a measurable set D in A".

Integral of the Derivative and the Measure of Images

For increasing function, we have a precise result that the outer measure of the image of the set of points, where the function has finite derivative, is equal to the Lebesgue integral of the derivative of the function, over the same set.

Theorem 13. Suppose A is a non-empty measurable subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is an increasing bounded function. Let B be a measurable subset of A and let $E = \{x \in B : {}_A Df(x) \text{ exists and is finite}\}$. Then ${}_A Df(x)$ is measurable on B and

$$\int_B {}_A Df(x) dx = \int_E {}_A Df(x) dx = m^*\big(f(E)\big) \le m^*\big(f(B)\big).$$

If f is absolutely continuous, then $\int_B {}_A Df(x) dx = m\big(f(B)\big)$.

Proof. Since A is measurable and the set of discontinuities of f is at most denumerable, f is continuous almost everywhere on a measurable set and so is a measurable function on A. Note that f is a measurable function of bounded variation on A. By Theorem 8 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, ${}_A Df(x)$ exists and is finite for almost all x in A. Therefore, E is measurable since B is measurable.

By Theorem 5, $m^*\left(f(E)\right) \le \int_E \left| {}_A Df(x)\right| dx = \int_E {}_A Df(x)dx$. To prove the opposite inequality, we proceed as in Theorem 6 to prove that $\int_E \left| {}_A Df(x)\right| dx = m^*\left(\nu_f(E)\right)$. The proof is almost word for word the same. For convenience we record it below. Let $E_n = E \cap [-n,n]$. Then E_n is measurable and bounded. Let $c = \inf E_n$ and $d = \sup E_n$ and so $E_n \subseteq [c,d]$. Now *f* is defined on E_n. We shall extend *f* to all of [*c*, *d*]. Firstly, we extend f to the boundary of E_n, ∂E_n. Let $x \in \partial E_n$. If $x \in E_n$, define $G(x) = f(x)$. Suppose $x \in \partial E_n - E_n$. If *x* is a left limit point of E_n, define $G(x) = \sup\left\{ f(y) : y < x, y \in E_n\right\}$. If *x* is not a left limit point of E_n, then it must be a right limit point of E_n, define $G(x) = \inf\left\{ f(y) : y > x, y \in E_n\right\}$. Thus, *G* so defined is increasing on $E_n \cup \partial E_n$, which is the closure of E_n. The closure of E_n, $\overline{E_n} = E_n \cup \partial E_n \subseteq [c,d]$. Note that both *c* and *d* belong to ∂E_n. Therefore, $[c,d] - \overline{E_n} = (c,d) - \overline{E_n}$ is at most a countable disjoint union of bounded open intervals. The end points of these open intervals are in ∂E_n. Now extend *G* to $[c,d] - \overline{E_n}$ by linear maps joining the values of *G* at the end points of these open intervals. Thus, *G* is defined and increasing on the closed interval, [*c*, *d*], and $G(x) = f(x)$ for *x* in E_n. Moreover, $G'(x) = {}_A Df(x)$ for *x* in $E_n = E \cap [-n,n]$.

By Theorem 15 of *Arc Length, Functions of Bounded Variation and Total Variation*, we can decompose *G* = *g* + *h*, where g is absolutely continuous and increasing on [*c*, *d*] and *h* is an increasing singular function on [*c*, *d*].

By Lemma 10 of *Functions of Bounded Variation and Johnson's Indicatrix*, $m^*(G(E_n)) \ge m^*\left(g(E_n)\right) = m\left(g(E_n)\right)$. By Theorem 11 of *Functions of Bounded Variation and Johnson's Indicatrix*, $\int_{E_n} g'(x)dx = m\left(g(E_n)\right)$ as *g* is increasing and absolutely continuous on [*c*, *d*]. Since $G'(x) = g'(x)$ almost everywhere on [*c*, *d*],

$$\int_{E_n} G'(x)dx = \int_{E_n} g'(x)dx = m\left(g(E_n)\right) \le m^*(G(E_n)) = m^*(f(E_n)).$$

It follows that $\int_{E_n} {}_A Df(x)dx = \int_{E_n} G'(x)dx \le m^*(f(E_n)) \le m^*(f(E))$.

Therefore, $\int_E {}_A Df(x)dx = \lim_{n\to\infty} \int_{E_n} {}_A Df(x)dx \le m^*(f(E))$ and so

$$m^*\left(f(E)\right) = \int_E {}_A Df(x)dx.$$

Thus, as $m(B-E)=0$,

$$\int_B {}_A Df(x)dx = \int_{E\cup(B-E)} {}_A Df(x)dx = \int_E {}_A Df(x)dx = m^*(f(E)) \le m^*(f(B)).$$

If *f* is absolutely continuous on *A*, then by Lemma 3, *f* is a Lusin function and by Corollary 11, *f* maps measurable subsets to measurable subsets.

$$m^*\big(f(B)\big) = m\big(f(B)\big) = m\big(f(E)\cup f(B-E)\big)$$

$$\le m\big(f(E)\big) + m\big(f(B-E)\big) = m\big(f(E)\big) + 0 = m\big(f(E)\big),$$

since $m\big(f(B-E)\big)=0$ because *B*–*E* is a null set. It follows that $m\big(f(E)\big)=m\big(f(B)\big)$. Therefore, $\int_B {}_A Df(x)dx = m\big(f(B)\big)$.

More generally, we have:

Theorem 14. Suppose *A* is a measurable subset of $\mathbb{R}$ and $f: A\to\mathbb{R}$ is a continuous function of bounded variation on *A*. Suppose *f* is a Lusin function on *A*. Then ${}_A Df(x)$ exists and is finite for almost all *x* in *A*. Moreover, ${}_A Df(x)$ is measurable on *A*. Suppose *B* is a measurable subset of *A*. Let $E=\{x\in B: {}_A Df(x) \text{ exists and is finite}\}$. Then

$$\int_B \left|{}_A Df(x)\right| dx = \int_E \left|{}_A Df(x)\right| dx = m\big(v_f(E)\big) = m\big(v_f(B)\big),$$

where v_f is the total variation of *f*.

Proof.

By Theorem 18 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, there is a null set $N\subseteq A$ such that $m(N)=m(f(N))=m\big(v_f(N)\big)=0$ and for all $x\in A-N$, ${}_A Df(x)$ and ${}_A Dv_f(x)$ exist finitely or infinitely and that $\left|{}_A Df(x)\right| = {}_A Dv_f(x)$. Moreover, the set $F=\{x\in A: {}_A Df(x)=\pm\infty\}$ has measure zero.

We may assume that $B\subseteq A-N$ and $E\subseteq A-N-F$, without affecting the conclusion of the theorem. As *f* is a Lusin function, by Theorem 10 of *Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*, v_f is also a Lusin function. Hence, $\left|{}_A Df(x)\right| = {}_A Dv_f(x)$ and is finite for all *x* in *E*. By

Theorem 5, ${}_A Df(x)$ is measurable on *A–N–F* and hence on *A*. It follows that ${}_A D\nu_f(x)$ is measurable on *A*. Note that

$$m(B\cap N)=m\big(f(B\cap N)\big)=m\big(\nu_f(B\cap N)\big)=0\,.$$

We may assume that $B\subseteq A-N$ and $E\subseteq A-N-F$, without affecting the conclusion of the theorem. Note that both ${}_A Df(x)$ and ${}_A D\nu_f(x)$ are measurable on *B*. Since *f* is continuous, by Theorem 13 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, ν_f is continuous on *A*. Therefore, by Theorem 10, $\nu_f(E)$ and $\nu_f(B)$ are measurable.

By Theorem 13,

$$\int_E {}_A D\nu_f(x)dx=m^*\big(\nu_f(E)\big)=m\big(\nu_f(E)\big)\,.$$

Thus, $\int_E\left|{}_A Df(x)\right|dx=m\big(\nu_f(E)\big)$. Since $m(B-E)=0$ and ν_f is a Lusin function,

$$\int_B\left|{}_A Df(x)\right|dx=\int_E\left|{}_A Df(x)\right|dx=m\big(\nu_f(E)\big)=m\big(\nu_f(B)\big)\,.$$

Remark. 1. If *A* is a bounded measurable subset of $\mathbb{R}$ and $f:A\to\mathbb{R}$ is an absolutely continuous function on *A*, then by Lemma 2 and Lemma 3, *f* satisfies the hypothesis of Theorem 14. Hence, the conclusion of the theorem holds for an absolutely continuous function on a measurable bounded domain.

2. If we assume that $f:A\to\mathbb{R}$ is a just a function of bounded variation on a measurable *A*, the same proof as in Theorem 14 gives the conclusion that

$$\int_B\left|{}_A Df(x)\right|dx=\int_E\left|{}_A Df(x)\right|dx=m^*\big(\nu_f(E)\big)\le m^*\big(\nu_f(B)\big)\,.$$

This is essentially, the conclusion of Theorem 6.

Theorem 15. Suppose *I* is an interval not necessarily bounded. *I* may be open, half open, or closed. Suppose $f:I\to\mathbb{R}$ is a continuous function of bounded variation. Then *f* is absolutely continuous if, and only if, *f* is a Lusin function.

Proof.

If *f* is absolutely continuous, then by Lemma 3, *f* is a Lusin function.

Suppose *f* is a Lusin function. Then by Theorem 14, $f'(x)$ exists and is finite for almost all *x* in *I*. Moreover, $f'(x)$ is Lebesgue integrable. Let *B* be any measurable set in *I*. If $E=\{x\in B: f'(x) \text{ exists and is finite.}\}$, then $m(B-E)=0$, and

$$\int_B\left|f'(x)\right|dx=\int_E\left|f'(x)\right|dx=m^*(\nu_f(E))\,.$$

Now. $m^*(v_f(E)) = m^*(v_f(B))$ because *f* is a Lusin function and so by Theorem 9 of *Function of Bounded Variation on Arbitrary Subset and Johnson's Indicatrix*, v_f is also a Lusin function. Therefore, $\int_B |f'(x)|dx = m^*(v_f(B))$.

By Proposition 9, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any measurable set $E \subseteq I$, with $m(E) < \delta$, then $\int_E |f'(x)|\,dx < \varepsilon$.

Take any sequence of non-overlapping intervals $\{[a_i, b_i]\}$ with end points, $a_i, b_i \in I$, such that $\sum_i |b_i - a_i| < \delta$. Let $I_i = [a_i, b_i]$. Since the total variation function, v_f , is also continuous, for each *i*, $|v_f(b_i) - v_f(a_i)| = m(v_f(I_i))$.

For each *i*,

$$|f(b_i) - f(a_i)| \le |v_f(b_i) - v_f(a_i)| = m(v_f(I_i)) = \int_{I_i} |f'(x)|\,dx.$$

Therefore, since $\sum_i |b_i - a_i| = m\left(\bigcup_i I_i\right) < \delta$, we have then

$$\sum_i |f(b_i) - f(a_i)| \le \sum_i m^*(v_f(I_i)) = \sum_i \int_{I_i} |f'(x)|\,dx = \int_{\bigcup_i I_i} |f'(x)|\,dx < \varepsilon.$$

Hence, *f* is absolutely continuous on *I*.

The next theorem, when applied to strictly monotone function with domain, a closed and bounded interval, is known as the Zarecki Theorem. It affords a quick way of deciding when a continuous function of bounded variation is absolutely continuous, just by examining the measure of the image of the set of points, where *f* is not differentiable finitely.

Necessary and Sufficient Condition for Absolute Continuity

Theorem 16. Suppose *A* is a non-empty closed and bounded subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a continuous function of bounded variation. Then *f* is absolutely continuous if, and only if, $m\left(f\left(\{x \in A : {}_A Df(x) = \pm\infty\}\right)\right) = 0$.

Proof. Suppose *f* is absolutely continuous on *A*. Then by Lemma 3, *f* is a Lusin function. By Theorem 8 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, ${}_{A}Df(x)$ exists and is finite for almost all *x* in *A*. Therefore, the set $\{x \in A : {}_{A}Df(x) = \pm\infty\}$ is of measure zero. Hence, $f\left(\{x \in A : {}_{A}Df(x) = \pm\infty\}\right)$ is a null set and so $m\left(f\left(\{x \in A : {}_{A}Df(x) = \pm\infty\}\right)\right) = 0$.

Conversely, suppose $m\left(f\left(\{x \in A : {}_{A}Df(x) = \pm\infty\}\right)\right) = 0$. By Theorem 18 of *Functions of Bounded Variation and de La Valléee Poussin's Theorem*, there is a subset *N* of *A* such that $m(N) = m(f(N)) = m\left(v_f(N)\right) = 0$ and for all *x* in *A*–*N*, ${}_{A}Df(x)$ exists finitely or infinitely. Let $E = \{x \in A : {}_{A}Df(x) = \pm\infty\}$. Then, *f* is a Lusin function on $A - N - E$. Note that

$$m^*(f(N \cup E)) \le m^*(f(E)) + m(f(N)) = m^*(f(E)) = 0.$$

Therefore, $m(f(N \cup E)) = 0$. It follows that *f* is a Lusin function on *A*. By, Theorem 4, *f* is absolutely continuous.

This completes the proof of Theorem 16.

For continuous function known to satisfy the Lusin condition and is differentiable almost everywhere, we may also examine the integral of the derivative for deciding the absolute continuity of the function.

Theorem 17. Suppose $f : [a,b] \to \mathbb{R}$ is a continuous Lusin function. Suppose $f'(x)$ exists and is finite almost everywhere on $[a,b]$. Then *f* is absolutely continuous on $[a,b]$ if, and only if, f' is Lebesgue integrable on $[a,b]$.

Proof. Suppose *f* is absolutely continuous on [*a*, *b*]. By Lemma 2, *f* is of bounded variation on [*a*, *b*]. By Theorem 6, f' is Lebesgue integrable on [*a*, *b*].

Conversely, suppose f' is Lebesgue integrable on $[a,b]$. Then $|f'|$ is Lebesgue integrable. By Proposition 9, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any set $E \subseteq [a,b]$, with $m(E) < \delta$, we have that $\int_E |f'(x)|\,dx < \varepsilon$.

Note that since *f* is a continuous Lusin function, by Theorem 10, *f* maps measurable subsets of [*a*, *b*] to measurable sets. Take any non-overlapping sequence of closed intervals, $\{I_i\}$ with end points in $[a,b]$. For each integer $i \geq 1$, let $I_i = [a_i, b_i]$. Then

$|f(b_i) - f(a_i)| \leq m\left(f(I_i)\right)$, since f is continuous on $[a_i, b_i]$,

$= m\left(f(E_i)\right)$, where $E_i = \{x \in [a_i, b_i] : f \text{ is differentiable at } x \text{ and is finite.}\}$,

since f is differentiable finitely almost everywhere on $[a, b]$ and f is a Lusin function,

$\leq \int_{E_i} |f'(x)|\, dx$, by Theorem 5. ---------------------------------- (1)

Therefore, if $\sum_i |b_i - a_i| = m\left(\bigcup_i I_i\right) < \delta$, then letting $E = \bigcup_i E_i$,

$$m(E) = m\left(\bigcup_i E_i\right) = m\left(\bigcup_i I_i\right) = \sum_i |b_i - a_i| < \delta \text{ and}$$

$$\sum_i |f(b_i) - f(a_i)| \leq \sum_i \int_{E_i} |f'(x)|\, dx = \int_E |f'(x)|\, dx < \varepsilon .$$

Hence, f is absolutely continuous on $[a, b]$.

Remark.

Theorem 17 is for function on closed and bounded interval, that is, a path connected closed and bounded domain. It is easy to find function with non-closed domain, continuous with finite derivative almost everywhere and satisfies the Lusin condition and yet be non-absolutely continuous.

Corollary 18.

Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function. Suppose $f'(x)$ exists and is finite except for x in a denumerable set in $[a,b]$. Then f is absolutely continuous on $[a,b]$ if, and only if, f' is Lebesgue integrable on $[a,b]$.

Proof. By Theorem 17 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, f is a Lusin function on $E = \{x \in [a,b] : f'(x) \text{ exists and is finite}\}$.

By hypothesis, $[a,b]-E$ is denumerable and so $f([a,b]-E)=0$. It follows that *f* is a Lusin function on [*a*, *b*]. Thus, *f* is a continuous Lusin function. Therefore, by Theorem 17, *f* is absolutely continuous on $[a,b]$ if, and only if, f' is Lebesgue integrable on $[a,b]$.

Corollary 19. Suppose $f:[a,b]\to\mathbb{R}$ is a continuous function. Suppose $f'(x)$ exists and is finite for every *x* in [*a*, *b*]. Suppose f' is Lebesgue integrable or $f'(x)$ is bounded for all *x* in $[a,b]$. Then *f* is absolutely continuous on [*a*, *b*] and $f(x)=f(a)+\int_a^x f'(x)dx$ for *x* in $[a,b]$.

Proof. Since $f'(x)$ exists (finitely) for every *x* in [*a*, *b*], *f* is continuous on [*a*, *b*]. By Theorem 17 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, *f* is a Lusin function on [*a*, *b*]. Hence, *f* is a continuous Lusin function. By Theorem 5, $f'(x)$ is measurable. Suppose $|f'(x)|\le M$ for all *x* in [*a*, *b*]. Then $\int_{[a,b]}|f'(x)|dx\le M(b-a)<\infty$ and so f' is Lebesgue integrable on $[a,b]$. Therefore, by Theorem 16, *f* is absolutely continuous on [*a*, *b*]. By Theorem 8, $f(x)=f(a)+\int_a^x f'(x)dx$ for *x* in $[a,b]$.

Some Observation on Cantor Function.

The generalized Cantor Lebesgue function, $f_{C_1}:[0,1]\to[0,1]$, is continuous, increasing and maps C_1 onto *I* = [0,1], where C_1 is the Cantor set of measure zero. (For the notation, please refer to *The Generalized Cantor Set, Generalized Cantor Lebesgue Function, Canonical Function Mapping Cantor Set to Another, Absolute Continuity, Arc Length and Singular Functions*.) Since it is obviously not a Lusin function, it is not absolutely continuous. Therefore, by Theorem 16, $m\left(f_{C_1}\left(\left\{x\in[0,1]: f_{C_1}{}'(x)=+\infty\right\}\right)\right)\ne 0$. Since $f_{C_1}{}'(x)=0$ for *x* in $[0,1]-C_1$, $m\left(f_{C_1}\left(\left\{x\in C_1: f_{C_1}{}'(x)=+\infty\right\}\right)\right)\ne 0$. It follows that the set of points in C_1, where the derivative is infinite, is non-denumerable. Moreover, the set of points in C_1, where f_{C_1} is not differentiable finitely or infinitely is of the power of the continuum according to Eidswick in "*A Characterization of The Non-differentiability Set of The Cantor Function*, Proc Amer Math Soc, Vol 42, p214-217", although it gets mapped to a set of measure zero.

Take the Lebesgue Cantor like function, $g_{D_{\delta_1},D_{\delta_2}}:[0,1]\to[0,1]$, which maps the Cantor set D_{δ_1} of measure $1-\delta_1$ onto the Cantor set D_{δ_2} of measure $1-\delta_2$. This function is defined in *The Generalized Cantor Set, Generalized Cantor Lebesgue Function, Canonical Function Mapping Cantor Set to Another, Absolute Continuity, Arc Length and Singular Functions*. It is strictly increasing and continuous on [0,1]. With $\delta_1 = 1$ and $0<\delta_2<1$, $g_{D_1,D_{\delta_2}}$ is not absolutely continuous. Given any $0<\varepsilon<1$, we can take $1-\varepsilon<\delta_2<1$. Then the measure of D_{δ_2} is less than ε. We deduce as before that

$$m*\left(g_{D_1,D_{\delta_2}}\left(\left\{x\in D_1 : g_{D_1,D_{\delta_2}}{}'(x)=+\infty\right\}\right)\right)\neq 0 .$$

Note that $m*\left(g_{D_1,D_{\delta_2}}\left(\left\{x\in D_1 : g_{D_1,D_{\delta_2}}{}'(x)=+\infty\right\}\right)\right)\le m(D_{\delta_2})<\varepsilon$. Thus, we can find strictly increasing, continuous but non-absolutely continuous function that maps the set of points of infinite derivative to a set with prescribed arbitrary small but non-zero measure. We conclude that the set of points in D_1, where the derivative is infinite, is non-denumerable. Note that D_1 is compact, nowhere dense, and is its own boundary points. For the Cantor function f_{C_1} it is known that it is not differentiable finitely in C_1. We can show similarly as in "*R. B. Darst, Some Cantor Sets and Cantor Functions, Mathematics Magazine, vol. 45(1972), pp. 2- 7*", that $g_{D_1,D_{\delta_2}}$ is not differentiable finitely in D_1. It is not clear how one can characterize the set of points in D_1, where $g_{D_1,D_{\delta_2}}$ is not differentiable finitely or infinitely. We may ask the question: "Is it of the power of the continuum?".