Chapter 11 The Elementary Functions.

The term "elementary function" refers to any function formed from a set of basic functions by specified rules. The set of basic functions, which are themselves elementary functions consists of polynomials, rational functions, power functions, the exponential function and its inverse the natural logarithmic function, the trigonometric and inverse trigonometric functions and the hyperbolic and inverse hyperbolic functions. To form other elementary functions, we are allowed to add, subtract, multiply, divide and use the rule of composition. So the rules are the rules of arithmetic and the rule of composition. If f and g are elementary functions and if f and g are composable, then the composite $f \circ g$ is also an elementary function. Properties of polynomial and rational functions are well known and we shall not repeat them here. We shall now examine the most important of these elementary functions, the exponential, logarithmic, sine and cosine functions. We can now make use of power series to define analytically the exponential function as well as the sine and cosine functions. We shall then define the natural logarithmic function as the inverse of the exponential function. We shall establish the well known properties of these functions.

11.1 The Exponential and Logarithmic Functions.

The Exponential Function

Definition 1.

We have assumed the exponential function is defined as the inverse of the natural logarithmic function, $\ln(x)$ which is defined in terms of the Riemann integral. That is, the natural logarithmic function $\ln: (0, \infty) \to \mathbf{R}$ is defined by $\ln(x) = \int_1^x \frac{1}{t}dt$ for $x > 0$ and the exponential function, $\exp : \mathbf{R} \to (0, \infty)$ is then defined as the inverse $\ln^{-1} : \mathbf{R} \to (0, \infty)$. The properties of the exponential function are then deduced via the properties of the natural logarithmic function. (see for example, Chapter 10 of Calculus, an Introduction by Ng Tze Beng). In Chapter 7 Example 2 (1) we give its definition as power series for the first time. In Chapter 7 Example 8(2) we establish its radius of convergence. In Example 9 Chapter 8 we establish its differentiability and determine its derivative. In Example 12 Chapter 8, we show that it is the solution to the differential equation, $f' = f$ with initial condition $f(0) = 1$. We shall now show that the power series definition or analytic definition of the exponential function is indeed the exponential function by proving all the properties it enjoys. We shall then define the natural logarithmic function as the inverse of the exponential function.

Define the exponential function, $\exp : \mathbf{R} \to (0, \infty)$, by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n \text{ for any } x \text{ in } \mathbf{R}.$$

We have by a simple ratio test shown that the power series converges for all x. Therefore, its radius of convergence is $+\infty$. (See Theorem 18 or 19 Chapter 7 for a formula for the radius of convergence.)By Theorem 10, Chapter 7, exp is continuous on $\mathbf{R}$.

Properties of Exponential Function.

By Theorem 11 Chapter 8 exp is differentiable and we can differentiate it term by term. Therefore,

$$\exp'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\frac{1}{n!}x^n = \sum_{n=1}^{\infty} \frac{1}{n!}nx^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}x^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

Thus, for all x in **R**,

$$\exp'(x) = \exp(x). \quad \text{-----------------------------------} \quad (1)$$

We shall now establish its well known properties.

Let $g(x) = \exp(x)\exp(-x)$ for all x in **R**. Then by the Product Rule and Chain Rule, for all x in **R**,

$$\begin{aligned} g'(x) &= \exp'(x)\exp(-x) + \exp(x)\exp'(-x)(-1) \\ &= \exp(x)\exp(-x) + \exp(x)\exp(-x)(-1) = 0, \qquad \text{by (1)} \end{aligned}$$

Hence $g(x)$ = a constant C for all x in **R**. (See Theorem 16 Chapter 4). Therefore, evaluating g at x = 0 gives us the constant $C = g(0) = \exp(0)\exp(-0) = 1$. Therefore, for all x in **R**,

$$\exp(x)\exp(-x) = 1\ (> 0) \quad \text{-------------------------------} \quad (2)$$

Hence, $\exp(x) \neq 0$ for all x in **R**. In particular $\exp(x) > 0$. This is because if $x \geq 0$, $\exp(x) > 0$ by definition and if $x < 0$, $\exp(-x) > 0$ and so by (2), $\exp(x) > 0$. It follows then from (2) that for all x in **R**,

$$\exp(-x) = \frac{1}{\exp(x)} \quad \text{--------------------------------------} \quad (3)$$

Since $\exp'(x) = \exp(x) > 0$ for all x in **R**, exp is strictly increasing on **R** (see Theorem 19 Chapter 4).

Let y be a fixed point in **R**. For each x in **R**, define $h(x) = \dfrac{\exp(x)\exp(y)}{\exp(x+y)}$. Then h is differentiable on **R** and by the Quotient Rule, for each x in **R**,

$$\begin{aligned} h'(x) &= \frac{\exp'(x)\exp(y)\exp(x+y) - \exp(x)\exp(y)\exp'(x+y)}{(\exp(x+y))^2} \\ &= \frac{\exp(x)\exp(y)\exp(x+y) - \exp(x)\exp(y)\exp(x+y)}{(\exp(x+y))^2} = 0 \end{aligned}$$

Thus h is a constant function, say $h(x) = C$ for all x in **R**. Therefore, $C = h(0) = \dfrac{\exp(0)\exp(y)}{\exp(0+y)} = 1$. This means that for all x and for all y in **R**,

$$\exp(x+y) = \exp(x)\exp(y). \quad \text{--------------------------------} \quad (4)$$

Plainly, for $x > 0$, $\exp(x) = 1 + x + \frac{x^2}{2!} + \cdots > x$. Hence, by Theorem 46 (1) of Chapter 3, $\exp(x) \to \infty$ as $x \to \infty$. Therefore, by Theorem 47 (1) Chapter 3 $\exp(-x) = \dfrac{1}{\exp(x)} \to 0$ as $x \to \infty$.

Thus we have,

$$\exp(x) \to \infty \text{ as } x \to \infty \quad \text{and} \quad \exp(x) \to 0 \text{ as } x \to -\infty. \quad \text{---------------} \quad (5)$$

Similarly since $\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ is convergent for any x in **R**, for $x > 0$ and for any integer $n \geq 1$,

$$\begin{aligned} \frac{\exp(x)}{x^n} &= \frac{1}{x^n}\sum_{n=0}^{\infty}\frac{1}{n!}x^n = \frac{1}{x^n} + \frac{1}{x^{n-1}} + \frac{1}{2!x^{n-2}} + \cdots + \frac{1}{n!} + \frac{x}{(n+1)!} + \cdots \\ &> \frac{x}{(n+1)!} \end{aligned}$$

Therefore, since $\frac{x}{(n+1)!} \to \infty$ as $x \to \infty$, by Theorem 46 (1) of Chapter 3, for any integer $n \ge 1$,

$$\frac{\exp(x)}{x^n} \to \infty \text{ as } x \to \infty.$$

Because $\lim_{x\to\infty} \exp(x) = \infty$ and $\lim_{x\to-\infty} \exp(x) = 0$ by (5) and exp is a continuous function on **R**, by the Intermediate Value Theorem, the range of the exponential function is (0, ∞). We can deduce this as follows. Let a be any point in (0, ∞). Since $\lim_{x\to\infty} \exp(x) = \infty$, there exists a real number $K > 0$ such that $x \ge K \Rightarrow \exp(x) > a$. Since $\lim_{x\to-\infty} \exp(x) = 0$, there exists a real number $L < 0$ such that $x \le L \Rightarrow \exp(x) < a$. Therefore, $\exp(L) < a < \exp(K)$. Since exp is continuous on the interval $[L, K]$, by the Intermediate Value Theorem (Theorem 13 Chapter 3), there exists a point x_0 in (L, K) such that $\exp(x_0) = a$. Therefore, the range of exp is (0, ∞).

We summarize the above proceeding as follows.

Theorem 2. The exponential function exp : **R** → (0, ∞) is a strictly increasing differentiable bijective function satisfying the following properties,

(1) $\exp' = \exp$
(2) $\exp(x+y) = \exp(x)\exp(y)$ for all x, y in R,
(3) $\exp(-x) = \frac{1}{\exp(x)}$ for all x in R,
(4) $\frac{\exp(x)}{x^n} \to \infty$ as $x \to \infty$ for any integer $n \ge 0$ and
(5) $\exp(x) \to 0$ as $x \to -\infty$.

The Natural Logarithmic Function

Definition 3. Since exp is a bijective function it therefore has an inverse function $\exp^{-1} : (0, \infty) \to \mathbf{R}$ defined by $\exp^{-1}(x) = y \Leftrightarrow \exp(y) = x$. Since exp is a strictly increasing function its inverse is a strictly increasing continuous function (by Theorem 23 of Chapter 3). Moreover, since exp' is non-zero, by Theorem 34 Chapter 4 , $\exp^{-1}$ is differentiable and

$$(\exp^{-1})'(y) = \frac{1}{\exp'(\exp^{-1}(y))} = \frac{1}{\exp(\exp^{-1}(y))} \quad \text{since } \exp' = \exp$$
$$= \frac{1}{y}$$

for all y in (0, ∞). We define the *natural logarithmic function* ln: (0, ∞) → **R** to be $\exp^{-1}$. Hence $\ln'(y) = \frac{1}{y}$ for $y > 0$.

Properties of logarithmic Function.

Note that $\lim_{x\to\infty} \ln(x) = +\infty$. This is because given any real number $K > 0$, take $L = \exp(K)$. Then since ln is strictly increasing for any $x > L$, $\ln(x) > \ln(L) = \ln(\exp(K)) = K$. Hence by Definition 41 Chapter 3 $\lim_{x\to\infty} \ln(x) = +\infty$. Now given any real number $L < 0$, let $\delta = \exp(L)$. Then for any x such that $0 < x < \delta$, $\ln(x) < \ln(\delta) = \ln(\exp(L)) = L$. Thus, by Definition 40 Chapter 3 $\lim_{x\to 0^+} \ln(x) = -\infty$.

Next we claim that for all x, y in (0, ∞),

$$\ln(x\,y) = \ln(x) + \ln(y) \quad \text{---------------------------------} \quad (6)$$

We can show this by evaluating (6) on both sides by exp.

$$\exp(\ln(x) + \ln(y)) = \exp(\ln(x))\exp(\ln(y)) \quad \text{by Theorem 2(2)}$$
$$= x\,y = \exp(\ln(x\,y)).$$

It follows that $\ln(x\,y) = \ln(x) + \ln(y)$ since exp is injective.
Using (6), we have that for any $x > 0$, $\ln(1) = \ln(x \cdot \frac{1}{x}) = \ln(x) + \ln(\frac{1}{x})$. But $\ln(1) = \exp^{-1}(1) = 0$ and so for any $x > 0$,

$$\ln(\frac{1}{x}) = -\ln(x) \quad \text{---} \ (7)$$

By L'Hôpital's Rule (Theorem 37 Chapter 4), we have that for any integer $n \geq 1$,

$$\frac{\ln(x)}{x^n} \to 0 \quad \text{as } x \to \infty .$$

Hence we have in summary:

Theorem 4. The *natural logarithmic function* $\ln: (0, \infty) \to \mathbf{R}$ is a strictly increasing differentiable bijective function satisfying the following properties,

(1) $\ln'(x) = \frac{1}{x}$ for $x > 0$,
(2) $\ln(x\,y) = \ln(x) + \ln(y)$ for all $x, y > 0$,
(3) $\ln(\frac{1}{x}) = -\ln(x)$ for all $x > 0$,
(4) $\lim_{x\to\infty} \ln(x) = +\infty$ as $x \to \infty$ and 3 $\lim_{x\to 0^+} \ln(x) = -\infty$
(5) $\frac{\ln(x)}{x^n} \to 0$ as $x \to \infty$.

Since $\frac{1}{x}$ is Riemann integrable on $[1, x]$ for $x \geq 1$ and on $[x, 0]$ if $0 < x < 1$, by the Fundamental Theorem of Calculus (see Theorem 42 Chapter 5), for any $x > 0$,

$$\int_1^x \frac{1}{t}dt = \int_1^x \ln'(t)dt = \ln(x) - \ln(1) = \ln(x),$$

i.e.,

$$\ln(x) = \int_1^x \frac{1}{t}dt.$$

This is the usual definition of natural logarithm function. Gregory of St. Vincent was aware of this possible definition of the logarithm and his pupil, the Belgian Jesuit A. de Sarasa (1618-67) had observed that the area (the Riemann integral) can be interpreted as logarithm in his *Solutio Problematis a Mersenno Propositi* (1649). Newton too knew of this connection and included this relation in his *Method of Fluxions*. He expanded $\frac{1}{1+x}$ by the binomial theorem and integrate term by term to obtain

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Of course since the radius of convergence is 1, we can only integrate term by term within the interval of convergence. More precisely,

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{for} \quad |x| < 1$$

The series on the right is convergent for $|x| < 1$ and diverges for $|x| \geq 1$. By Theorem 9 Chapter 9, we can integrate the power series term by term to obtain the integral $\ln(1+x) = \int_0^x \frac{1}{1+t}dt$ for $|x| < 1$ as a power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$. By Abel's Theorem this power series gives the value of ln(2). (See Example 21 Chapter 8 for details.)

Powers.

We first introduce the well known Euler constant e as exp(1). Then for any integer $n \geq 1$,

$$e = \exp(1) = \exp(n \cdot \frac{1}{n}) = \exp(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}})$$

$$= \left(\exp(\frac{1}{n})\right)^n \quad \text{by Theorem 2 (2).}$$

Therefore, we have for any integer $n \geq 1$,

$$\exp(\frac{1}{n}) = e^{\frac{1}{n}} .$$

Hence for any positive integers m and n ,

$$\exp(\frac{m}{n}) = \exp(\underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m \text{ times}}) = \left(\exp(\frac{1}{n})\right)^m \quad \text{by Theorem 2 (2),}$$

$$= \left(e^{\frac{1}{n}}\right)^m = e^{\frac{m}{n}}. \qquad \text{------------------------------ (9)}$$

Also for any positive integers m and n,

$$\exp(\frac{-m}{n}) = \frac{1}{\exp(\frac{m}{n})} \quad \text{by Theorem 2 (3),}$$

$$= \frac{1}{e^{\frac{m}{n}}} \qquad \text{by (9)}$$

$$= e^{-\frac{m}{n}}$$

It follows then that for any rational number q,

$$\exp(q) = e^{q} \qquad \text{--(10)}$$

Definition 5.

We shall now extend the usual definition of power to include irrational exponent.

Let a be a positive real number. Define, for any real number x, the power

$$a^x = \exp(x \ln(a)) \qquad \text{-- (11)}$$

Then the assignment $x \mapsto a^x$ is a continuous function on **R** because it is a composition of multiplication by ln(a) followed by exp.

Note that if $a = 1$, then since ln(1) = 0, a^x is a constant function taking the value 1.

We shall show that this is the extension of the usual power for rational exponent. Note that first of all, for any integer $n \geq 1$ and any real number $y > 0$,

$$\ln(y) = \ln((y^{\frac{1}{n}})^n) = \underbrace{\ln(y^{\frac{1}{n}}) + \ln(y^{\frac{1}{n}}) + \cdots + \ln(y^{\frac{1}{n}})}_{n \text{ times}} \quad \text{by Theorem 4 (2)}$$

$$= n \ln(y^{\frac{1}{n}}).$$

Hence, for any integer $n \geq 1$ and any real number $y > 0$,

$$\ln(y^{\frac{1}{n}}) = \frac{1}{n} \ln(y).$$

Therefore, for any integers m, $n \geq 1$ and $y > 0$,

$$\ln(y^{\frac{m}{n}}) = \underbrace{\ln(y^{\frac{1}{n}}) + \ln(y^{\frac{1}{n}}) + \cdots + \ln(y^{\frac{1}{n}})}_{m \text{ times}} = \underbrace{\frac{1}{n} \ln(y) + \cdots + \frac{1}{n} \ln(y)}_{m \text{ times}}$$

$$= m \cdot \frac{1}{n} \ln(y) = \frac{m}{n} \ln(y) \qquad \text{---------------------------------- (12)}$$

For any integers m, $n \geq 1$ and $y > 0$,

$$\ln(y^{-\frac{m}{n}}) = \ln\left(\frac{1}{y^{\frac{m}{n}}}\right) = -\ln(y^{\frac{m}{n}}) \qquad \text{by Theorem 4 (3)}$$

$$= -\frac{m}{n}\ln(y) \quad \text{by (12)}$$

Note that for q = 0, $\ln(y^q) = \ln(1) = 0 = 0\ln(y)$. Therefore, for any rational number q and $y > 0$,

$$\ln(y^q) = q\ln(y) \qquad \text{-- (13)}$$

Thus, for any rational q and $a > 0$,

$$\exp(q\ln(a)) = \exp(\ln(a^q)) = a^q$$

by (13).

Observe that the right hand side is the usual meaning of power but the left hand side is the new definition. Therefore, the new definition coincides with the usual power when q is rational.

Observe that this definition of power satisfies (13) as well. That is, for any x in **R** and any $a > 0$,

$$\ln(a^x) = \ln(\exp(x\ln(a))) = x\ln(a) \qquad \text{-------------------------------- (14)}$$

We also have the following behaviour of the exponential function with respect to power.

For any real number r,

$$\exp(r x) = \exp(r\ln(\exp(x)))$$

$$= (\exp(x))^r . \qquad \text{--------------------------------- (15)}$$

by the definition of power (see (11).

Hence we have proved the following:

Proposition 6. For any real numbers x and r, any $a > 0$,

(1) $\ln(a^x) = x\ln(a)$

(2) $\exp(r x) = (\exp(x))^r$

The significance of Proposition 6 is that the power in the statement is the extended power function.

We can now use this to write exponential function in a more convenient form.

Observe,

$$\exp(x) = \exp(x \cdot 1) = \exp(x \cdot \ln(\exp(1)) = \exp(x \cdot \ln(e)) = e^x \quad \text{---------- (16)}$$

Thus, the properties of the exponential function can then be stated simply as follows:

For all, x, y and r in R,

1. $e^{x+y} = e^x e^y$
2. $e^{-x} = \dfrac{1}{e^x}$

3. $(e^x)^r = e^{rx}$

and

4. $\frac{d}{dx}e^x = e^x$

5. $\dfrac{e^x}{x^n} \to \infty$ as $x \to \infty$ for any integer $n \geq 0$ and
6. $e^x \to 0$ as $x \to -\infty$.

Hence e^x is a very convenient notation for the exponential function.

Other logarithmic function

Note that if $a > 0$ and $a \neq 1$, then the multiplication by $\ln(a)$ is a bijective function from **R** onto **R**. Let $M_a : \mathbf{R} \to \mathbf{R}$ denote this multiplication map, i.e., $M_a(x) = \ln(a)\, x$ for x in **R**. Hence if $a > 0$ and $a \neq 1$, $a^x = \exp(x\ln(a)) = \exp(M_a(x)) = \exp \circ M_a(x)$ for x in **R**. Therefore, the power function a^x is equal to $\exp \circ M_a$, a composite of two bijective functions and so is bijective. The *logarithmic function to the base a*, is defined as the inverse of the power function a^x . If we denote logarithmic function to the base a by $\log_a :(0, \infty) \to \mathbf{R}$, then

$$\log_a = (M_a)^{-1} \circ \exp^{-1}$$

and $\log_a(x) = (M_a)^{-1} \circ \exp^{-1}(x) = (M_a)^{-1}(\ln(x)) = \dfrac{\ln(x)}{\ln(a)}$ for $x > 0$ since the inverse of multiplication by $\ln(a)$ is division by $\ln(a)$. Therefore, $\log_a$ has similar properties as the natural logarithmic function.

11. 2 The Sine and Cosine Functions.

We now give an analytic definition of the sine and cosine functions and show that they coincide with the usual trigonometric ratios and prove their periodicity.

We define sine and cosine by the following power series.

$$\sin(x) = x - \frac{x3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{------------------------------------} \quad (1)$$

and $$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots.$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{------------------------------------} \quad (2)$$

Both series have radius of convergence equal to $+\infty$ by a simple ratio test (see Example 8(6) of Chapter 7). By Theorem 10 Chapter 7 both series define continuous functions on **R**. Thus sine and cosine functions are continuous on **R**. Now for any x in **R**,

$$\cos(-x) = \sum_{n=0}^{\infty}(-1)^n \frac{(-x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = \cos(x).$$

Thus cosine is an even function. Also we have for any x in **R**,

$$\sin(-x) = \sum_{n=0}^{\infty}(-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{(-1)x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = -\sin(x).$$

It follows that sine is an odd function.
By Theorem 11 Chapter 8, we can differentiate both sine and cosine functions term by term. We obtain easily, that for all x in **R**,

$$\sin'(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = \cos(x) \text{ -------------- } (3)$$

and

$$\cos'(x) = \sum_{n=1}^{\infty}(-1)^n \frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = -\sin(x) \text{ -------- } (4)$$

The Fundamental Identities

We shall establish the usual identities for the sine and cosine functions.
Define g: $\mathbf{R} \to \mathbf{R}$ by $g(x) = \sin^2(x) + \cos^2(x)$ for x in **R**. Then it follows from (3) and (4) and the Chain Rule that for all x in **R**,

$$g'(x) = 2\sin(x)\cos(x) + 2\cos(x)(-\sin(x)) = 0.$$

Hence, by Theorem 16 Chapter 4, $g(x) = C$ for all x in **R** for some constant C. Evaluating g at $x = 0$, $C = g(0) = 0 + 1^2 = 1$. Therefore, $g(x) = 1$ for all x in **R**. Thus we have established that for all x in **R**,

$$\sin^2(x) + \cos^2(x) = 1. \quad \text{------------------------------------}$$

(5)
Next we shall derive the addition formulae.
Let a be a point in **R**. Define $h : \mathbf{R} \to \mathbf{R}$ by $h(x) = \sin(x)\cos(a-x)+\cos(x)\sin(a-x)$ for x in **R**. Then h is differentiable on **R** and for all x in **R**, by the Product Rule, Chain Rule, (3) and (4), we have

$$h'(x) = \cos(x)\cos(a-x)+\sin(x)(-\sin(a-x))(-1)-\sin(x)\sin(a-x)+\cos(x)\cos(a-x)(-1) = 0.$$

Therefore, h is a constant function, i.e., $h(x) = D$ for all x in **R** for some constant D. Evaluating h at $x = 0$, we get $D = h(0) = \sin(0)\cos(a)+\cos(0)\sin(a) = \sin(a)$. Therefore, we have that for any a in **R** and all x in **R**,

$$\sin(a) = \sin(x)\cos(a-x)+\cos(x)\sin(a-x). \text{ ---------------------------- } (6)$$

Let $a = x + y$ so that $a - x = y$. We then obtain from (6) the addition formula for sine, that for all x, y in **R**,

$$\sin(x + y) = \sin(x)\cos(y)+\cos(x)\sin(y). \quad \text{---------------------------}$$

(7)
Now fix y and differentiating both sides of (7) with respect to x, regarding both sides of (7) as functions of (x), we obtain

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y) \quad \text{---------------------------}$$

(8)
the addition formula for cosine.

Definition of π

Before we show the periodicity of sine and cosine function, we need to introduce an important number **π**. We shall give a definition that is function theoretic in nature, basically from the cosine function itself.

Recall cos(0) =1. We claim that cos(2) < 0.

$$\begin{aligned}\cos(2) &= \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}\\ &= 1-2+\frac{16}{24}-2^6\sum_{n=3}^{\infty}(-1)^{n-1}\frac{2^{2n-6}}{(2n)!}\\ &= -\frac{1}{3}-2^6\left\{\frac{1}{6!}\left(1-\frac{2^2}{7\cdot 8}\right)+\frac{2^4}{10!}\left(1-\frac{2^2}{11\cdot 12}\right)+\cdots\right\}\\ &= -\frac{1}{3}-2^6\left\{\sum_{k=1}^{\infty}(-1)^{2k}\frac{2^{4k-4}}{(4k+2)!}+\sum_{k=1}^{\infty}(-1)^{2k+1}\frac{2^{4k-2}}{(4k+4)!}\right\}\\ &= -\frac{1}{3}-2^6\left\{\sum_{k=1}^{\infty}\frac{2^{4k-4}}{(4k+2)!}\left(1-\frac{2^2}{(4k+3)(4k+4)}\right)\right\}<0\ .\end{aligned}$$

Thus, since cosine is continuous on [0, 2], by the Intermediate Value Theorem (Theorem 13 Chapter 3), cosine has a zero somewhere between 0 and 2. Following Richard Baltzer (1818-1887), we define π by defining $\frac{\pi}{2}$ to be the smallest positive zero of the cosine function (defined by its power series), i.e.,

$$\frac{\pi}{2}=\inf\{x : x>0 \text{ and } \cos(x)=0\}.$$

We note that $\cos(\frac{\pi}{2})=0$. This is because by the definition of infimum, there exists a sequence (x_n) in {x: x > 0 and cos(x) = 0} such that $x_n \to \frac{\pi}{2}$. (For instance, for any integer $n \geq 1$, there exists x_n in {x: x > 0 and cos(x) = 0} such that $\frac{\pi}{2}\leq x_n<\frac{\pi}{2}+\frac{1}{n}$ since $\frac{\pi}{2}$ is the infimum of {x: x > 0 and cos(x) = 0}). Therefore, by the continuity of cosine at $\frac{\pi}{2}$,

$$\cos(\frac{\pi}{2})=\lim_{n\to\infty}\cos(x_n)=\lim_{n\to\infty}0=0.$$

Next using the identity (5), we have then, $\sin^2(\frac{\pi}{2})=1-\cos^2(\frac{\pi}{2})=1$. Thus $\sin(\frac{\pi}{2})=1\text{or}-1$. Now cos($x$) is positive on $[0,\frac{\pi}{2})$. We can deduce this as follows: Suppose there exists a point y in $[0,\frac{\pi}{2})$ such that cos(y) ≤ 0. Then by the Intermediate Value Theorem there exists a point x such that $0<x\leq y<\frac{\pi}{2}$ such that cos(x) = 0. Hence , $\frac{\pi}{2}\leq x$ since $\frac{\pi}{2}$ is the infimum of {x: x > 0 and cos(x) = 0}). This contradicts that $x<\frac{\pi}{2}$.

Therefore, as sin'(x) = cos(x) > 0 in $[0,\frac{\pi}{2})$, sin(x) is strictly increasing on $[0,\frac{\pi}{2}]$. It follows that, $\sin(\frac{\pi}{2})>\sin(0)=0$. Hence $\sin(\frac{\pi}{2})=1$. We have thus established that

$$\sin(\frac{\pi}{2})=1 \text{ and } \cos(\frac{\pi}{2})=0. \quad \text{------------------------} (9)$$

By (7), $\sin(\pi)=2\sin(\frac{\pi}{2})\cos(\frac{\pi}{2})=0.$ And by induction, for any integer $n \geq 1$, $\sin(n\pi)=0$ and $\sin(-n\pi)=-\sin(n\pi)=0$. Thus for any integer n,

$$\sin(n\pi)=0. \quad \text{---------------------------} (10)$$

Note that by (8) $\cos(\pi)=\cos(\frac{\pi}{2}+\frac{\pi}{2})=\cos^2(\frac{\pi}{2})-\sin^2(\frac{\pi}{2})=-1$ ------------------ (11)

Next we have that for any integer n,

$$\sin((2n+1)\frac{\pi}{2})=(-1)^n. \quad \text{----------------------------} (12)$$

We shall prove (12) by induction on the non-negative n. For $n = 0$, $\sin((2n+1)\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1 = (-1)^0$ so (12) is true for $n = 0$. Assume it is true for $n \geq 0$. Then

$$\begin{aligned}\sin((2n+3)\frac{\pi}{2}) &= \sin((2n+1)\frac{\pi}{2}+\pi)\\ &= \sin((2n+1)\frac{\pi}{2})\cos(\pi)+\cos((2n+1)\frac{\pi}{2})\sin(\pi) && \text{by (7)}\\ &= \sin((2n+1)\frac{\pi}{2})(-1) && \text{by (10) and (11)}\\ &= (-1)^n(-1) = (-1)^{n+1}\end{aligned}$$

by induction hypothesis.

Hence $\sin((2n+1)\frac{\pi}{2}) = (-1)^n$ is true for all integer $n \geq 0$. Now for integer $n \leq -1$,

$$\sin((2n+1)\frac{\pi}{2}) = -\sin(-(2n+1)\frac{\pi}{2}) = -\sin((2(-n-1)+1)\frac{\pi}{2}) = -(-1)^{-n-1} = (-1)^n .$$

This shows that (12) is true for all integer n.

Observe then that for any integer n,

$$\begin{aligned}\cos(n\pi+\frac{\pi}{2}) &= \cos(n\pi)\cos(\frac{\pi}{2})-\sin(n\pi)\sin(\frac{\pi}{2}) && \text{by (8)}\\ &= \cos(n\pi)\cdot 0 - 0\cdot\sin(\frac{\pi}{2}) = 0\end{aligned}$$

by (9) and (10).

That is, for any integer n,

$$\cos(n\pi+\frac{\pi}{2}) = 0 \qquad \text{------------------------} \quad (13)$$

Starting from (12),

$$\begin{aligned}(-1)^n &= \sin((2n+1)\frac{\pi}{2}) = \sin(n\pi+\frac{\pi}{2}) = \sin(n\pi)\cos(\frac{\pi}{2})+\cos(n\pi)\sin(\frac{\pi}{2}) && \text{by (7)}\\ &= 0+\cos(n\pi) = \cos(n\pi)\end{aligned}$$

by (9).

Thus, for any integer n,

$$\cos(n\pi) = (-1)^n . \qquad \text{----------------------------------} \quad (14)$$

Therefore, for any x in **R** and for any integer n,

$$\begin{aligned}\sin(x+2n\pi) &= \sin(x)\cos(2n\pi)+\cos(x)\sin(2n\pi) = \sin(x)(-1)^{2n} && \text{by (7), (14) and (10)}\\ &= \sin(x) && \text{-----------------------------------} \quad (15)\end{aligned}$$

and

$$\begin{aligned}\cos(x+2n\pi) &= \cos(x)\cos(2n\pi)-\sin(x)\sin(2n\pi) = \cos(x)(-1)^{2n} && \text{by (8), (14) and (10)}\\ &= \cos(x) && \text{------------------------------------} \quad (16)\end{aligned}$$

Thus, we have established the periodicity of the sine and cosine functions. To show that the period is 2π, it is sufficient to show that the zero's of sine and cosine are precisely those given by (10) and (13).

Lemma 7. The points in the set $\{n\pi+\frac{\pi}{2} : n \in \mathbf{Z}\}$ are the only zero's of the cosine function.

Proof. Suppose $\cos(a) = 0$ and for some integer k, $k\pi+\frac{\pi}{2} < a < (k+1)\pi+\frac{\pi}{2}$. Then $-\frac{\pi}{2} < a-(k+1)\pi < \frac{\pi}{2}$. It follows that

$$\begin{aligned}\cos(a-(k+1)\pi) &= \cos(a)\cos((k+1)\pi)+\sin(a)\sin((k+1)\pi) && \text{by (8)}\\ &= \cos(a)(-1)^{k+1}+\sin(a)\cdot 0 && \text{by (14) and (10)}\\ &= 0.\end{aligned}$$

But by definition of $\frac{\pi}{2}$, there are no zero's of cosine in $[0,\frac{\pi}{2})$ and since cosine is even, i.e., $\cos(-x) = \cos(x)$ for all x, there are no zero's of cosine in $(-\frac{\pi}{2},0]$. This means there are no zero's of cosine in $(-\frac{\pi}{2}, \frac{\pi}{2})$. We have just shown that $-\frac{\pi}{2} < a-(k+1)\pi < \frac{\pi}{2}$ and $a-(k+1)\pi$ is a zero of cosine thus giving rise to a contradiction. This means that *a* must be of the form $k\pi + \frac{\pi}{2}$ for some integer *k*. We have already shown that the points in the set $\{n\pi + \frac{\pi}{2} : n \in \boldsymbol{Z}\}$ are zero's of cosine (see (13)) and so these points are the only zero's of cosine.

Lemma 8. The points in the set $\{n\pi : n \in \boldsymbol{Z}\}$ are the only zero's of the sine function.

Proof.
If $\sin(a) = 0$, then $\cos(a+\frac{\pi}{2}) = \cos(a)\cos(\frac{\pi}{2}) - \sin(a)\sin(\frac{\pi}{2}) = 0$. Therefore, by Lemma 7 , $a + \frac{\pi}{2} = k\pi + \frac{\pi}{2}$ for some integer *k*. Hence, $a = k\pi$ for some integer *k*. Thus, any zero of the sine function must be of the form $k\pi$ for some integer *k*. But by (10) the points in the set $\{n\pi : n \in \boldsymbol{Z}\}$ are zero's of the sine function and so are precisely the zero's of the sine function.

Lemma 9. The sine function sin: $\mathbf{R} \to \mathbf{R}$ is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and strictly decreasing on $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

Proof. We have previously shown that $\cos(x) > 0$ for *x* in $[0, \frac{\pi}{2})$ and since cos is even, it follows that $\cos(x) > 0$ in $(-\frac{\pi}{2}, 0]$. Therefore, $\cos(x) > 0$ for *x* in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, $\sin'(x) = \cos(x) > 0$ for *x* in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Since $\sin(x)$ is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, by Theorem 19 Chapter 4, $\sin(x)$ is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This proves the first assertion. Now since $\sin(\frac{\pi}{2}) = 1 > 0$ and there are no zero's of $\sin(x)$ in $(0, \pi)$, $\sin(x) > 0$ for *x* in $(0, \pi)$. [If there exists a point *y* such that $0 < y < \pi$ and $\sin(y) < 0$, then by the Intermediate Value Theorem, there is a point y_0 between *y* and $\frac{\pi}{2}$ such that $\sin(y_0) = 0$ contradicting that there are no zero's of $\sin(x)$ in $(0, \pi)$. Thus $\sin(x) > 0$ for all *x* in $(0, \pi)$.] Hence $\cos(\frac{\pi}{2}+x) = \cos(\frac{\pi}{2})\cos(x) - \sin(\frac{\pi}{2})\sin(x) = -\sin(x) < 0$ for all *x* in $(0, \pi)$. This means that $\sin'(x) = \cos(x) < 0$ for all *x* such that $\frac{\pi}{2} < x < \frac{3\pi}{2}$. Hence by Theorem 19 Chapter 4, $\sin(x)$ is strictly decreasing on $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

Lemma 10. The cosine function cos: $\mathbf{R} \to \mathbf{R}$ is strictly decreasing on $[0, \pi]$ and strictly increasing on $[-\pi, 0]$.

Proof. We have shown in the proof of Lemma 9 that $\sin(x) > 0$ for *x* in $(0, \pi)$. Therefore, $\cos'(x) = -\sin(x) < 0$ for *x* in $(0, \pi)$. Thus since cosine is a continuous function, by Theorem 19 Chapter 4, $\cos(x)$ is strctly decreasing on $[0, \pi]$. Since $\sin(x)$ is an odd function, for *x* in $(-\pi, 0)$ $\sin(x) = -\sin(-x) < 0$. Therefore, $\cos'(x) = -\sin(x) > 0$ for *x* in $(-\pi, 0)$. We deduce similarly that cos(x) is strictly increasing on $[-\pi, 0]$.

Lemma 11. The range of the sine and cosine function is $[-1, 1]$.

Proof. Since for all x in **R**, $\sin^2(x) + \cos^2(x) = 1$, $|\sin(x)|, |\cos(x)| \le 1$ for all x in **R**. Therefore, for all x in **R**, $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$. Now because $\sin(\frac{\pi}{2}) = 1$, $\sin(-\frac{\pi}{2}) = -1$ and that $\sin(x)$ is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, by the Intermediate Value Theorem, for any y in $[-1, 1]$ there exists a point x_0 in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\sin(x_0) = y$. This means the range of $\sin(x)$ is $[-1, 1]$. Similarly, since $\cos(0) = 1$ and $\cos(\pi) = -1$, by the Intermediate Value Theorem, the range of $\cos(x)$ is also $[-1, 1]$.

Now, $\sin(-\frac{\pi}{2}) = \sin(2\pi - \frac{\pi}{2}) = \sin(\frac{3\pi}{2}) = -1$ and by Lemma 9, there is no point x in $(-\frac{\pi}{2}, \frac{3\pi}{2})$ with $\sin(x) = -1$, the period of sine function is precisely $\frac{3\pi}{2} - (-\frac{\pi}{2}) = 2\pi$ by (15). Similarly, since $\cos(-\pi) = \cos(-\pi+2\pi) = \cos(\pi) = -1$ and by Lemma 10, there is no point in $(-\pi, \pi)$ with value equals to -1, the period of the cosine function is precisely 2π by (16). Thus we have shown that the sine and cosine functions defined analytically satisfy the same fundamental identities as the trigonometric sine and cosine functions. We shall show that indeed they are the same.

Relation with the Trigonometric Ratio

We have observed that the sine function restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is strictly increasing and maps $[-\frac{\pi}{2}, \frac{\pi}{2}]$ onto $[-1, 1]$. Therefore, its inverse function,

$$\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

is also strictly increasing and continuous. Since $\sin'(x) = \cos(x) \ne 0$ for x in $(-\frac{\pi}{2}, \frac{\pi}{2})$, by Theorem 34 Chapter 4, $\sin^{-1}$ is differentiable on $(-1, 1)$ and

$$\begin{aligned}(\sin^{-1})'(x) &= \frac{1}{\sin'(\sin^{-1}(x))} = \frac{1}{\cos(\sin^{-1}(x))} \\ &= \frac{1}{\sqrt{1-\sin^2(\sin^{-1}(x))}} \\ &= \frac{1}{\sqrt{1-x^2}} \qquad \text{---------------------------} \quad (17)\end{aligned}$$

for $|x| < 1$.

Now consider the unit circle centred at the origin in the Cartesian plane $\mathbf{R}^2$.

The equation of the upper circle is given by $f(x) = \sqrt{1-x^2}\,, -1 \le x \le 1$. Fix an x such that $0 < x < 1$. Consider the arc length P_0P_1 , where $P_0 = (0, 1)$ and P_1 is the point $(x, f(x))$. This is the limit of the length of the polygonal curve $P_0Q_1Q_2\ldots P_1$ as the norm of the partition Δ of $[0, x]$ gets smaller, where partition Δ is given by $\Delta : 0 = x_0 < x_1 < x_2 < \ldots < x_n = x$ for the interval $[0, x]$, $Q_i = (x_i , f(x_i))$, $0 \le i \le n$ and $P_0 = Q_0$ and $Q_n = P_1$. Then the length of the polygonal curve $Q_0Q_1\ldots Q_n$ is an approximation of the arc length P_0P_1. It is given by

$$|Q_0Q_1| + |Q_1Q_2| + \cdots + |Q_{n-1}Q_n| \quad \text{or} \quad \sum_{i=1}^{n} |Q_{i-1}Q_i|.$$

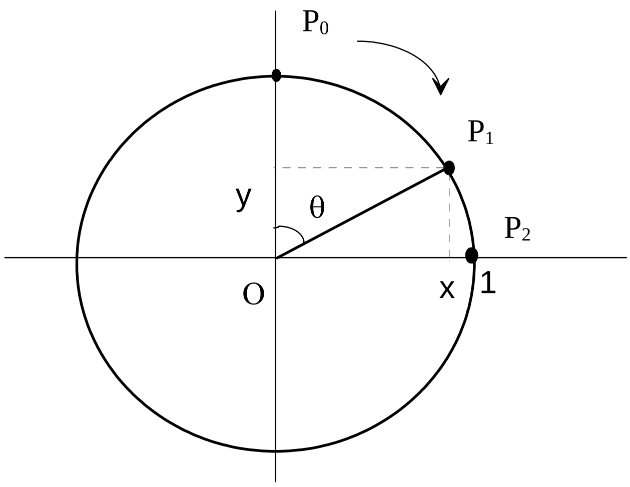
Now the length of each line segment $|Q_{i-1}Q_i|$ is the length of the line joining $(x_{i-1} , f(x_{i-1}))$ to $(x_i , f(x_i))$. Thus by the Pythagorean Theorem,

$$|Q_{i-1}Q_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}\,,\ \ 1 \le i \le n.$$

Therefore, the length of the polygonal curve $Q_0Q_1\ldots Q_n$ is then given by

$$\begin{aligned}
&\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\
&= \sum_{i=1}^{n} \sqrt{\left(1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2\right)}\,(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \sqrt{(1 + (f'(\eta_i))^2)}\,(x_i - x_{i-1}) \quad \text{---------------------------- (18)}
\end{aligned}$$

for some $\eta_i \in [x_{i-1}, x_i]$ by the Mean Value Theorem (Theorem 15 Chapter 4).



The expression (18) is then a Riemann sum for the function $\sqrt{1+(f'(t))^2}$ with respect to the partition $\Delta : 0 = x_0 < x_1 < x_2 < \ldots < x_n = x$. Therefore, as the norm of the partition $\|\Delta\| = \max\{|x_i - x_{i-1}| : 1 \le i \le n\}$ tends to 0, the expression (18) tends to the Riemann integral $\int_0^x \sqrt{1+(f'(t))^2}\,dt$ (See Theorem 36 Chapter 5) provided $\sqrt{1+(f'(t))^2}$ is Riemann integrable on [0, x]. Thus the arc length P_0P_1 is given by the Riemann integral

$$\int_0^x \sqrt{1+(f'(t))^2}\,dt.$$

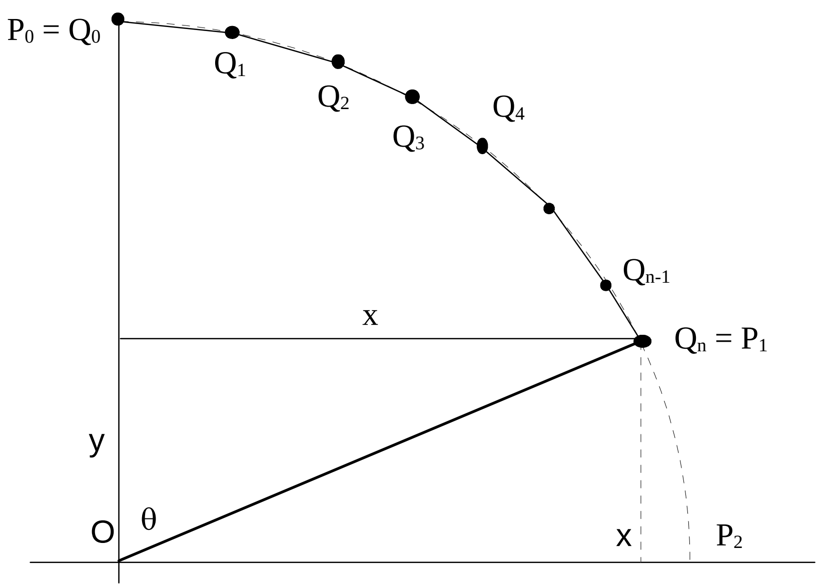
Now, for $0 \le t < 1$, $f'(t) = \dfrac{-t}{\sqrt{1-t^2}}$ and so $\sqrt{1+(f'(t))^2} = \dfrac{1}{\sqrt{1-t^2}}$ is Riemann integrable on [0, x] for $0 \le x < 1$ since it is continuous on [0, x]. Hence the arc length P_0P_1 is given by

$$\begin{aligned}\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt &= \int_0^x (\sin^{-1})'(t)\,dt \quad \text{by (17)}\\ &= \sin^{-1}(x) - \sin^{-1}(0)\\ &\qquad \text{by Darboux Fundamental Theorem of Calculus}\\ &\qquad \text{(Theorem 32 Chapter 5)}\\ &= \sin^{-1}(x).\end{aligned}$$

Now parametrize the angle θ subtended by the arc P_0P_1 at the centre of the unit circle by the arc length P_0P_1, then for $0 \le x < 1$,

$$\theta = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = \sin^{-1}(x) \qquad \text{------------------} \quad (19)$$

For x =1, the arc length of the quarter circle P_0P_2 is given by the improper Riemann integral,



$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt &= \lim_{x\to 1^-} \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt\\ &= \lim_{x\to 1^-} \sin^{-1}(x)\\ &= \sin^{-1}(1) \quad \text{since } \sin^{-1} \text{ is continuous at } x = 1,\end{aligned}$$

$$= \frac{\pi}{2} \qquad \text{--------------------- (20)}$$

Thus from (19) and (20) we see that

$$x = \sin(\theta),\ 0 \le \theta \le \frac{\pi}{2}.$$

Thus, $\sin(\theta) = \dfrac{x}{1} = \dfrac{opposite}{hypotenuse}$ = the trigonometric ratio. Note that, if y denotes $f(x)$, then

$$y = f(x) = \sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta),$$

since $0 \le \theta \le \frac{\pi}{2}$ so that $\cos(\theta) \ge 0$.

Hence, $\cos(\theta) = \dfrac{y}{1} = \dfrac{adjacent}{hypotenuse}$, the familiar trigonometric ratio.

Remark. Our definition of π introduced via $\frac{\pi}{2}$ as the smallest positive number for which cos(x) = 0 is due to Richard Baltzer at Giessen, who was a friend of Kronecker. Edmund Landau (1877-1938) advocated and published this approach in his Göttingen lectures and his *Einführung in die Differentialrechnung und Integralrechnung* (Verlag Nordorff, Groningen) (1934). This approach was attacked as un-German and he was dismissed on racial ground. This surely was a disgraceful episode full of injustice of that time and place and has nothing to do with mathematics. The number π has been a source of mystique and intrigue. For instance the House of Representative of the State of Indiana in USA unanimously passed in 1897 an "Act introducing a new math for all", which proposed two values for π, 4 and 3.2. Fortunately, the senate of Indiana postponed "indefinitely" the adoption of this act.

All the trigonometric functions for complex argument are much more richer. Restricting to real θ, we have the following remarkable formula of Euler,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

We can easily deduce this identity from the complex exponential series, $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ which converges for all z in **C**. Substituting $i\theta$ for z and noting that $\sum_{n=0}^{\infty} \frac{1}{n!}(i\theta)^n$ converges if and only if the real and imaginary parts converge we can easily derive the right hand side of the identity. A particularly beautiful equation emerges from this identity, namely,

$$e^{i\pi} + 1 = 0,$$

linking five of the most important numbers in mathematics.

Exercises 12.

1. Prove that $e^x \ge 1 + x$ for any x in **R**. When does strict inequality hold? Show that the equation $2e^x = (1+x)^2$ has exactly one solution in **R**.
2. Evaluate the following limits without the use of L'Hôpital's rule.

   (a) $\lim_{x\to 0} \dfrac{1-e^{(x^3)}}{x^2\sin(x)}$ (b) $\lim_{x\to 0} \dfrac{x}{\sin(x)}$ (c) $\lim_{x\to 0} \dfrac{1-\cos(x)}{x^2}$ (d) $\lim_{x\to 0} \dfrac{x-\sin(x)}{x^3}$.
3. Prove that the following equation has exactly one solution in **R**.

$$e^{2x} + \cos(x) + x = 0.$$

4. Find the maximum and minimum points of the set $\{\sin(x) + \cos(x) : x \in \mathbf{R}\}$.

5. Let a and b be such that $a^2 + b^2 = 1$. Prove that there exists exactly one number x in the interval $[0, 2\pi)$ such that $\cos(x) = a$ and $\sin(x) = b$.

6. Show that for $a > 0$, $\lim_{n\to\infty} n(a^{1/n} - 1) = \ln(a)$.

7. Using Mean Value Theorem show that there is a number c in the open interval $(1, e)$

   such that $1 = \ln(e) - \ln(1) = (e - 1)/c$. Hence deduce that $e > 2$.

8. Using the definition of the derivative of the natural logarithm, show that

$$\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n}) - \ln(1)}{\frac{1}{n}} = \lim_{n\to\infty} n\ln(1+\frac{1}{n}) = 1.$$

9. Using the definition of power and the continuity of the exponential function show that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$

10. Using only the definition of derivative , prove that

    (a) $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and (b) $\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0.$

11. Show that the *Kepler's equation* $x = a\sin(x) + b$, $|a| < 1$ has exactly one solution in **R**.

12. Which is the larger number, e^{π} or π^{e} ? Justify your answer without using tables or calculators. (Hint: Consider $x - e\ln(x)$.)

13. Suppose g is a function defined for all $x > 0$ and satisfies $g(xy) = g(x) + g(y)$ for all $x, y > 0$ and that $\lim_{x\to 0} \frac{g(1+x)}{x} = 1$. Show that $g(x) = \ln(x)$ for all $x > 0$.