Change of Variable Theorem for Riemann Integral

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Kestelman gave the most general form of the change of variable theorem for Riemann integral. Note that in Kestelman's theorem, the change of variable function is an indefinite integral of a Riemann integrable function. It is, of course, an absolutely continuous function. We present here a slightly general change of variable theorem (Theorem 1 part (1) below) not requiring the derivative of the change of variable function to be Riemann integrable. This makes the application of this version readily available.

Theorem 1. Let [*a*, *b*] be a non-trivial interval. Suppose $\varphi:[a,b]\to[c,d]$ is a continuous onto function and $f:[c,d]\to\mathbb{R}$ is a bounded function. Suppose φ maps its set of non-differentiability into a set of measure zero. That is, if $P=\{x\in[a,b]:\varphi \text{ is not differentiable or has infinite derivative at } x\}$, then the Lebesgue measure $m(\varphi(P))=0$. Define $\varphi^*:[a,b]\to\mathbb{R}$ by

$$\varphi^*(x)=\begin{cases}\varphi'(x), & \text{if } \varphi \text{ is differentiable finitely at } x,\\ 0, & \text{if } \varphi \text{ is not differentiable finitely or is infinite at } x\end{cases}.$$

(1) Suppose *f* is Riemann integrable on [*c*, *d*]. If $f(\varphi(x))\varphi^*(x)$ is Riemann integrable on [*a*, *b*], then

$$\int_a^b f(\varphi(x))\varphi^*(x)dx=\int_{\varphi(a)}^{\varphi(b)} f(x)dx .$$

(2) Suppose φ^* is Riemann integrable on [*a*, *b*]. Then $f(\varphi(x))\varphi^*(x)$ is Riemann integrable on [*a*, *b*] if, and only if, *f* is Riemann integrable on [*c*, *d*].

Remark.
1. If φ is continuous and differentiable on (*a*, *b*), then it satisfies the hypothesis of Theorem 1. The function φ satisfying the condition of Theorem 1 need not be of bounded variation but it is a Lusin function. If φ is absolutely continuous, then it satisfies the condition of Theorem 1.
2. Kestelman change of variable theorem follows from Theorem 1 part (2).

To prove Theorem 1, we shall need the following result.

Theorem 2. Let [*a*, *b*] be a non-trivial interval. Suppose $\varphi:[a,b]\to[c,d]$ is a continuous onto function and $f:[c,d]\to\mathbb{R}$ is a bounded function. Suppose φ maps its set of non-differentiability into a set of measure zero. That is, if $P=\{x\in[a,b]:\varphi \text{ is not differentiable or has infinite derivative at } x\}$, then the Lebesgue measure $m(\varphi(P))=0$. Define $\varphi^*:[a,b]\to\mathbb{R}$ by

$$\varphi^*(x)=\begin{cases}\varphi'(x), & \text{if } \varphi \text{ is differentiable finitely at } x,\\ 0, & \text{if } \varphi \text{ is not differentiable finitely at } x \text{ or } \varphi'(x) \text{ is infinite}\end{cases}.$$

Suppose φ^* is Riemann integrable on [*a*, *b*].

Suppose A is a subset of $[a, b]$ of measure zero such that φ^* is continuous on $[a, b] - A$. Then for x in $[a, b] - A$, $f(\varphi(t))\varphi^*(t)$ is continuous at x, if, and only if, $\varphi^*(x) = 0$ or f is continuous at $\varphi(x)$.

Proof.
Take $x \in [a, b] - A$. Then φ^* is continuous at x. If additionally, $\varphi^* = 0$, then $f(\varphi(t))\varphi^*(t)$ is continuous at x since f is bounded on $\varphi([a, b])$. If f is continuous at $\varphi(x)$, then $f(\varphi(t))$ is continuous at x since φ is continuous at x and it follows that $f(\varphi(t))\varphi^*(t)$ is continuous at x.

Suppose now $f(\varphi(t))\varphi^*(t)$ is continuous at $x \in [a,b] - A$. $\varphi^*(x)$ is either equal to zero or is non zero. Suppose $\varphi^*(x) \neq 0$. Since $f(\varphi(t))\varphi^*(t)$ is continuous at x, $f(\varphi(t))$ is continuous at x. Since $\varphi^*(x) \neq 0$, φ is differentiable at x $\varphi'(x) = \varphi^*(x) \neq 0$. Without loss of generality, we may assume that x is in the interior of $[a, b]$. Suppose $\varphi^*(x) > 0$. Then by continuity of φ^* at x, there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq (a,b)$ and $t \in (x-\delta, x+\delta)$ implies that $\left|\varphi^*(t) - \varphi^*(x)\right| < \frac{1}{2}\varphi^*(x)$ so that $\varphi^*(t) > \frac{1}{2}\varphi^*(x) > 0$. Hence, for $t \in (x-\delta, x+\delta)$, $\varphi^*(t) = \varphi'(t) > \frac{1}{2}\varphi^*(x) > 0$. Therefore, φ is differentiable on $(x-\delta, x+\delta)$ and $\varphi'(t) > 0$. Hence, φ is a strictly increasing continuous function on $(x-\delta, x+\delta)$. Therefore, the restriction of φ to $(x-\delta, x+\delta)$ has a strictly continuous inverse g. Moreover, $\lim_{y \to \varphi(x)} g(y) = g(\varphi(x)) = x$. Therefore,
$\lim_{y \to \varphi(x)} f(y) = \lim_{y \to \varphi(x)} \left(f \circ \varphi\right) \circ g(y) = \left(f \circ \varphi\right)(x) = f(\varphi(x))$. Therefore, f is continuous at $\varphi(x)$.
We can show similarly that if $\varphi^*(x) < 0$, then f is continuous at $\varphi(x)$.

Proof of Theorem 1 part (1).
Suppose f is Riemann integrable on $[c, d]$. Let $F : [c,d] \to \mathbb{R}$ be defined by $F(x) = \int_c^x f(t)dt$ for $x \in [c,d]$. Then F is absolutely continuous and is differentiable almost everywhere on $[c, d]$. Thus, there exists a subset $E \subseteq [c,d]$ such that $m(E) = 0$ and $F'(x) = f(x)$ for $x \in [c,d] - E$. Note that and $m(\varphi(P)) = 0$.
Let $B = P \cup \varphi^{-1}(E)$. Then $m(\varphi(B)) = 0$. Since the Lebesgue measure is regular, we can take a measurable subset $C \subseteq [c,d]$ such that $B \subseteq C$ and $m(C - B) = 0$. For $x \in [c,d] - C$, φ is differentiable at x and F is differentiable at $\varphi(x)$ and $F'(\varphi(x)) = f(\varphi(x))$. Thus, $F \circ \varphi$ is differentiable on $[c,d] - C$ and $\left(F \circ \varphi\right)'(x) = f(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi^*(x)$ for $x \in [c,d] - C$. As $\varphi(B) \subseteq E \cup \varphi(P)$ and $m(\varphi(P)) = m(E) = 0$, $m(\varphi(B)) = 0$. As φ is a Lusin function, $m(\varphi(C)) = 0$. As F is also a Lusin function $m(F \circ \varphi(C)) = 0$. Therefore, as $\left(F \circ \varphi\right)'(x) \leq f(\varphi(x))\varphi^*(x)$ for $x \notin C$ and $f(\varphi(x))\varphi^*(x)$ is Riemann integrable, by Theorem 1 of my article, "*When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous? The answer and application to generalized change of variable for Lebesgue integral*", $F \circ \varphi$ is absolutely continuous and so is of bounded variation and differentiable almost everywhere on $[a, b]$. By Theorem 2 of

"Change of Variables Theorem", $\left(F\circ\varphi\right)'(x)=0$ almost everywhere on $B=P\cup\varphi^{-1}(E)$.

Note that $f(\varphi(x))\varphi^*(x)=0$ for x in P. It follows that $\left(F\circ\varphi\right)'(x)=f(\varphi(x))\varphi^*(x)$ almost everywhere for x in P. Observe that φ is differentiable on $\varphi^{-1}(E)-P$ and since $m(\varphi(\varphi^{-1}(E)-P))=0$. By Theorem 2 of "Change of Variables Theorem", $\varphi'=0$ almost everywhere on $\varphi^{-1}(E)-P$ and we have $\left(F\circ\varphi\right)'(x)=f(\varphi(x))\varphi^*(x)$ almost everywhere for x in $\varphi^{-1}(E)-P$. Hence, $\left(F\circ\varphi\right)'(x)=f(\varphi(x))\varphi^*(x)$ almost everywhere on C. Thus, $\left(F\circ\varphi\right)'(x)=f(\varphi(x))\varphi^*(x)$ almost everywhere on [c, d]. Since F is absolutely continuous, $\int_{\varphi(a)}^{\varphi(b)}f(t)dt=F(\varphi(b))-F(\varphi(a))$. As $F\circ\varphi$ is absolutely continuous,

$$F(\varphi(b))-F(\varphi(a))=F\circ\varphi(b)-F\circ\varphi(a)=\int_a^b(F\circ\varphi)'(t)dt=\int_a^b f(\varphi t))\varphi^*(t)dt\,.$$

Proof of Theorem 1 part (2).

Suppose φ^* is Riemann integrable on [a, b]. Then there exists a subset A of measure zero such that φ^* is continuous on [a, b] – A.

Suppose $f(\varphi(x))\varphi^*(x)$ is Riemann integrable on [a, b]. Then $f(\varphi(x))\varphi^*(x)$ is continuous almost everywhere on [a, b] – A. For $x\in[a,b]-A$, by Theorem 2, $f(\varphi(x))\varphi^*(x)$ is not continuous at x if, and only if, $\varphi^*(x)\neq 0$ and f is not continuous at $\varphi(x)$. Let $C=\{t\in[a,b]-A:\ f$ is not continuous at $\varphi(t)\}$ and

$$D=\{t\in[a,b]-A:\varphi^*(t)\neq 0\}$$
$$=\{t\in[a,b]-A:\varphi \text{ is differentiable at } t \text{ and } \varphi'(t)\neq 0\}.$$

Thus, for $x\in[a,b]-A$, $f(\varphi(x))\varphi^*(x)$ is not continuous at x if, and only if, $x\in C\cap D$. Since $f(\varphi(x))\varphi^*(x)$ is continuous almost everywhere on [a, b] – A, $m(C\cap D)=0$. As φ is a Lusin function, $m\left(\varphi(C\cap D)\right)=0$. Let $D^*=\{t\in[a,b]-A:\varphi^*(t)=0\}$. Then $[a,b]-A$ is a disjoint union $D\cup D^*$. Note that $C=(C\cap D)\cup(C\cap D^*)$. Observe that D^* is a union of $\{t\in[a,b\}-A:\varphi$ is differentiable at t and $\varphi'(t)=0\}$ and $\{t\in[a,b\}-A:\varphi$ is not differentiable at t or $\varphi'(t)=\pm\infty\}$.

By hypothesis, $m\left(\varphi\left(\{t\in[a,b\}-A:\varphi \text{ is not differentiable at } t \text{ or } \varphi'(t)=\pm\infty\}\right)\right)=0$. By Theorem 3 of my article "*Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*", $m\left(\varphi\left(\{t\in[a,b]-A:\varphi \text{ is differentiable at } t \text{ and } \varphi'(t)=0\}\right)\right)=0$. Therefore, $m(\varphi(D^*))=0$. It follows that $m\left(\varphi(C)\right)=0$. Since A is of measure zero, $m\left(\varphi(A)\right)=0$. Suppose $y\in[c,d]-\varphi(A)$ and f is not continuous at y. Then there exists $t\in[a,b]-A$ such that $y=\varphi(t)$ and so $t\in C$ and $y\in\varphi(C)$ which is a set of measure zero. Hence,

$$E=\{y\in[c,d]:\ f \text{ is not continuous at } y\}\subseteq\varphi(A)\cup\varphi(C).$$

Therefore, $m(E)=0$ and so f is continuous almost everywhere and bounded and so is Riemann integrable.

Suppose f : [c, d] $\to$ **R** is Riemann integrable. Then f is continuous almost everywhere on [c, d]. Hence, there exists a subset E in [c, d] of measure zero such that f is continuous on [c, d]–

E. Note that f is bounded on [c, d] and φ^* is bounded on [a, b]. It follows that $f(\varphi(t))\varphi^*(t)$ is bounded on [a, b].
Suppose $x \in [a,b]-A$ and $\varphi^*(x)=0$. Since the function φ^* is continuous at x and f is bounded so that $f\circ\varphi$ is also bounded, $\lim_{y\to x} f(\varphi(y))\varphi^*(y)=\lim_{y\to x}\varphi^*(y)=\varphi^*(x)=0$. Hence $f(\varphi(t))\varphi^*(t)$ is continuous at x for $x\in\{t\in[a,b]-A:\varphi^*(t)=0\}$.
Let $L=\{t\in[a,b]-A:\varphi^*(t)\neq 0\}=\{t\in[a,b]-A:\varphi'(t)\neq 0\}$. It remains to show that $f(\varphi(t))\varphi^*(t)$ is continuous almost everywhere in L. Let $B=\varphi^{-1}(E)$. For $x\in L-B$, $\varphi(x)\notin E$ x so that f is continuous at $\varphi(x)$ and since φ is continuous at x, it follows that $f(\varphi(t))$ is continuous at x. Therefore, $f(\varphi(t))\varphi^*(t)$ is continuous on $L-B$.
By Theorem 2 of *Change of Variables Theorems*, since $m\left(\varphi\left(B\cap L\right)\right)=0$ because $m\left(\varphi(B)\right)=0$, $\varphi'(t)=0$ almost everywhere on $B\cap L$. This means $\varphi^*(t)=0$ almost everywhere on $B\cap L$. It follows that $f(\varphi(t))\varphi^*(t)$ is continuous almost everywhere on $B\cap L$. Hence, $f(\varphi(t))\varphi^*(t)$ is continuous almost everywhere on L and so on $[a,b]-A$ and as $m(A)$ =0, it is continuous almost everywhere on [a, b]. This means that $f(\varphi(t))\varphi^*(t)$ is Riemann integrable on [a, b].

Corollary 3. Let [a, b] be a non-trivial interval. Suppose $\varphi:[a,b]\to[c,d]$ is a continuous onto function and $f:[c,d]\to\mathbb{R}$ is a bounded function. Suppose φ is differentiable finitely except for a denumerable subset in [a, b]. Define $\varphi^*:[a,b]\to\mathbb{R}$ by

$$\varphi^*(x)=\begin{cases}\varphi'(x), & \text{if } \varphi \text{ is differentiable finitely at } x,\\ 0, & \text{if } \varphi \text{ is not differentiable finitely or is infinite at } x\end{cases}.$$

(1) Suppose f is Riemann integrable on [c, d]. If $f(\varphi(x))\varphi^*(x)$ is Riemann integrable on [a, b], then $\int_a^b f(\varphi(x))\varphi^*(x)dx=\int_{\varphi(a)}^{\varphi(b)} f(x)dx$.
(2) Suppose φ^* is Riemann integrable on [a, b]. Then $f(\varphi(x))\varphi^*(x)$ is Riemann integrable on [a, b] if, and only if, f is Riemann integrable on [c, d].

Proof. If φ is differentiable finitely except for a denumerable subset in [a, b], then it satisfies the condition in Theorem 1 and the Corollary follows from Theorem 1.

Corollary 3 applies to common situation when $\varphi:[a,b]\to[c,d]$ is continuous and differentiable or continuous and piecewise differentiable on (a, b).

Example 4.
(1) Here is an example of seemingly difficult improper Riemann integral.

Let g:[0, 1] $\to$ **R** be defined by $g(x)=\begin{cases} x^2\sin^2\left(\dfrac{\pi}{2x}\right), & x>0 \\ 0, & x=0 \end{cases}$ and let f:[0, 1] $\to$ **R** be defined by $f(x)=\begin{cases} \dfrac{1}{2\sqrt{x}}, & x>0 \\ 0, & x=0 \end{cases}$. Then *f* is not bounded on [0, 1]. The function *f* is Lebesgue integrable and the integral is given by the improper Riemann integral $\int_0^1 f(x)dt=\lim_{t\to 0^+}\int_t^1 f(x)dt=\lim_{t\to 0^+}F(t)$, where F : (0, 1] $\to$ **R** defined by $F(t)=\int_t^1 f(x)dx=1-\sqrt{t}$. Observe that

$$f(g(x))=\begin{cases} \dfrac{1}{2x\left|\sin\left(\frac{\pi}{2x}\right)\right|}, & x>0 \text{ and } x\neq \dfrac{1}{2k} \text{ , integer } k\geq 1 \\ 0, & x=0 \text{ or } x=\dfrac{1}{2k} \text{ , integer } k\geq 1 \end{cases}$$

Note that $f\circ g$ is not bounded on [0, 1], hence it is not Riemann integrable on [0, 1].

The function *g* is differentiable on [0, 1] and

$$g'(x)=\begin{cases} 2x\sin^2\left(\dfrac{\pi}{2x}\right)-\pi\sin\left(\dfrac{\pi}{2x}\right)\cos\left(\dfrac{\pi}{2x}\right), & x>0 \\ 0, & x=0 \end{cases}.$$

Note that g' is bounded and continuous almost everywhere on [0, 1]. Hence, it is Riemann integrable on [0, 1]. However,

$$f(g(x))g'(x)=\begin{cases} \left|\sin\left(\dfrac{\pi}{2x}\right)\right|-\dfrac{\pi}{2}\cdot\dfrac{\sin\left(\frac{\pi}{2x}\right)}{\left|\sin\left(\frac{\pi}{2x}\right)\right|}\cdot\dfrac{\cos\left(\frac{\pi}{2x}\right)}{x}, x>0 \text{ and } x\neq\dfrac{1}{2k} \text{ , integer } k\geq 1 \\ 0, x=0 \text{ or } x=\dfrac{1}{2k}, \text{ integer } k\geq 1 \end{cases}$$

and is not Lebesgue integrable on [0, 1]. Moreover, $f(g(x))g'(x)$ is unbounded on [0, 1] and so it is not Riemann integrable on [0, 1]. However, $f(g(x))g'(x)$ is Riemann integrable on the interval, [t, 1], for $0<t<1$.
We can apply Theorem 1 in this case to the integrals on [t, 1] for $0<t<1$, giving

$$\int_t^1 f(g(x))\cdot g'(x)dx=\int_{g(t)}^{g(1)} f(x)dx=\int_{g(t)}^1 f(x)dx=1-\sqrt{g(t)}.$$

Therefore, the improper Riemann integral,

$$\int_0^1 f(g(x))\cdot g'(x)dx=\lim_{t\to 0^+}\int_t^1 f(g(x))\cdot g'(x)dx=\lim_{t\to 0^+}\left(1-\sqrt{g(t)}\right)=1.$$

This shows that $f(g(x))g'(x)$ is improperly Riemann integrable on [0, 1] and the integral is equal to 1.
Note that one can deduce by applying an alternating series test for improper integrals, that $f(g(x))g'(x)$ is improperly Riemann integrable on [0, 1].

(2) Let g:[0, 1] $\to$ **R** be defined by $g(x)=\begin{cases} x\sin\left(\dfrac{\pi}{2x}\right), & x>0 \\ 0, & x=0 \end{cases}$. The function g is continuous on [0, 1] and differentiable on (0,1] with $g'(x)=\sin\left(\dfrac{\pi}{2x}\right)-\dfrac{\pi}{2x}\cos\left(\dfrac{\pi}{2x}\right)$ for $x>0$. Note that g' is not Lebesgue integrable on [0, 1] and so is not Riemann integrable on [0, 1]. Let $g^*(x)=\begin{cases} g'(x), & \text{if } x>0, \\ 0, & \text{if } x=0 \end{cases}$. Let $f(x)=\dfrac{\pi}{2}\sin\left(\dfrac{\pi}{2}x\right)$, Then f is Riemann integrable on [0, 1] and

$$f(g(x))g^*(x)=\begin{cases} \dfrac{\pi}{2}\sin\left(\dfrac{\pi}{2}x\sin\left(\dfrac{\pi}{2x}\right)\right)\left(\sin\left(\dfrac{\pi}{2x}\right)-\dfrac{\pi}{2x}\cos\left(\dfrac{\pi}{2x}\right)\right), x>0 \\ 0, x=0 \end{cases}.$$

Note that $f(g(x))g^*(x)$ is bounded on [0, 1] and is continuous on (0, 1]. Therefore, $f(g(x))g^*(x)$ is Riemann integrable on [0, 1]. By Theorem 1, part (1),

$$\int_0^1 f(g(x))g^*(x)dx=\int_{g(0)}^{g(1)} f(x)dx=\int_0^1 \frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right)dx=1.$$