In this article, we aim to prove that if a function of bounded variation maps a subset to a set of measure zero, then its total variation function will map the same set to a set of measure zero too. We use the same technique used as before in my article, *Functions of Bounded Variation and Johnson’s Indicatrix*.

We use the notation and definitions of terms given in *Functions of Bounded Variation and de La Vallée Poussin’s Theorem and Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*.

Suppose \( f : A \to \mathbb{R} \) is a function of bounded variation and \( A \) is a subset of \( \mathbb{R} \).

We shall assume initially that \( A \) is a bounded subset and extend the definition to arbitrary unbounded subset \( A \).

Take \( a_1, a_2 \in A \) with \( a_1 < a_2 \) and consider the closed interval \( I = [a_1, a_2] \). Let \( \tilde{I} = [a_1, a_2] \cap A \). Let \( \nu_f = \nu_{f, \tilde{I}} : A \to \mathbb{R} \) be the total variation function of \( f \) on \( A \).

Then there exists a sequence, \( \{P_n\} \), of partitions of the closed interval, \( \tilde{I} \) by points in \( A \) such that \( P_n \subseteq P_{n+1} \subseteq \cdots \) and

\[
\lim_{n \to \infty} \sum_{\tilde{P}_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| = \text{total variation of } f \text{ on } \tilde{I} = \nu_f(a_2) - \nu_f(a_1) = M,
\]

where \( P_n : a_1 = x_{0,n} < x_{1,n} < \cdots < x_{k_n,n} = a_2 \),

\[
\sum_{\tilde{P}_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| = \sum_{j=1}^{k_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right|.
\]

Observe that by definition of the total variation over \( \tilde{I} \), given any positive integer, \( n \), there exists a partition, \( P_n \), such that

\[
M - \frac{1}{n} \sum_{\tilde{P}_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| \leq M
\]

and there exists a partition, \( Q \), such that
\[ M - \frac{1}{n+1} < \sum_{Q} |f(x_{j,n}) - f(x_{j-1,n})| \leq M. \]

We can then choose \( P_{n+1} \) to be the refinement \( P_n \cup Q \). Starting with \( n = 1 \), we can then construct such a sequence \( \{P_n\} \) and plainly, \( \lim_{n \to \infty} \sum_{P_n} |f(x_{j,n}) - f(x_{j-1,n})| = M \).

For each partition, \( P_n \), we can define an indicatrix as follows.

For \( 1 \leq j \leq k_n \), let \( S_{j,n} \) be the closed intervals with \( f(x_{j,n}) \) and \( f(x_{j-1,n}) \) as end points, i.e., \( S_{j,n} = [f(x_{j,n}), f(x_{j-1,n})] \) or \( [f(x_{j-1,n}), f(x_{j,n})] \). Let \( \chi(S_{j,n}) \) be the characteristic function of \( S_{j,n} \). Then, plainly, \( \chi(S_{j,n}) \) is Lebesgue integrable and

\[ \int_{-\infty}^{\infty} \chi(S_{j,n}) = |f(x_{j,n}) - f(x_{j-1,n})| \quad \text{for} \quad 1 \leq j \leq k_n \]

For the partition, \( P_n \), let \( T_n = \sum_{j=1}^{k_n} \chi(S_{j,n}) \). Then \( T_n \) is a measurable function. In particular,

\[ \int_{-\infty}^{\infty} T_n(y) dy = \sum_{j=1}^{k_n} \int_{-\infty}^{\infty} \chi(S_{j,n}) = \sum_{j=1}^{k_n} |f(x_{j,n}) - f(x_{j-1,n})| = \sum_{P_n} |f(x_{j,n}) - f(x_{j-1,n})| \).

Since \( P_{n+1} \) refines \( P_n \), plainly, \( T_{n+1} \geq T_n \). It follows that \( \{T_n\} \) is an increasing sequence of non-negative measurable and integrable functions. We now define for this sequence of partitions, \( \{P_n\} \) for \( \tilde{I} \),

\[ T_{\tilde{I}} = T_{[a_1,a_2] \cap A} = \lim_{n \to \infty} T_n. \]

By the Monotone Convergence Theorem, the function \( T_{\tilde{I}} \) is Lebesgue integrable and

\[ \int_{-\infty}^{\infty} T_{\tilde{I}}(y) dy = \lim_{n \to \infty} \int_{-\infty}^{\infty} T_n(y) dy = \lim_{n \to \infty} \sum_{P_n} |f(x_{j,n}) - f(x_{j-1,n})| = \text{total variation of over } \tilde{I}, \]

\[ = v_f(a_2) - v_f(a_1). \]

\[ \text{-----------------------------}(1) \]

**Definition 1.** We define the *indicatrix* of \( f \), the restriction of \( f \) to \( \tilde{I} = [a_1, a_2] \cap A \) to be \( T_{\tilde{I}} \). Note that the function \( T_{\tilde{I}} \) depends on the sequence of
partitions \( \{P_n\} \) used to define it. Nevertheless, \( T_j \) is unique up to a set of measure zero. That is to say, if \( \tilde{T}_j \) is defined using another sequence of partitions, \( \{Q_n\} \), then \( \tilde{T}_j = T_j \) almost everywhere on \( \tilde{I} = [a_1, a_2] \cap A \).

**Lemma 2.** With notation as above, \( T_j \) is unique up to a set of measure zero.

**Proof.**

Denote \( T_{I,(P_n)} \) to be the indicatrix function defined by the sequence of partitions \( \{P_n\} \) and \( T_{I,(Q_n)} \) to be the indicatrix function defined by the sequence of partitions \( \{Q_n\} \). Let \( \{R_n\} \) be the common refinement of \( \{P_n\} \) and \( \{Q_n\} \), with \( R_n = P_n \cup Q_n \). Then \( T_{I,(R_n)} = \lim_{n \to \infty} T_{I,(P_n)} \), where \( T_{I,(R_n)} \) is the function defined using the partition, \( R_n \) and

\[
\int_{\infty}^{\infty} T_{I,(R_n)}(y)dy = \lim_{n \to \infty} \int_{\infty}^{\infty} T_{I,(P_n)}(y)dy = \lim_{n \to \infty} \sum_{R_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| = \nu_f(a_2) - \nu_f(a_1).
\]

Since \( R_n \) is a refinement of both \( P_n \) and \( Q_n \), \( T_{I,(P_n)} \geq T_{I,(R_n)}, T_{I,(Q_n)} \). Passing to the limit we have then \( T_{I,(R_n)} \geq T_{I,(P_n)}, T_{I,(Q_n)} \). We claim that \( T_{I,(R_n)} = T_{I,(P_n)} \) almost everywhere on \( \tilde{I} = [a_1, a_2] \cap A \). Suppose there exists a subset \( E \) of positive measure in \( \tilde{I} = [a_1, a_2] \cap A \) such that \( T_{I,(R_n)} > T_{I,(P_n)} \), then \( \int_{\infty}^{\infty} T_{I,(R_n)}(y)dy > \int_{\infty}^{\infty} T_{I,(P_n)}(y)dy \). But

\[
\int_{\infty}^{\infty} T_{I,(R_n)}(y)dy = \int_{\infty}^{\infty} T_{I,(P_n)}(y)dy = \nu_f(a_2) - \nu_f(a_1)
\]

gives a contradiction. Hence, we have that \( T_{I,(R_n)} = T_{I,(P_n)} \) almost everywhere on \( \tilde{I} \). Similarly, \( T_{I,(R_n)} = T_{I,(Q_n)} \) almost everywhere on \( \tilde{I} \). Therefore, \( T_{I,(P_n)} = T_{I,(Q_n)} \) almost everywhere on \( \tilde{I} \).

**Johnson’s Indicatrix**

**Definition 3.**

Suppose \( f: A \to \mathbb{R} \) is a function of bounded variation and \( A \) is a bounded subset of \( \mathbb{R} \). Let \( a = \inf A \) and \( b = \sup A \). Then \( A \subseteq [\inf A, \sup A] = [a, b] \).
Suppose \( \inf A \) and \( \sup A \) belongs to \( A \). Then the indicatrix function \( T_A \) is given by Definition 1.

Suppose \( a \notin A \) and \( b \in A \). Take a sequence \( (a_n) \) in \( A \) such that \( a_n \searrow a \). Define the indicatrix function on \( A \)

\[
T_A = \lim_{n \to \infty} T_{[a_n, b]} \cap A.
\]

Note that \( \{T_{[a_n, b]} \cap A\} \) is an increasing sequence of functions and so is Lebesgue integrable. Moreover,

\[
\int_{-\infty}^{\infty} T_A(y) \, dy = \lim_{n \to \infty} \int_{-\infty}^{\infty} T_{[a_n, b]} \cap A(y) \, dy = \lim_{n \to \infty} \nu_f(b) - \nu_f(a) = \text{the total variation of } f \text{ on } A.
\]

Note that \( \nu_f(b) - \nu_f(a) \) does not depend on the anchor point used to define the variation function. Note that \( T_A \) does not depend on the sequence \( (a_n) \). If \( (\tilde{a}_n) \) is another sequence in \( A \) such that \( \tilde{a}_n \searrow a \), then we can find a subsequence \( (\tilde{a}_{n_k}) \) such that \( \tilde{a}_{n_k} < a_n \). Observe that since \( (\tilde{a}_{n_k}) \) is a subsequence of \( (\tilde{a}_n) \),

\[
\lim_{n \to \infty} T_{[\tilde{a}_n, b]} \cap A = \lim_{k \to \infty} T_{[\tilde{a}_{n_k}, b]} \cap A \quad \text{almost everywhere on } A.
\]

Therefore, \( \lim_{n \to \infty} T_{[\tilde{a}_n, b]} \cap A \geq \lim_{k \to \infty} T_{[\tilde{a}_{n_k}, b]} \cap A \) almost everywhere on \( A \). By the same reasoning we can show that \( \lim_{n \to \infty} T_{[a_n, b]} \cap A \geq \lim_{k \to \infty} T_{[\tilde{a}_n, b]} \cap A \) almost everywhere on \( A \). Hence, \( \lim_{n \to \infty} T_{[a_n, b]} \cap A = \lim_{k \to \infty} T_{[\tilde{a}_n, b]} \cap A \) almost everywhere on \( A \). Thus, \( T_A = \lim_{n \to \infty} T_{[a_n, b]} \cap A \) is defined.

Suppose \( a \in A \) and \( b \notin A \). Take a sequence \( (b_n) \) in \( A \) such that \( b_n \nearrow b \). Define the indicatrix function on \( A \)

\[
T_A = \lim_{n \to \infty} T_{[a_n, b_n]} \cap A.
\]

Note that

\[
\int_{-\infty}^{\infty} T_A(y) \, dy = \lim_{n \to \infty} \int_{-\infty}^{\infty} T_{[a_n, b_n]} \cap A(y) \, dy = \lim_{n \to \infty} \nu_f(b) - \nu_f(a) = \text{the total variation of } f \text{ on } A.
\]

Suppose \( a \notin A \) and \( b \notin A \). Take a sequence \( (a_n) \) in \( A \) such that \( a_n \searrow a \) and a sequence \( (b_n) \) in \( A \) such that \( b_n \nearrow a \). Consider \( (a_n, b_n] \cap A \). We have just defined
\[ T_{(a,b)\cap d} = \lim_{k \to \infty} T_{[a_k,b_k] \cap d} \] and that
\[ \int_{-\infty}^{\infty} T_{(a,b)\cap d} (y) dy = \lim_{k \to \infty} \int_{-\infty}^{\infty} T_{[a_k,b_k] \cap d} (y) dy = \lim_{k \to \infty} \nu_f (b_k) - \nu_f (a_k) = \text{total variation of } f \text{ on } (a,b) \cap A. \]
Define \( T_{(a,b)\cap d} = \lim_{n \to \infty} T_{(a,b)\cap d} \). Then
\[ \int_{-\infty}^{\infty} T_{d} (y) dy = \int_{-\infty}^{\infty} T_{(a,b)\cap d} (y) dy = \lim_{n \to \infty} \int_{-\infty}^{\infty} T_{[a_k,b_k] \cap d} (y) dy = \lim_{n \to \infty} \left( \nu_f (b_k) - \nu_f (a_k) \right) \]
\[ = \text{total variation of } f \text{ on } (a,b) \cap A = A. \]

Suppose \( f : A \to \mathbb{R} \) is a function of bounded variation and \( A \) is an unbounded subset of \( \mathbb{R} \). If it is not bounded above but bounded below, then we can take a sequence \( (b_n) \) in \( A \) such that \( b_n \nearrow \infty \) and define the indicatrix function \( T_d \) as in the above procedure and \( \int_{-\infty}^{\infty} T_{d} (y) dy = \text{total variation of } f \text{ on } A. \) Similarly, we can define \( T_d \) when \( A \) is not bounded below but bounded above. Finally, we can define \( T_d \) when \( A \) is not bounded above and below in a similar fashion. Moreover, \( \int_{-\infty}^{\infty} T_{d} (y) dy = \text{total variation of } f \text{ on } A. \)

The next result is an immediate consequence of the definition of the indicatrix function.

**Lemma 4.** Suppose \( f : A \to \mathbb{R} \) is a function of bounded variation and \( A \) is a subset of \( \mathbb{R} \). Suppose \( a, b \in A \) and \( a < b \). Let \( I = [a,b] \cap A \). Suppose
\[ y \notin \left[ \inf \{ f(x) : x \in I \}, \sup \{ f(x) : x \in I \} \right]. \]
Then \( T_I (y) = 0. \)

**Lemma 5.** Suppose \( f : A \to \mathbb{R} \) is a function of bounded variation. Suppose \( \{ I_j \} \) is a sequence of pairwise disjoint closed intervals with end points in \( A \). Then \( T_d (y) \geq \sum_j T_{I_j} (y) \) almost everywhere on \( A \).

**Proof.**
We prove the inequality for a finite number of the sequence \( \{I_j\} \). Note that these are pairwise disjoint subsets. Let \( \tilde{I}_j = I_j \cap A \). Take \( k \) of these sets, \( \tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_k \). Suppose \( I_j = [a_j, b_j] \).

Let \( a = \min\{a_1, a_2, \ldots, a_k\} \) and \( b = \max\{b_1, b_2, \ldots, b_k\} \) and \( I = [a, b] \). Take typical sequences of partitions for \( \tilde{I} = [a, b] \cap A \), and \( \tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_k \) for the definition of the indicatrix functions. Refine the sequence of partitions for \( \tilde{I} = [a, b] \cap A \) to include all the partitions for \( \tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_k \). Denote the new sequence of partitions for \( \tilde{I} \) by \( \{R_n\} \) and the sequences of partitions for \( \tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_k \) by \( \{P_{1,n}\}, \{P_{2,n}\}, \ldots, \{P_{k,n}\} \).

Observe that the members of the collection of sequences are each collection of disjoint sets, i.e., for each integer \( n \), \( \{P_{1,n}, P_{2,n}, \ldots, P_{k,n}\} \) is a collection of disjoint sets. Hence, \( T_{I,R_n}(y) \geq \sum_{j=1}^{k} T_{I_j,P_{j,n}}(y) \). Then passing to the limit we have then

\[
T_{I,R_n}(y) \geq \sum_{j=1}^{k} T_{I_j,P_{j,n}}(y) \text{ almost everywhere on } \tilde{I}.
\]

Therefore, by Lemma 4,

\[
T_{A}(y) \geq T_{I,R_n}(y) \geq \sum_{j=1}^{k} T_{I_j,P_{j,n}}(y) \text{ almost everywhere on } A. \]

It follows that

\[
T_{A}(y) \geq \lim_{n \to \infty} \sum_{j=1}^{k} T_{I_j,P_{j,n}}(y) = \sum_{j=1}^{k} T_{I_j,P_{j,n}}(y) \text{ almost everywhere on } A.
\]

Dropping the reference to the partitions used to defined the indicatrix functions, we obtain, \( T_{A}(y) \geq \sum_{j=1}^{k} T_{I_j}(y) \) almost everywhere on \( A \).

The next result gives a bound to the image, under the total variation function, of the points of \( A \) in a closed interval, with end points in \( A \), by the integral of the indicatrix function. This is a crucial inequality used to limit the bound of the image of the total variation function.

**Lemma 6.** Suppose \( f : A \to \mathbb{R} \) is a function of bounded variation and \( A \) is a subset of \( \mathbb{R} \). Suppose \( a, b \in A \) and \( a < b \). Let \( \tilde{I} = [a, b] \cap A \). Then

\[
m^*(v_f(\tilde{I})) = m^*(v_f([a, b] \cap A)) \leq \int_{\tilde{I}} T_{I}(y)dy,
\]

where \( m^* \) is the Lebesgue outer measure.
Proof.

\[ m^*(\nu_f(\tilde{I})) = m^*(\nu_f([a, b] \cap A)) \leq \nu_f(b) - \nu_f(a) = \int_{-\infty}^{\infty} T_j(y) dy. \]

**Lemma 7.** Suppose \( f : A \to \mathbb{R} \) is a function of bounded variation and \( A \) is a subset of \( \mathbb{R} \). Suppose \( \{I_j\} \) is a sequence of pairwise disjoint closed intervals with end points in \( A \). Let \( I_j = [a_j, b_j], \ a_j < b_j, \ a_j, b_j \in A \). Let \( S = \bigcup_j I_j \), \( \tilde{I}_j = [a_j, b_j] \cap A \) and \( \tilde{S} = \left( \bigcup_j I_j \right) \cap A = \bigcup_j \tilde{I}_j \). Suppose \( E \) is a measurable subset of \( \mathbb{R} \) such that \( \left[ \inf \{ f(x) : x \in I_j \cap A \}, \sup \{ f(x) : x \in I_j \cap A \} \right] = \left[ \inf \{ f(\tilde{I}_j) \}, \sup \{ f(\tilde{I}_j) \} \right] \subseteq E \) for each integer \( j \). Then

\[ m^*(\nu_f(\tilde{S})) = m^*(\nu_f(S \cap A)) \leq \int_E \sum_j T_j(y) dy \leq \int_E T_A(y) dy. \]

**Proof.**

\[ m^*(\nu_f(\tilde{S})) \leq \sum_j m^*(\nu_f(\tilde{I}_j)) \leq \sum_j \int_{-\infty}^{\infty} T_j(y) dy, \text{ by Lemma 6,} \]

\[ \leq \sum_j \int_E T_j(y) dy, \text{ by Lemma 4,} \]

since \( \left[ \inf \{ f(x) : x \in I_j \cap A \}, \sup \{ f(x) : x \in I_j \cap A \} \right] = \left[ \inf \{ f(\tilde{I}_j) \}, \sup \{ f(\tilde{I}_j) \} \right] \subseteq E, \)

\[ \leq \int_E T_A(y) dy, \text{ by Lemma 5.} \]

The next result is useful for the approach to using finite union of subsets before passing to infinite union of subsets.

**Lemma 8.** Suppose \( \{A_j\} \) is a sequence of subsets of \( \mathbb{R} \), uniformly bounded. Then there exists an integer \( k \) such that

\[ m^* \left( \bigcup_{n=1}^{k} A_n \right) \geq \frac{1}{2} m^* \left( \bigcup_{n=1}^{\infty} A_n \right), \]

where \( m^* \) is the Lebesgue outer measure.
Proof. Note that $\bigcup_{n=1}^{\infty} A_n$ is bounded and so $m^*\left(\bigcup_{n=1}^{\infty} A_n\right)$ is finite.

If $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$, then we have nothing to prove since both sides of the inequality are zero.

Suppose now $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) > 0$. Then by the continuity from below property of Lebesgue outer measure,

$$\lim_{j \to \infty} m^*\left(\bigcup_{n=1}^{j} A_n\right) = m^*\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Therefore, there exists an integer $k > 0$, such that for all $j \geq k$, we have that

$$m^*\left(\bigcup_{n=1}^{k} A_n\right) \leq \frac{1}{2} m^*\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Hence, $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) > m^*\left(\bigcup_{n=1}^{\infty} A_n\right) - \frac{1}{2} m^*\left(\bigcup_{n=1}^{k} A_n\right) = \frac{1}{2} m^*\left(\bigcup_{n=1}^{\infty} A_n\right)$.

It is easier to prove the result we stated at the outset on set where the function is continuous. We formulate the special case in the next theorem.

**Theorem 9.** Suppose $f: A \to \mathbb{R}$ is a function of bounded variation and $A$ is a subset of $\mathbb{R}$. Suppose $E$ is a subset of $A$ such that $f$ is continuous at every point of $E$ and that the measure of its image under $f$, $m(f(E))$, is zero. Then $m(\nu_f(E)) = 0$.

**Proof.**

We may assume that every point of $E$ is a two-sided limit point of $A$ because isolated points and one-sided only limit points constitute at most a denumerable set.

Since $m(f(E)) = 0$ and $f(E)$ is bounded, for any positive integer, $n$, there exists a bounded open set $U_n$ such that $f(E) \subseteq U_n$ and $m(U_n) \leq \frac{1}{n}$.
Take \( e \in E \). Then \( f(e) \in U_a \) and so there exists \( \varepsilon > 0 \) such that 
\[
(f(e) - \varepsilon, f(e) + \varepsilon) \subseteq U_a.
\] As \( f \) is continuous at \( e \), there exists \( \delta > 0 \) such that 
\[
f((e - \delta, e - \delta) \cap A) \subseteq \left( f(e) - \frac{\varepsilon}{2}, f(e) + \frac{\varepsilon}{2} \right).
\] Since \( e \) is a two-sided limit point of \( A \), there exists \( a_e \in (e - \frac{\delta}{2}, e) \cap A \) and \( b_e \in (e, e + \frac{\delta}{2}) \cap A \). Let \( I_e = [a_e, b_e] \). Then

\[
f(I_e \cap A) \subseteq \left( f(e) - \frac{\varepsilon}{2}, f(e) + \frac{\varepsilon}{2} \right) \subseteq U_a.
\]

Therefore,

\[
\left[ \inf f(I_e \cap A), \sup f(I_e \cap A) \right] \subseteq \left[ f(e) - \frac{\varepsilon}{2}, f(e) + \frac{\varepsilon}{2} \right] \subseteq (f(e) - \varepsilon, f(e) + \varepsilon) \subseteq U_a.
\]

The collection \( \Gamma = \{ (a_e, b_e) : e \in E \} \) is an open cover for \( E \). Therefore, by Lindelöf Theorem, \( \Gamma \) has a countable subcover, \( \mathscr{C} = \left\{ \text{int } I_{e_i} : i = 1, 2, \ldots \right\} \).

We claim that

\[
m^* \left( v_f \left( \bigcup_{i=1}^\infty I_{e_i} \cap A \right) \right) = m^* \left( v_f \left( \bigcup_{i=1}^\infty \tilde{I}_{e_i} \right) \right) \leq 2 \int_{U_a} T_d(y) dy, \text{ where } \tilde{I}_{e_i} = I_{e_i} \cap A. \quad \text{------ (*)}
\]

By Lemma 8, \( \frac{1}{2} m^* \left( v_f \left( \bigcup_{i=1}^\infty \tilde{I}_{e_i} \right) \right) \leq m^* \left( v_f \left( \bigcup_{i=1}^k \tilde{I}_{e_i} \right) \right) \) for some positive integer \( k \).

Thus,

\[
m^* \left( v_f \left( \bigcup_{i=1}^\infty \tilde{I}_{e_i} \right) \right) \leq 2 m^* \left( v_f \left( \bigcup_{i=1}^k \tilde{I}_{e_i} \right) \right). \quad \text{---------- (1)}
\]

Note that \( \bigcup_{i=1}^k I_{e_i} \) is a finite collection of closed intervals. Hence \( \bigcup_{i=1}^k I_{e_i} \) is a finite disjoint collection of closed intervals, say, \( C_1, C_2, \ldots, C_j \). Each \( C_j \) is a union of a finite number of closed intervals in \( \{ I_{e_i} : i = 1, 2, \ldots, k \} \), say, \( I_{e_{i_1}}, I_{e_{i_2}}, \ldots, I_{e_{i_j}} \), where the union \( \bigcup \{ I_{e_{i_1}}, I_{e_{i_2}}, \ldots, I_{e_{i_j}} \} \) cannot be partitioned into two disjoint collections. It follows that the corresponding collections

\[
\left\{ \left[ \inf f(I_i \cap A), \sup f(I_i \cap A) \right], i = 1, 2, \ldots, n_j \right\},
\]

also have the property that their union cannot be partitioned into two disjoint collections. We deduce this as follows. Suppose
\[
[\inf f(I_1 \cap A), \sup f(I_1 \cap A)] \cap [\inf f(I_2 \cap A), \sup f(I_2 \cap A)] = \emptyset.
\]

Then, \((I_1 \cap A) \cap (I_2 \cap A) = \emptyset\), for if \((I_1 \cap A) \cap (I_2 \cap A)\) were to be non-empty then there exists \(a \in (I_1 \cap A) \cap (I_2 \cap A)\) and 
f\((a) \in [\inf f(I_1 \cap A), \sup f(I_1 \cap A)]\) and \(f(a) \in [\inf f(I_2 \cap A), \sup f(I_2 \cap A)]\),

contradicting that \([\inf f(I_1 \cap A), \sup f(I_1 \cap A)] \cap [\inf f(I_2 \cap A), \sup f(I_2 \cap A)] = \emptyset\).

Because each \([\inf f(I_j \cap A), \sup f(I_j \cap A)] \subseteq U_n\), it follows that

\[
\left[ \min_{E \subseteq I_i} \{\inf f(I_i)\}, \max_{E \subseteq I_i} \{\sup f(I_i)\} \right] \subseteq [\inf f(C_j \cap A), \sup f(C_j \cap A)] \subseteq U_n.
\]

Hence, by Lemma 7,

\[
m^* \left( \bigcup_{i=1}^{k} I_i \right) = m^* \left( \bigcup_{i=1}^{j} C_i \right) \leq \int_{U_n} T_A(y)dy.
\]

Therefore, it follows from inequality (1) that

\[
m^* \left( \bigcup_{i=1}^{\infty} I_i \right) \leq 2 m^* \left( \bigcup_{i=1}^{k} I_i \right) \leq 2 \int_{U_n} T_A(y)dy.
\]

This proves the claim.

Since \(E \subseteq \bigcup_{i=1}^{\infty} I_i\), \(m^*(E) \leq m^* \left( \bigcup_{i=1}^{\infty} I_i \right) \leq 2 \int_{U_n} T_A(y)dy\).

Since \(m(U_n) \to 0\), \(\lim_{n \to \infty} \int_{U_n} T_A(y)dy = 0\). It follows that \(m^*(E) = 0\).

This completes the proof of Theorem 9.

Finally, we state our main theorem as follows.

**Theorem 10.** Suppose \(f : A \to \mathbb{R}\) is a function of bounded variation and \(A\) is a subset of \(\mathbb{R}\). Suppose \(E\) is a subset of \(A\) such that \(m(f(E)) = 0\). Then
\(m(v_f(E)) = 0\).

**Proof.** By Theorem 4 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, the set \(D\) of discontinuities of \(f\) is at most denumerable. It
follows that \( m(f(D)) = m(\nu_f(D)) = 0 \). Since \( m(f(E)) = 0 \), \( m(f(E - D)) = 0 \). Note that \( f \) is continuous at every point of \( E - D \). Therefore, by Theorem 9, \( m(\nu_f(E - D)) = 0 \). Hence, \( m^*(\nu_f(E)) \leq m^*(\nu_f(E - D)) + m^*(\nu_f(E \cap D)) = 0 + 0 = 0 \). It follows that \( m(\nu_f(E)) = 0 \).