Chapter 14 Improper Integral and Lebesgue Integral

Introduction.

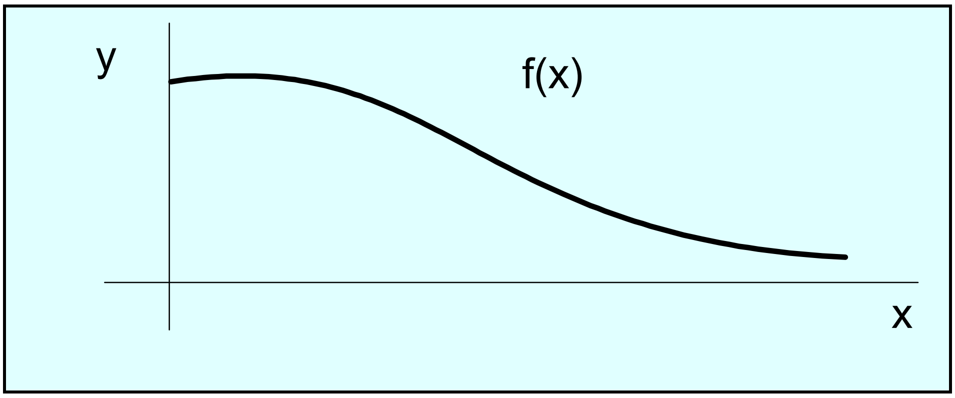
The concept of Riemann integral of a function presupposes that the function is defined on a bounded subset of **R** , usually a connected interval and the function is necessarily bounded on this bounded and connected interval. Since the function is bounded, by arbitrarily assigning values at the boundary or end points of the connected interval, we may always assume that the domain of the function is a closed and bounded interval, say $[a, b]$. Lebesgue's Theorem then asserts that a function $f : [a, b] \to \mathbf{R}$ is Riemann integrable if and only if it is continuous except on a set of measure zero in $[a, b]$. Hence we have two obvious situations when a function is not Riemann integrable. The first is when f is discontinuous on a set of positive measure even though f is bounded. The second is when either f is not bounded or the domain is not a bounded subset of **R**. We shall be concern with this later case. Obviously this presents infinite possibilities and situations. We shall describe the more manageable ones here.

We shall first consider functions on unbounded domains. Domains are assumed to be connected.

Suppose $f :[0, \infty) \to \mathbf{R}$ is such that for any $b > 0$, the restriction of f to $[0, b]$ is Riemann integrable. Cauchy's definition of the improper integral is defined to be

$$\int_0^{+\infty} f = \lim_{b\to+\infty} \int_0^b f(x)dx \ .$$

We shall see later that this definition is equivalent to the Lebesgue integral $L\int_{[0,\infty)} f$ if f is non-negative and the limit exists. We say f is *improperly integrable* if the limit $\lim_{b\to+\infty} \int_0^b f(x)dx$ exists and is finite.



Note that by Lebesgue'sTheorem f is integrable on $[0, b]$ if and only if f is continuous except on a set of measure zero in $[0, b]$. Thus if f is Riemann integrable on $[0, b]$ for any $b > 0$, then f is continuous on $[0, \infty)$ except on a set of measure zero. We can deduce this as follows. Take b to be any integer n and let S_n to be the set of measure zero in $[0, b]$ on which f is discontinuous. Then the set S on which f is discontinuous is a countable union $\cup \{S_n : n \in \boldsymbol{P}\}$ and is a set of measure zero, since each S_n is of measure zero and the measure of S, $\mu(S) \le \sum \mu(S_n) = 0$, where μ is the Lebesgue measure. This means f is continuous almost everywhere on $[0, \infty)$ and so f is (Lebesgue) measurable on $[0, \infty)$.

Similarly, for function $f:(-\infty, 0] \to \mathbf{R}$ such that f is Riemann integrable on $[a, 0]$ for any $a < 0$, we can define the improper integral.

$$\int_{-\infty}^{0} f = \lim_{a \to -\infty} \int_{a}^{0} f(x)dx .$$

And if further more f is non-negative and if the limit exists, then the improper integral is also the Lebesgue integral.

Plainly, for non-positive function f satisfying *either* f is Riemann integrable on $[0, b]$ for any $b > 0$ *or* f is Riemann integrable on $[a, 0]$ for any $a < 0$, the improper integral $\int_0^{+\infty} f$ or $\int_{-\infty}^{0} f$, if it exists is also the Lebesgue integral.

For non-negative function f defined on the whole of $\mathbf{R}$, which is the only interval unbounded on both sides, we can define the improper integral as

$$\int_{-\infty}^{\infty} f = \lim_{t \to \infty} \int_{-t}^{t} f ,$$

if f is Riemann integrable on $[-t, t]$ for each $t > 0$.

We would also like to use the convergence theorems in Lebesgue integration theory to deduce statements about improper Riemann integrals. The notion of Lebesgue integrability is actually a notion about absolute integrability, meaning f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable. Thus if a function f is such that the improper integral of f exists but the improper integral of $|f|$ does not exist, then it is not necessary that the improper Riemann integral is equal to the Lebesgue integral. For instance, take the function f defined by $f(x) = \sin(x)/x$ for $x \neq 0$ and $f(0) = 1$, the improper Riemann integral $\int_0^{+\infty} f$ exists and equals $\pi/2$. But the improper integral $\int_0^{+\infty} |f|$ does not exist. We can deduce this as follows. First note that $\int_{(k-1)\pi}^{k\pi} \left|\frac{\sin(x)}{x}\right| dx \ge \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin(x)| dx = \frac{2}{k\pi}$ for each integer $k > 0$. Thus we get,

$$\int_0^{n\pi} |f(x)| = \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \left|\frac{\sin(x)}{x}\right| dx \ge \sum_{k=1}^{n} \frac{2}{k\pi} .$$

And so since $\sum_{k=1}^{\infty} \frac{2}{k\pi}$ is divergent, $\int_0^{+\infty} |f|$ does not exist. It follows that f is not Lebesgue integrable.

14.1 Improper Integral on Unbounded Domain

Definition 1.

1. Suppose $f:[a, \infty) \to \mathbf{R}$ is such that f is Riemann integrable on every closed sub-interval of $[a, \infty)$. We define the improper Riemann integral
$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^{t} f(x)dx.$$
2. If $f:(-\infty, b] \to \mathbf{R}$ is Riemann integrable on every closed sub-interval of $(-\infty, b]$, then we define the improper integral
$$\int_{-\infty}^{b} f(x)dx = \lim_{t \to -\infty} \int_{t}^{b} f(x)dx.$$

3. If $f: \mathbf{R} \to \mathbf{R}$ is such that f is Riemann integrable on every closed sub-interval of $\mathbf{R}$, then we define the improper integral,
$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{t\to-\infty}\int_t^c f(x)dx + \lim_{t\to+\infty}\int_c^t f(x)dx$$
for some c in $\mathbf{R}$.

In each case if the limit exists, then we say the improper integral is *convergent*, otherwise it is *divergent*.

Plainly, if $\int_a^{+\infty} f$ exists, then $\int_r^{+\infty} f$ exists for any $r \geq a$. Similar conclusion for the other two types of improper integral is valid.

In general, convergence of the integral $\int_a^{+\infty} f(x)dx$ (respectively $\int_{-\infty}^{b} f(x)dx$, $\int_{-\infty}^{+\infty} f(x)dx$) does not imply the convergence of $\int_{-\infty}^{+\infty} |f(x)|dx$ (respectively $\int_{-\infty}^{b} |f(x)|dx$, $\int_{-\infty}^{+\infty} |f(x)|dx$).
If $\int_{-\infty}^{+\infty} |f(x)|dx$ (respectively $\int_{-\infty}^{b} |f(x)|dx$, $\int_{-\infty}^{+\infty} |f(x)|dx$) exists, then we say the improper integral is *absolutely convergent*. Absolute convergence amounts to the function f being Lebesgue integrable.

First we have a convergence theorem.

Theorem 2. Suppose $f:[a, \infty) \to \mathbf{R}$ is Riemann integrable on every closed sub-interval of $[a, \infty)$. Then the improper integral $\int_a^{+\infty} f(x)dx$ exists if and only if for every $\varepsilon > 0$, there exists a number $N > 0$ (depending on ε) such that for all s and $t > N$,
$$\left|\int_s^t f(x)dx\right| < \varepsilon.$$

Proof. Suppose $I = \int_a^{+\infty} f(x)dx$ exists. Then given any $\varepsilon > 0$ there exsits $N > 0$ such that
$$\left|I - \int_a^r f(x)dx\right| < \frac{\varepsilon}{2} \text{ for } r > N.$$
Therefore, $\left|\int_s^t f(x)dx\right| = \left|\int_a^t f(x)dx - \int_a^s f(x)dx\right| = \left|\int_a^t f(x)dx - I + I - \int_a^s f(x)dx\right|$
$$\leq \left|I - \int_a^t f(x)dx\right| + \left|I - \int_a^s f(x)dx\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
for $r, s > N$.
Conversely, assume the condition is satisfied, that is, for every $\varepsilon > 0$, there exists a number $N > 0$ such that for all s and $t > N$, $\left|\int_s^t f(x)dx\right| < \varepsilon$.
Let (a_n) be a sequence in $[a, \infty)$ such that $a_n \to \infty$. Then there exists an integer $M > 0$ such that $n \geq M \Rightarrow a_n > N$. Hence the sequence
$$\left(\int_a^{a_n} f(x)dx\right)$$
satisfies that $n, m \geq M \Rightarrow \left|\int_a^{a_n} f(x)dx - \int_a^{a_m} f(x)dx\right| = \left|\int_{a_n}^{a_m} f(x)dx\right| < \varepsilon$.
Therefore, $\left(\int_a^{a_n} f(x)dx\right)$ is convergent and so by the Cauchy Princoiple of Convertgence, it is convergent. Suppose $\int_a^{a_n} f(x)dx \to A$. We shall show that $\int_a^{\infty} f(x)dx = A$. We shall show that for any sequence (b_n) in $[a, \infty)$ such that $b_n \to \infty$, $\int_a^{b_n} f(x)dx \to A$. Suppose $\int_a^{b_n} f(x)dx \to B$. For any $\varepsilon > 0$, there exists an integer N_1 such that

$$n \geq N_1 \Rightarrow \left| \int_a^{b_n} f(x)dx - B \right| < \frac{\varepsilon}{3}. \quad \text{------------------------- (1)}$$

Since $\int_a^{a_n} f(x)dx \to A$, there exists an integer N_2 such that

$$n \geq N_2 \Rightarrow \left| \int_a^{a_n} f(x)dx - A \right| < \frac{\varepsilon}{3}. \quad \text{------------------------- (2)}$$

By assumption, there exists a number $N > 0$ such that for all s and $t > N$,

$$\left| \int_s^t f(x)dx \right| < \frac{\varepsilon}{3}$$

Since $a_n \to \infty$ and $b_n \to \infty$. and there exists an integer $M > 0$ such that

$$n \geq M \Rightarrow a_n \text{ , } b_n > N.$$

Therefore, $$n \geq M \Rightarrow \left| \int_{a_n}^{b_n} f(x)dx \right| < \frac{\varepsilon}{3}. \quad \text{--------------------------- (3)}$$

Therefore, for $n \geq \max(N_1, N_2, M)$,

$$\begin{aligned}|B - A| &= \left| B - \int_a^{b_n} f(x)dx + \int_{a_n}^{b_n} f(x)dx + \int_a^{a_n} f(x)dx - A \right| \\ &\leq \left| B - \int_a^{b_n} f(x)dx \right| + \left| \int_{a_n}^{b_n} f(x)dx \right| + \left| \int_a^{a_n} f(x)dx - A \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $B = A$. This shows that $\int_a^{\infty} f(x)dx$ is convergent. This completes the proof.

We have a similar result for the integral $\int_{-\infty}^{b} f(x)dx$

Theorem 3. Suppose $f : (-\infty, b] \to \mathbf{R}$ is Riemann integrable on every closed sub-interval of $(-\infty, b]$. Then the improper integral $\int_{-\infty}^{b} f(x)dx$ exists if and only if for every $\varepsilon > 0$, there exists a number $N < 0$ such that for all s and $t < N$,

$$\left| \int_s^t f(x)dx \right| < \varepsilon$$

The proof of Theorem 3 is similar to that of Theorem 2.

Theorem 2 and Theorem 3 then gives a convergence criterion for integrals of the type $\int_{-\infty}^{+\infty} f(x)dx$.

For non-negative function $f : \mathbf{R} \to \mathbf{R}$ we have the following results concerning convergence.

Theorem 4. If $f : \mathbf{R} \to \mathbf{R}$ is Riemann integrable on every closed sub-interval of $\mathbf{R}$ and is non-negative, then

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{t \to \infty} \int_{-t}^{t} f(x)dx \text{ .}$$

Proof. Suppose $\int_{-\infty}^{+\infty} f(x)dx$ is convergent. Then $\lim_{t \to \infty} \int_{-t}^{0} f(x)dx$ and $\lim_{t \to \infty} \int_{0}^{t} f(x)dx$ are convergent. Therefore,

$$\begin{aligned}\lim_{t \to \infty} \int_{-t}^{t} f(x)dx &= \lim_{t \to \infty} \left(\int_{-t}^{0} f(x)dx + \lim_{t \to \infty} \int_{0}^{t} f(x)dx \right) \\ &= \lim_{t \to \infty} \int_{-t}^{0} f(x)dx + \lim_{t \to \infty} \int_{0}^{t} f(x)dx \\ &= \int_{-\infty}^{+\infty} f(x)dx.\end{aligned}$$

Conversely, suppose $\lim_{t \to \infty} \int_{-t}^{t} f(x)dx$ is convergent. Then since f is non-negative, $g(t) = \int_0^t f(x)dx$ is an increasing function on $[0, \infty)$. In particular ,

$$g(t) = \int_0^t f(x)dx \leq \int_{-t}^{t} f(x)dx \leq \lim_{t \to \infty} \int_{-t}^{t} f(x)dx.$$

Hence, g is bounded above and so $\lim_{t\to\infty}\int_0^t f(x)dx = \sup\{g(t) : t \in [0,\infty)\}$ by the completeness of **R**. It can be shown similarly that $h(t) = \int_{-t}^0 f(x)dx$ is an increasing function bounded above by $\lim_{t\to\infty}\int_{-t}^t f(x)dx$. Thus $\lim_{t\to\infty}\int_{-t}^0 f(x)dx = \sup\{h(t) : t \in [0,\infty)\}$. This means $\int_{-\infty}^{+\infty} f(x)dx$ is convergent. By the above discussion, $\int_{-\infty}^{+\infty} f(x)dx = \lim_{t\to\infty}\int_{-t}^t f(x)dx$. This completes the proof.

Suppose $f : \mathbf{R} \to \mathbf{R}$ is Riemann integrable on every closed sub-interval of **R**. We can define the *principal value* of $\int_{-\infty}^{+\infty} f(x)dx$ to be $\lim_{t\to\infty}\int_{-t}^t f(x)dx$. Note that though $\int_{-\infty}^{+\infty} f(x)dx$ may be divergent its principal value may exist. For example, take $f(x) = x$. Then plainly, $\int_{-\infty}^{+\infty} f(x)dx$ is divergent but its principal value is 0 as $\int_{-t}^t f(x)dx = 0$.

Remark. Theorem 4 simply states that for a non-negative function $f : \mathbf{R} \to \mathbf{R}$, which is Riemann integrable on every closed sub-interval of **R**, the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ is equal to its principal value. The proof of Theorem 4 also shows that if we know that $\int_{-\infty}^{+\infty} f(x)dx$ is convergent, then $\int_{-\infty}^{+\infty} f(x)dx$, the improper integral, is equal to its principal value. Thus very often we just need to compute the principal value in order to obtain the improper integral once convergence is ascertained.

Example 5.

(1) Let $f(x) = 1/(1+x^2)$. Then for any $t > 0$, $\int_0^t f(x)dx = [\tan^{-1}(x)]_0^t = \tan^{-1}(t)$. Thus, $\int_0^\infty f(x)dx = \lim_{t\to\infty}\tan^{-1}(t) = \frac{\pi}{2}$. Also $\int_{-t}^t f(x)dx = [\tan^{-1}(x)]_{-t}^t = 2\tan^{-1}(t)$ and so

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{t\to\infty} 2\tan^{-1}(t) = \pi.$$

(2) $\int_0^\infty x e^{-x} dx$.

$$\int_0^t x e^{-x}dx = [-e^{-x}x]_0^t + \int_0^t e^{-x}dx \quad \text{by integration by parts}$$

$$= [-e^{-x}x - e^{-x}]_0^t = -e^{-t}t - e^{-t} + 0 + 1 = 1 - \tfrac{t}{e^t} - \tfrac{1}{e^t}.$$

Therefore,

$$\int_0^\infty x e^{-x}dx = \lim_{t\to+\infty}\int_0^t x e^{-x}dx = \lim_{t\to+\infty}\left[1 - \tfrac{t}{e^t} - \tfrac{1}{e^t}\right].$$

Now $\lim_{t\to+\infty}\frac{t}{e^t} = \lim_{t\to+\infty}\frac{1}{e^t} = 0$ by L' Hôpital's Rule. Thus $\int_0^\infty x e^{-x}dx = 1 - 0 - 0 = 1$.

(3) $\int_0^{+\infty} x dx$ is divergent since $\lim_{t\to+\infty}\int_0^t x dx = \lim_{t\to+\infty}\frac{t^2}{2} = +\infty$.

Definition 6. If $\int_a^{+\infty}|f(x)|dx$ is convergent, then we say $\int_a^{+\infty} f(x)dx$ is *absolutely convergent*. Similarly, if $\int_{-\infty}^b |f(x)|dx$ is convergent, then we say $\int_{-\infty}^b f(x)dx$ is *absolutely convergent*. Likewise, if $\int_{-\infty}^{\infty}|f(x)|dx$ is convergent, then we say $\int_{-\infty}^{\infty} f(x)dx$ is *absolutely convergent*.

Theorem 7. If $\int_a^{+\infty}|f(x)|dx$ **respectively,** $\int_{-\infty}^b|f(x)|dx$, $\int_{-\infty}^{\infty}|f(x)|dx$ is convergent, then $\int_a^{+\infty} f(x)dx$ **respectively,** $\int_{-\infty}^b f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$ is convergent.

Proof. Suppose $\int_a^{+\infty} |f(x)|dx$ is convergent. Then by Theorem 2, for every $\varepsilon > 0$, there exists a number $N > 0$ such that for all s and $t > N$,

$$\left|\int_s^t |f(x)|dx\right| < \varepsilon.$$

It follows that for all s and $t > N$, $\left|\int_s^t f(x)dx\right| \le \left|\int_s^t |f(x)|dx\right| < \varepsilon$. Hence, by Theorem 2, $\int_a^{+\infty} f(x)dx$ is convergent. For the case of $\int_{-\infty}^b |f(x)|dx$ and $\int_{-\infty}^{\infty} |f(x)|dx$, the proof proceeds in the same way.

Definition 8. If $\int_a^{+\infty} f(x)dx$ **respectively,** $\int_{-\infty}^b f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$ is convergent and $\int_a^{+\infty} |f(x)|dx$ **respectively,** $\int_{-\infty}^b |f(x)|dx$, $\int_{-\infty}^{\infty} |f(x)|dx$ is divergent, then we say $\int_a^{+\infty} f(x)dx$ **respectively,** $\int_{-\infty}^b f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$ is *conditionally convergent*.

Example 9. Let $f(x) = \begin{cases} \dfrac{\sin(x)}{x}, \text{ if } x \ne 0 \\ 1, \text{if } x = 0 \end{cases}$. Then $\int_0^{\infty} f(x)dx = \int_0^{\infty} \frac{\sin(x)}{x}dx$. We can establish that $\int_0^{\infty} \frac{\sin(x)}{x}dx$ is convergent as follows.

For $t > s > 0$,

$$\int_s^t \frac{\sin(x)}{x}dx = \left[-\frac{\cos(x)}{x}\right]_s^t - \int_s^t \frac{\cos(x)}{x^2}dx = \frac{\cos(s)}{s} - \frac{\cos(t)}{t} - \int_s^t \frac{\cos(x)}{x^2}dx$$

by integration by parts. Therefore,

$$\left|\int_s^t \frac{\sin(x)}{x}dx\right| \le \frac{1}{s} + \frac{1}{t} + \int_s^t \frac{1}{x^2}dx = \frac{2}{s}. \quad \text{--------------------------} \quad (1).$$

For any $\varepsilon > 0$, there exists a number $K > 0$ such that $2/K < \varepsilon$. It follows that for $t > s > K$,

$$\left|\int_s^t \frac{\sin(x)}{x}dx\right| \le \frac{2}{s} < \frac{2}{K} < \varepsilon .$$

Therefore, by Theorem 2, $\int_0^{\infty} \frac{\sin(x)}{x}dx$ is convergent. Actually $\int_0^{\infty} \frac{\sin(x)}{x}dx = \frac{\pi}{2}$. However, $\int_0^{\infty} \left|\frac{\sin(x)}{x}\right|dx$ is divergent. To see this, note that

$$\int_{(k-1)\pi}^{k\pi} \frac{|\sin(x)|}{x}dx \ge \frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin(x)|dx = \frac{2}{k\pi}$$

and so

$$\int_0^{n\pi} \frac{|\sin(x)|}{x}dx \ge \frac{2}{\pi}\sum_{k=1}^{n} \frac{1}{k}.$$

Therefore, since $\sum_{k=1}^{n} \frac{1}{k}$ is divergent, it follows that $\int_0^{\infty} \frac{|\sin(x)|}{x}dx$ is divergent. Hence $\int_0^{\infty} \frac{\sin(x)}{x}dx$ is conditionally convergent.

For non-negative function f we have also the following comparison test for convergence. Indeed we have used the following result in the proof of Theorem 4 implicitly.

Theorem 10. If $\int_a^{\infty} g(x)dx$ exists and $0 \le f(x) \le k\, g(x)$ for $x \ge a$ and $f:[a, \infty) \to \mathbf{R}$ is Riemann integrable on every closed sub-interval of $[a, \infty)$, then $\int_a^{\infty} f(x)dx$ exists.

Proof. By hypothesis, $h(t) = \int_a^t f(x)dx$ is an increasing function on $[a, \infty)$ and bounded above by $k\int_a^{\infty} g(x)dx$. Let $K = \sup\{ h(t) : t \in [a, \infty)\}$. Then given any

$\varepsilon > 0$, there exists a t_0 in $[a, \infty)\}$ such that $K - \varepsilon < h(t_0) \le K$. Now given any sequence (a_n) in $[a, \infty)$ such that $a_n \to \infty$, there exists an integer N such that $n \ge N \Rightarrow a_n > t_0$. Therefore,

$$n \ge N \Rightarrow h(a_n) \ge h(t_0) \Rightarrow K - \varepsilon < h(t_0) \le h(a_n) \le K \Rightarrow | h(a_n) - K| < \varepsilon.$$

This shows that for any sequence (a_n) in $[a, \infty)$ such that $a_n \to \infty$, $h(a_n) \to K$. Therefore, by definition, $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \int_a^t f(x)dx = K$. This means $\int_a^\infty f(x)dx$ exists.

Note that we have used the completeness property of **R** by using the supremum of $\{h(t) : t \in [a, \infty)\}$. The following is a technical observation which may come in handy.

Lemma 11. If for every $\varepsilon > 0$, there exists a number R such that $| f(x) - f(y) | < \varepsilon$ whenever $x, y > R$, then $f(x)$ tends to a limit as x tends to ∞.

Proof. Take any sequence (a_n) such that $a_n \to \infty$. Then there exists an integer N such that $n \ge N \Rightarrow a_n > R$. Thus $| f(a_n) - f(a_m) | < \varepsilon$. This means $(f(a_n))$ is a Cauchy sequence. Consequently $(f(a_n))$ is convergent. Suppose $f(a_n) \to L$. We shall show that for any sequence (b_n) such that $b_n \to \infty$, $f(b_n) \to L$. By assumption, there exists a number R_1 such that $| f(x) - f(y) | < \varepsilon/2$ whenever $x, y > R_1$. Plainly, there exists an integer M such that $n \ge M \Rightarrow a_n, b_n > R_1$. Since $f(a_n) \to L$, there exists an integer M_1 such that

$$n \ge M_1 \Rightarrow | f(b_n) - L | < \varepsilon/2.$$

Therefore, $n \ge max(M, M_1) \Rightarrow | f(b_n) - L | \le | f(b_n) - f(a_n) + f(a_n) - L |$

$$\le | f(b_n) - f(a_n) | + | f(a_n) - L \; < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that $f(b_n) \to L$. Hence $\lim_{x\to\infty} f(x) = L$.

Theorem 12. The following is equivalent to , $\int_{-\infty}^{\infty} f(x)dx = S$.
Given any $\varepsilon > 0$, there exists a number $R > 0$ such that $u, v \ge R \Rightarrow \left| \int_{-u}^{v} f - S \right| < \varepsilon$.

Proof.
(i) Suppose , $\int_{-\infty}^{\infty} f(x)dx = S$. Then given any $\varepsilon > 0$ there exists a number $R > 0$ such that

$$u, \ge R \Rightarrow \left| \int_{-u}^{u} f - S \right| < \frac{\varepsilon}{2} \text{ and that } \left| \int_{v}^{u} f \right| < \frac{\varepsilon}{2} \text{ and } \left| \int_{-v}^{-u} f \right| < \frac{\varepsilon}{2}.$$

Therefore, for u, $v \ge R$, we have that if $v > u$,

$$\left| \int_{-u}^{v} f - S \right| = \left| \int_{-u}^{u} f - S + \int_{u}^{v} f \right| \le \left| \int_{-u}^{u} f - S \right| + \left| \int_{u}^{v} f \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and if $v < u$,

$$\left| \int_{-u}^{v} f \right| = \left| \int_{-u}^{-v} f + \int_{-v}^{v} f - S \right| \le \left| \int_{-u}^{-v} f \right| + \left| \int_{-v}^{v} f - S \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, for u, $v \ge R$, $\left| \int_{-u}^{v} f - S \right| < \varepsilon$.
(ii) Conversely, suppose for any $\varepsilon > 0$, there exists a number $R > 0$ such that $u, v \ge R \Rightarrow \left| \int_{-u}^{v} f - S \right| < \varepsilon$.
Then for any $x, y > R$, $\left| \int_{x}^{y} f \right| = \left| \int_{-R}^{y} f - S + S - \int_{-R}^{x} f \right| \le \left| \int_{-R}^{y} f - S \right| + \left| S - \int_{-R}^{x} f \right| < 2\varepsilon.$

It follows by Theorem 2 that $\int_0^\infty f(x)dx$ is convergent. We deduce in the same way that $\int_{-\infty}^0 f(x)dx$ is convergent. Hence $\int_{-\infty}^\infty f(x)dx$ is convergent. Let $\int_{-\infty}^\infty f(x)dx = S'$. Then by part (i) there exists $u \geq R$ such that $\left|\int_{-u}^u f - S'\right| < \varepsilon$. But $\left|\int_{-u}^u f - S\right| < \varepsilon$. Hence

$$|S' - S| = \left|S' - \int_{-u}^u f + \int_{-u}^u f - S\right| \leq \left|\int_{-u}^u f - S'\right| + \left|\int_{-u}^u f - S\right| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since ε is arbitrary, $S' = S$.

We next have the following convergence criterion that we shall use later,

Theorem 13. Suppose there exist real numbers R, $M > 0$ such that for all $|x| > R$, $x^2|f(x)| \leq M$ and f is Riemann integrable on any closed interval in **R**. Then $\int_{-\infty}^\infty f(x)dx$ is absolutely convergent.

Proof. f is Riemann integrable on $[0, R]$. For $x > R > 0$, $|f(x)| \leq M/x^2$. Since $\int_R^\infty \frac{1}{x^2}dx$ is convergent, by Theorem 10, $\int_R^\infty |f(x)|dx$ is convergent. It follows that $\int_R^\infty f(x)dx$ is absolutely convergent and therefore convergent (Theorem 7). We show similarly that $\int_{-\infty}^{-R} f(x)dx$ is absolutely convergent. Therefore,

$$\int_0^\infty |f(x)|dx = \int_0^R |f(x)|dx + \int_R^\infty |f(x)|dx$$

is convergent. Likewise $\int_{-\infty}^0 |f(x)|dx = \int_{-\infty}^{-R} |f(x)|dx + \int_{-R}^0 |f(x)|dx$ is convergent. Therefore, $\int_{-\infty}^\infty f(x)dx$ is absolutely convergent.

Remark. Similarly result holds for $\int_a^\infty f(x)dx$ or $\int_{-\infty}^b f(x)dx$.

Example 14. $\int_1^\infty e^{-x}x^p dx$ converges for all p.

Note that $x^2 e^{-x}x^p = \frac{x^{p+2}}{e^x} \to 0$ as $x \to \infty$. Therefore, there exists a number $R > 0$ such that $x^2 |e^{-x}x^p| \leq 1$ for $x > R$. Since $e^{-x}x^p = \frac{x^{p+2}}{e^x}$ is continuous on [1, R] and so is Riemann integrable on [1, R], it follows from Theorem 13 that $\int_1^\infty e^{-x}x^p dx$ is convergent.

14.2 Improper Integrals on Bounded Domain, Part 1

Definition 15

(1) Suppose f is continuous on $(a, b]$ and that $\lim_{x\to a^+} f(x) = \pm\infty$, then
$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx \text{, if the limit exists.}$$
(2) Suppose f is continuous on $[a, b)$ and that $\lim_{x\to b^-} f(x) = \pm\infty$, then
$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx \text{, if the limit exists.}$$
(3) Suppose f is continuous on $[a, b] - \{c\}$, where $a < c < b$ and that $\lim_{x\to c} |f(x)| = +\infty$. Then $\int_a^b f(x)dx = \lim_{t\to c^-}\int_a^t f(x)dx + \lim_{s\to c^+}\int_s^b f(x)dx$, if the limits exist.

Example 16.

(1) $\int_0^1 x\ln(x)dx = \lim\limits_{t\to 0^+}\int_t^1 x\ln(x)dx$. Using integration by parts,

$$\int_t^1 x\ln(x)dx = [\tfrac{x^2}{2}\ln(x)]_t^1 - \int_t^1 \tfrac{x^2}{2}\cdot\tfrac{1}{x}dx = [\tfrac{x^2}{2}\ln(x)]_t^1 - \int_t^1 \tfrac{x}{2}dx$$
$$= [\tfrac{x^2}{2}\ln(x) - \tfrac{1}{4}x^2]_t^1 = -\tfrac{1}{4} - \tfrac{t^2}{2}\ln(t) + \tfrac{1}{4}t^2.$$

But $\lim\limits_{t\to 0^+} t^2\ln(t) = \lim\limits_{t\to 0^+}\frac{\ln(t)}{\frac{1}{t^2}} = \lim\limits_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{2}{t^3}} = \lim\limits_{t\to 0^+} -\frac{t^2}{2} = 0$.

Thus, $\int_t^1 x\ln(x)dx = -\frac{1}{4} - \frac{1}{2}t^2\ln(t) + \frac{1}{4}t^2 \to -\frac{1}{4}$ as $t\to 0^+$.

Therefore $\int_0^1 x\ln(x)dx = -\frac{1}{4}$.

(2) $\int_{-1}^1 \left|\frac{1}{x^{\frac{1}{3}}}\right| dx$

$$\int_0^1 \left|\frac{1}{x^{\frac{1}{3}}}\right| dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^{\frac{1}{3}}}dx = \lim_{t\to 0^+}[\tfrac{3}{2}x^{\frac{2}{3}}]_t^1 = \lim_{t\to 0^+}\left(\tfrac{3}{2} - \tfrac{3}{2}t^{\frac{2}{3}}\right) = \tfrac{3}{2}\ .$$

Similarly,

$$\int_{-1}^0 \left|\frac{1}{x^{\frac{1}{3}}}\right| dx = -\lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^{\frac{1}{3}}}dx = -\lim_{t\to 0^-}[\tfrac{3}{2}x^{\frac{2}{3}}]_{-1}^t = -\lim_{t\to 0^-}\tfrac{3}{2}\left(t^{\frac{2}{3}} - 1\right) = \tfrac{3}{2}\ .$$

Therefore, $\int_{-1}^1 \left|\frac{1}{x^{\frac{1}{3}}}\right| dx = \lim\limits_{t\to 0^+}\int_t^1 \frac{1}{x^{\frac{1}{3}}}dx - \lim\limits_{t\to 0^-}\int_{-1}^t \frac{1}{x^{\frac{1}{3}}}dx = \frac{3}{2} + \frac{3}{2} = 3$.

(3) $\int_0^2 \frac{1}{(x-1)^2}dx = \lim\limits_{t\to 1^-}\int_0^t \frac{1}{(x-1)^2}dx + \lim\limits_{t\to 1^+}\int_t^2 \frac{1}{(x-1)^2}dx$

$$= \lim_{t\to 1^-}[-\tfrac{1}{x-1}]_0^t + \lim_{t\to 1^+}[-\tfrac{1}{x-1}]_t^2\ .$$

But both limits do not exist, therefore the integral $\int_0^2 \frac{1}{(x-1)^2}dx$ is *divergent*.

In a later section we shall consider convergence criterion and relation with Lebesgue integral.

14.3 Lebesgue Measure and Lebesgue Integral

In this section we give a review of Lebesgue theory.

Definition 17. For any function f , define the following associated functions:

$f^+(x) = (\,|f(x)| + f(x)\,)/2 \geq 0,$

$f^-(x) = (\,|f(x)| - f(x)\,)/2 \geq 0.$

Then $f^+(x) = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ 0 \text{ if } f(x) < 0 \end{cases}$ and $f^-(x) = \begin{cases} -f(x) \text{ if } f(x) \leq 0 \\ 0 \text{ if } f(x) > 0 \end{cases}$

In particular, $f(x) = f^+(x) - f^-(x)$ and $|f(x)| = f^+(x) + f^-(x)$.

Here we shall present the definition of measurable function and the definition of (Lebesgue) measure on **R**. $|f|$ is measurable need not imply that f is measurable. For instance, if we assume the existence of non-measurable set E, then if we define $f(x) = 1$ for x in E and $f(x) = -1$ for x not in E, then $|f|$ is measurable but f is not. In view of this , we shall always assume or state the requirement that f be measurable.

Lebesgue Measure.

Suppose I is an interval, define $\lambda(I)$ to be $(b-a)$ if I is bounded and a and b are end points of I, with $a < b$. Define $\lambda(I)$ to be ∞ if I is unbounded.

Let Γ be the family of all countable collections of disjoint open intervals. Define a function on Γ into the extended non-negative real numbers

$$\lambda^* : \Gamma \to \overline{\mathbf{R}^+} = [0, \infty]$$

by $\lambda^*(\gamma) = \sum_{I\in\gamma} \lambda(I)$.

Suppose E is a subset of $\mathbf{R}$. Define the family of countable cover of E by disjoint open intervals to be $\mathcal{C}(E) = \{ \gamma : \gamma$ is a countable collection of disjoint open intervals covering $E\}$.

Define the *Lebesgue outer measure* of E by

$$\mu^*(E) = \inf \{\lambda^*(\gamma) : \gamma \in \mathcal{C}(E) \}.$$

Thus $\mu^*(E)$ is either finite or ∞.

Definition 18. A subset E of $\mathbf{R}$ is said to be Lebesgue measurable if and only if for all subset X of $\mathbf{R}$,

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X \setminus E),$$

equivalently if for all $X \subseteq \mathbf{R}$,

$$\mu^*(X) \geq \mu^*(X \cap E) + \mu^*(X \setminus E).$$

If E is measurable, the *Lebesgue measure* of E, $\mu(E)$ is defined to be the outer measure $\mu^*(E)$.

Properties 19.

1. $\varnothing$ and $\mathbf{R}$ are measurable,
2. If E is measurable, then its complement $\mathbf{R} \setminus E$ is also measurable.
3. Any open subset of $\mathbf{R}$ is measurable and so any closed subset of $\mathbf{R}$ is also measurable. Hence any interval is measurable.
4. Countable union of measurable subsets is measurable and so countable intersection of measurable subsets is also measurable.
5. If $\mu^*(E) = 0$, then E and any subset of E is also measurable.
6. If $E_1, E_2, \ldots$ is a countable collection of disjoint measurable sets, then
   $$\mu(\cup\{E_n : n = 1, 2, \cdots\}) = \sum_{n=1}^{\infty} \mu(E_n).$$

Measurable Function

Definition 20.

A function $f : E \to \mathbf{R}$ is *measurable* if and only if for any open subset U of $\mathbf{R}$, $f^{-1}(U)$ is measurable.

Hence it is necessary that E be measurable for f to be measurable.

The following is an immediate consequence of the definition.

Properties 21.

(1) Suppose E is measurable. Then every continuous function $f : E \to \mathbf{R}$ is *measurable*.

(2) Suppose $f : E \to \mathbf{R}$ is *measurable* and $f(E) \subseteq X$ and $g : X \to \mathbf{R}$ is continuous. Then the composite function $g \circ f : E \to \mathbf{R}$ is measurable.

(3) Suppose $f : E \to \mathbf{R}$ and $g : E \to \mathbf{R}$ are *measurable*, then $f + g$ and $f g$ are measurable.

Lemma 22. If $f : E \to \mathbf{R}$ is measurable, then $|f|$, f^+ and f^- are measurable.

Proof. If f is measurable, then $f \cdot f$ is measurable by Property 21 (3) and by Property (2), $|f| = \sqrt{(f \cdot f)}$ is measurable. Thus by Property 21 (3), $f^+ = (|f| + f)/2$ is measurable. Likewise, $f^- = (|f| - f)/2$ is measurable.

Simple Function and Lebesgue Integral

Definition 23.

A function $s : E \to \mathbf{R}$ is said to be *simple* if s is measurable and s takes on only a finite number of values.

For any subset A of $\mathbf{R}$, the characteristic function of A, χ_A is the function defined by

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}.$$

Thus if $s : E \to \mathbf{R}$ is simple, $s(E) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $A_i = s^{-1}(\alpha_i)$, $i = 1, \ldots, n$, then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

Note that each A_i is measurable since s is measurable.

For a non-negative simple function $s : E \to \mathbf{R}^+ = [0, \infty)$, we define the *Lebesgue integral* of s to be

$$\int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

where $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, $s(E) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $A_i = s^{-1}(\alpha_i)$, $i = 1, \ldots, n$. Hence $\int_E s d\mu$ is either finite or infinite.

If the Lebesgue measure of E, $\mu(E)$ is finite, we can define for a simple function $s : E \to \mathbf{R}$ (not necessarily non-negative), the Lebesgue integral of s to be

$$\int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i),$$

where $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, $s(E) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $A_i = s^{-1}(\alpha_i)$, $i = 1, \ldots, n$.

Note that we have then $\int_E s d\mu$ is finite since each $\mu(A_i)$ is finite.

We say the simple function s is *Lebesgue integrable* if its Lebesgue integral is finite. Thus a simple function defined on a set E of finite measure is Lebesgue integrable.

Definition 24.

Suppose $\mu(E)$ is finite, i.e., $\mu(E) < \infty$. Suppose $f : E \to \mathbf{R}$ is a bounded function.

We say f is *Lebesgue integrable* if and only if given ε > 0 there exists simple functions u and v on E such that

$$u \leq f \leq v$$

and $\int_E (v-u)d\mu < \varepsilon$. Following Darboux we define the Lebesgue integral

$$\int_E f\, d\mu = \sup\{\int_E \phi\, d\mu : \phi \text{ is a simple function on } E \text{ such that } \phi \leq f\}.$$

Notice the Lebesgue integral of u and v are analogous to the lower Darboux and upper Darboux sums for a bounded function defined by step functions defined for a partition of a finite interval.

Theorem 25.
Suppose μ(E) is finite, i.e., μ(E) < ∞. Suppose $f: E \to \mathbf{R}$ is a bounded function. Then f is Lebesgue integrable if and only if f is measurable if and only if $|f|$ is Lebesgue integrable. In particular, when f is Lebesgue integrable, f^+ and f^- are Lebesgue integrable and $\int_E f\, d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$.

Proof. Suppose μ(E) is finite, i.e., μ(E) < ∞. Suppose $f: E \to \mathbf{R}$ is a bounded function. We can define the *lower Lebesgue integral* to be

$$L\int_E f\, d\mu = \sup\{\int_E \phi\, d\mu : \phi \text{ is a simple function on } E \text{ such that } \phi \leq f\}$$

and the *upper Lebesgue integral* to be

$$U\int_E f\, d\mu = \inf\{\int_E \phi\, d\mu : \phi \text{ is a simple function on } E \text{ such that } f \leq \phi\}.$$

The condition in Definition 24 is equivalent to that the lower and upper Lebesgue integrals are the same. It can be shown that f is measurable if and only if the lower and upper Lebesgue integrals are the same. We omit the details here as it is a little tedious. (For a reference see Proposition 3, Chapter 4, Royden's "Real Analysis", Third Edition.)
Note that $f = f^+ - f^-$. Since f is measurable, f^+ and f^- are measurable and so their lower and upper Lebesgue integrals are the same. It follows that $\int_E f\, d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$.

Now we consider unbounded functions.

Definition 26. Lebesgue integral of non-negative function.
Suppose $f: E \to \mathbf{R}^+$ is a non-negative function and E is measurable. Define *the Lebesgue integral of* f to be

$$\int_E f\, d\mu = \sup\{\int_E s\, d\mu : s \text{ is a non-negative simple function on } E \text{ such that } 0 \leq s \leq f\}.$$

We say f is *Lebesgue integrable* (or summable) if the integral $\int_E f\, d\mu$ is finite.

The following is well known.

Lemma 27. Suppose $f: E \to \mathbf{R}^+$ is a non-negative measurable function. Then there exists an increasing sequence of non-negative measurable simple functions $\{s_n : E \to \mathbf{R}^+\}$ such that $s_1 \leq s_2 \leq \ldots \leq f$ and (s_n) converges pointwise to f.

Theorem 28. Lebesgue Monotone Convergence Theorem.
Suppose $\{ f_n : E \to \mathbf{R}^+ \}$ is a sequence of non-negative increasing measurable functions tending pointwise to $f : E \to \mathbf{R}^+$. Then f is measurable and $\int_E f_n d\mu \nearrow \int_E f\, d\mu$.

In view of Theorem 28 we have

Corollary 29. Suppose $f : E \to \mathbf{R}^+$ is a non-negative measurable function. Then there exists an increasing sequence of non-negative measurable simple functions $\{s_n : E \to \mathbf{R}^+ \}$ such that $s_n \le f$, (s_n) converges pointwise to f and $\int_E s_n d\mu \nearrow \int_E f\, d\mu$.

In some sense Theorem 25 motivates the next definition.

Definition 30. Lebesgue integral of a real valued functions.
Suppose $f : E \to \mathbf{R}$ is a measurable function. We say f is Lebesgue integrable if and only if $\int_E |f| d\mu < \infty$. Note that then we have $\int_E f^+ d\mu, \int_E f^- d\mu \le \int_E |f| d\mu < \infty$.
We define the Lebesgue integral of f to be $\int_E f\, d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$

Theorem 31. Lebesgue Dominated Convergence Theorem.
Suppose $\{ f_n : E \to \mathbf{R} \}$ is a sequence of measurable functions tending pointwise (almost everywhere) to $f : E \to \mathbf{R}$. Suppose further that there exists a Lebesgue integrable function g : $E \to \mathbf{R}$ such that $|f_n| \le$ g for all n in $\boldsymbol{P}$ *and* for almost all x in E (i.e., $|f_n| \le$ g for all n in $\boldsymbol{P}$ except perhaps on a set of measure zero). Then f_n and f are Lebesgue integrable and $\int_E f_n d\mu \to \int_E f\, d\mu$.

Remark. A reference for Lebesgue Monotone Convergence Theorem and Lebesgue Dominated Convergence Theorem is Royden's "Real Analysis, Pearson Prentice Hall Third Edition", Page 87 Theorem 10, page 91 Theorem 16. Lemma 27 is well known and is often stated without proof (see for example, Proposition 7 in Chapter 11 page 260 of Royden's "Real Analysis".) One can produce the required simple functions by an appropriate dissection of the subinterval [0, n] in [0, ∞) for each n and use the pre-image of the subintervals of the dissection to define the simple function for each n.

For bounded function defined on a bounded interval the following is well known.

Theorem 32. Lebesgue.
Suppose f:[a, b] $\to \mathbf{R}$ is a bounded function on a closed and bounded interval [a, b].

If f is Riemann integrable, then f is Lebesgue integrable, hence measurable and the Lebesgue integral.

$$\int_{[a,b]} f\, d\mu = \int_a^b f(x) dx,$$

where the right integral is the Riemann integral of f on [a, b].

The following is a characterization of Riemann integrable functions in terms of continuity.

Theorem 33. Lebesgue.
Suppose $f:[a, b] \to \mathbf{R}$ is a bounded function on a closed and bounded interval $[a, b]$. f is Riemann integrable if and only if f is continuous except perhaps on a set of measure zero, i.e., f is continuous almost everywhere.

14. 4 Improper Integral and Lebesgue integral.

Theorem 34. Suppose $f:[a, \infty) \to \mathbf{R}^+$ is a non-negative function which is Riemann integrable on every closed sub-interval of $[a, \infty)$. Then f is measurable. f is Lebesgue integrable if and only if the improper (Riemann) integral $\int_a^{+\infty} f(x)dx$ exists. When f is Lebesgue integrable, the Lebesgue integral

$$\int_{[a,\infty)} f\,d\mu = \int_a^{+\infty} f(x)dx .$$

Proof.
Since f is Riemann integrable on every sub interval of $[a, \infty)$, for each integer n in $\boldsymbol{P}$, define $f_n : [a, \infty) \to \mathbf{R}^+$ by $f_n(x) = \begin{cases} f(x), & a \le x \le a+n \\ 0, & x > a+n \end{cases}$. Then (f_n) is a monotone increasing sequence converging pointwise to f. Each f_n is Riemann integrable on $[a, a+n]$ and 0 on $(a+n, \infty)$. Therefore, f_n restricted to $[a, a+n]$ is measurable by Theorem 32 and so for any open U not containing 0, $f_n^{-1}(U) = (f_n\,|_{[a,a+n]})^{-1}(U)$ is measurable. For any open U containing 0, $f_n^{-1}(U) = (f_n\,|_{[a,a+n]})^{-1}(U) \cup (a+n, \infty)$ which is a union of measurable sets and so is measurable. This shows that f_n is measurable. By definition 26 and the fact that $[a,\infty)$ is a disjoint union of $[a,a+n]$ and $(a+n, \infty)$ the Lebesgue integral

$$\int_{[a,\infty)} f_n\,d\mu = \int_{[a,a+n]} f_n d\mu + \int_{[a+n,\infty)} f_n d\mu = \int_{[a,a+n]} f_n d\mu + 0 = \int_a^{a+n} f(x)dx$$

since the Lebesgue integral $\int_{[a,a+n]} f_n d\mu = \int_a^{a+n} f(x)dx$ by Theorem 32.
By the Lebesgue Monotone Convergence Theorem (Theorem 28), f is measurable and

$$\int_{[a,\infty)} f_n\,d\mu \to \int_{[a,\infty)} f\,d\mu.$$

That is, $\int_{[a,\infty)} f\,d\mu = \lim_{n\to\infty} \int_a^{a+n} f(x)dx = \lim_{t\to\infty} \int_a^{t} f(x)dx.$
f is Lebesgue integrable if and only if $\int_{[a,\infty)} f\,d\mu$ is finite if and only if the sequence $\left(\int_{[a,\infty)} f_n\,d\mu\right) = \left(\int_a^{a+n} f(x)dx\right)$ has a limit if and only if the improper integral $\int_a^{+\infty} f(x)dx$ exists.
Thus when f is Lebesgue integrable, $\int_{[a,\infty)} f\,d\mu = \int_a^{+\infty} f(x)dx$.

Similarly we can prove the following analogous results.

Theorem 35. Suppose $f : (-\infty, a] \to \mathbf{R}^+$ is a non-negative function which is Riemann integrable on every closed sub-interval of $(-\infty, a]$ Then f is measurable. f is Lebesgue integrable if and only if the improper (Riemann) integral $\int_{-\infty}^{a} f(x)dx$ exists. When f is Lebesgue integrable, the Lebesgue integral

$$\int_{(-\infty,a]} f\,d\mu = \int_{-\infty}^{a} f(x)dx \;\; (= \lim_{t\to -\infty} \int_t^{a} f(x)dx) .$$

Theorem 36. Suppose $f:(-\infty,\infty)\to\mathbf{R}^+$ is a non-negative function which is Riemann integrable on every closed sub-interval of $(-\infty,\infty)$ Then f is measurable. f is Lebesgue integrable if and only if the improper (Riemann) integral $\int_{-\infty}^{\infty} f(x)dx$ exists. When f is Lebesgue integrable, the Lebesgue integral

$$\int_{(-\infty,\infty)} f\,d\mu = \int_{-\infty}^{\infty} f(x)dx (=\lim_{t\to\infty}\int_{-t}^{t} f(x)dx) .$$

Remark. The proof of Theorem 36 is the same as for Theorem 34 by defining f_n to be equal to f on $[-n, n]$ and 0 elsewhere and making use of the fact that the improper (Riemann) integral $\int_{-\infty}^{+\infty} f(x)dx = \lim_{t\to\infty}\int_{-t}^{t} f(x)dx$ from Theorem 4.

Theorem 37. Suppose $f:[a,\infty)\to\mathbf{R}$ is Riemann integrable on every closed sub-interval of $[a,\infty)$. Then f is Lebesgue integrable if and only if the improper (Riemann) integral $\int_a^{+\infty}|f(x)|dx$ exists. When f is Lebesgue integrable, the Lebesgue integral

$$\int_{[a,\infty)} f\,d\mu = \int_a^{+\infty} f(x)dx ,$$

where the integral on the right is the improper Riemann integral.

Proof. By Definition 30, f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable if and only if the improper integral $\int_a^{+\infty}|f(x)|dx$ exists by Theorem 34. If f is Lebesgue integrable,

$$\int_{[a,\infty)} f\,d\mu = \int_{[a,\infty)} f^+d\mu - \int_{[a,\infty)} f^-d\mu.$$

Note that $|f|$ is Lebesgue integrable implies both f^+ and f^- are Lebesgue integrable. By Theorem 34, $\int_{[a,\infty)} f^+d\mu = \int_a^{+\infty} f^+(x)dx$ and $\int_{[a,\infty)} f^-d\mu = \int_a^{+\infty} f^-(x)dx$. Therefore, when f is Lebesgue integrable,

$$\begin{aligned}\int_{[a,\infty)} f\,d\mu &= \int_{[a,\infty)} f^+d\mu - \int_{[a,\infty)} f^-d\mu\\ &= \int_a^{\infty} f^+(x)dx - \int_a^{\infty} f^-(x)dx\\ &= \int_a^{\infty}(f^+(x)-f^-(x))dx = \int_a^{\infty} f(x)dx,\end{aligned}$$

where all the improper integrals are convergent.

The following result is similar to Theorem 37 and is proved in exactly the same way.

Theorem 38. Suppose $f:(-\infty, a]\to\mathbf{R}$ is Riemann integrable on every closed sub-interval of $(-\infty, a]$. Then f is Lebesgue integrable if and only if the improper (Riemann) integral $\int_{-\infty}^{a}|f(x)|dx$ exists. Moreover when f is Lebesgue integrable, the Lebesgue integral

$$\int_{(-\infty,a]} f\,d\mu = \int_{-\infty}^{a} f(x)dx ,$$

where the integral on the right is the improper Riemann integral.

In view of Theorem 37 we can phrase absolute convergence of improper integral with Lebesgue integrability.

Definition 39. In view of Definition 6 of absolute convergence, if $f:[a,\infty)\to\mathbf{R}$ is Riemann integrable on every closed sub-interval of $[a,\infty)$, then $\int_a^{+\infty} f(x)dx$ is *absolutely convergent* if and only if $\int_a^{+\infty} |f(x)|dx$ is convergent if and only if f is Lebesgue integrable on $[a,\infty)$. If $\int_a^{+\infty} f(x)dx$ is conditionally convergent (see Definition 8), then this means $\int_a^{+\infty} f(x)dx$ is convergent but $\int_a^{+\infty} |f(x)|dx$ is divergent. Hence conditional convergent implies non Lebesgue integrability.
Similarly, if $f:(-\infty,a]\to\mathbf{R}$ is Riemann integrable on every closed sub-interval of $(-\infty,a]$, then $\int_{-\infty}^{a} f(x)dx$ is *absolutely convergent* if and only if $\int_{-\infty}^{a} |f(x)|dx$ is convergent if and only if f is Lebesgue integrable on $(-\infty,a]$. $\int_{-\infty}^{a} f(x)dx$ is conditionally convergent (see Definition 8), means that $\int_{-\infty}^{a} f(x)dx$ exists but $\int_{-\infty}^{a} |f(x)|dx$ is divergent or equivalently f is not Lebesgue integrable but the improper integral is convergent.

Recall from Definition 8 that for $f:(-\infty,\infty)\to\mathbf{R}$ which is Riemann integrable on every closed sub-interval of $(-\infty,\infty)$, the improper integral $\int_{-\infty}^{\infty} f(x)dx$ *converges absolutely* if and only if for some a in $\mathbf{R}$ (hence any a) the improper integrals $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ converge absolutely if and only if f is Lebesgue integrable on $(-\infty,a]$ and on $[a,\infty)$ if and only if f is Lebesgue integrable. Hence absolute convergence of f is equivalent to f being Lebesgue integrable.

$\int_{-\infty}^{\infty} f(x)dx$ converges conditionally if either one of the improper integrals $\int_a^{\infty} f(x)dx$ or $\int_{-\infty}^{a} f(x)dx$ converges conditionally which implies that f is not Lebesgue integrable on $\mathbf{R}$.

Example 40.
Let $f(x)=\begin{cases}\dfrac{\sin(x)}{x}, \text{ if } x\neq 0\\ 1, \text{ if } x=0\end{cases}$. Then $\int_0^{\infty} f(x)dx=\int_0^{\infty}\dfrac{\sin(x)}{x}dx=\dfrac{\pi}{2}$. We have shown in Example 9 that f is conditionally convergent hence f is not Lebesgue integrable on $[0,\infty)$.

14.5 Improper Integrals on Bounded Domain, Part 2

If $f:[a,b]\to\mathbf{R}$ is unbounded, then f is not Riemann integrable. We shall consider the case when f has only one singularity at one of the end points of $[a,b]$.

Definition 41. Suppose $f:[a,b]\to\mathbf{R}$ is such that f is Riemann integrable on any subinterval in $(a,b]$.
(1) Suppose there is a sequence of points (x_n) in $[a,b]$ such that $|f(x_n)|\to\infty$ as $x_n\to a$. Then if the limit

$$\lim_{t\to a^+}\int_t^b f(x)dx$$

exists, this is defined to be the *improper Riemann integral* of f on $[a,b]$.

(2) Suppose there is a sequence of points (y_n) in $[a,b]$ such that $|f(y_n)|\to\infty$ as $y_n\to b$. Then if the limit

$$\lim_{t\to b^-}\int_a^t f(x)dx$$

exists, this is defined to be the *improper Riemann integral* of f on $[a, b]$.

(3) Suppose there is a sequence of points (x_n) in $[a, b]$ such that $|f(x_n)| \to\infty$ as $x_n \to a$ and a sequence of points (y_n) in $[a, b]$ such that $|f(y_n)| \to\infty$ as $y_n \to b$. If for some c in (a, b) the improper integrals, $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ exist, then the improper integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

We have a convergence criterion similar to Theorem 2.

Theorem 42. Suppose $f:[a, b] \to \mathbf{R}^+$ is such that f is Riemann integrable on any subinterval in $(a, b]$. Then the improper integral $\int_a^b f(x)dx$ exists if and only if for every $\varepsilon > 0$, there exists a number $k > a$ (depending on ε) such that for all $a < s < t < k$,

$$\int_s^t f(x)dx < \varepsilon.$$

The proof is similar to Theorem 2.

We now consider non-negative function $f: [a, b] \to \mathbf{R}^+$, Suppose f is measurable. What is the relation between Lebesgue integral on $[a, b]$ and improper Cauchy Riemann integral? It will be very similar to the case of unbounded domain.

Theorem 43. Suppose $f:[a, b] \to \mathbf{R}^+$ is such that f is Riemann integrable on any subinterval in (a, b). Suppose f is either unbounded at a or at b or even at both points. Then f is Lebesgue integrable on $[a, b]$ if and only if the limit of the sequence of Riemann integrals $\left(\int_a^b f_n(x)dx\right)$, where $f_n = \min(f, n)$, exists.

Proof. For each integer n in $\boldsymbol{P}$, $f_n:[a, b] \to \mathbf{R}^+$ is defined by

$$f_n(x) = \begin{cases} f(x), \text{ if } f(x) \le n \\ n, \text{ if } f(x) > n \end{cases}.$$

where $f_n = \min(f, n)$. Take a decreasing sequence (x_n) in $(a, (a+b)/a)$ such that $x_n \to a$ and an increasing sequence (y_n) in $((a+b)/a, b)$ such that and $y_n \to b$. Then f is Riemann integrable on $[x_n, y_n]$ and so $f|_{[x_n, y_n]}$ is continuous except on a set A_n of measure zero. Therefore, f is continuous except perhaps on the set $E = \cup\{A_n : n$ in $\boldsymbol{P}\}$ whose measure is also zero since each A_n has measure zero. Since $f_n = \min(f, n) = (f + n - |f - n|)/2$, f_n is continuous except perhaps on a set of measure zero. Since each f_n is plainly bounded, f_n is Riemann integrable and so is measurable and Lebesgue integrable by Theorem 32. Plainly (f_n) is an increasing sequence and so is a sequence of increasing non-negative measurable functions converging pointwise to f. Therefore, by the Lebesgue Monotone Convergence Theorem (Theorem 28), f is measurable and the Lebesgue integrals,

$$\int_{[a,b]} f_n d\mu \nearrow \int_{[a,b]} f d\mu.$$

Since $\int_{[a,b]} f_n d\mu$ is equal to the Riemann integral $\int_a^b f_n(x)dx$,

$$\int_a^b f_n(x)dx \nearrow \int_{[a,b]} f\, d\mu.$$

Thus f is Lebesgue integrable if and only if the limit $\lim_{n\to\infty} \int_a^b f_n(x)dx$ exists. This completes the proof.

Now we shall apply the above proof to a different sequence of functions.

Theorem 44. Suppose $f:[a, b] \to \mathbf{R}^+$ is such that f is Riemann integrable on any subinterval in $(a, b]$. Suppose there is a strictly decreasing sequence (x_n) in (a, b) such that $x_n \to a$ but $f(x_n) \to \infty$, that is, f is unbounded in any neighbourhood containing a. Then f is Lebesgue integrable on $[a, b]$ if and only if the improper Cauchy Riemann integral $\int_a^b f(x)dx$ exists.

Proof. For each integer n in $\boldsymbol{P}$, define $f_n:[a, b] \to \mathbf{R}^+$ by

$$f_n(x) = \begin{cases} f(x), \text{ if } x \geq x_n \\ 0, \text{if } a < x < x_n \\ f(a) \text{ if } x = a \end{cases}.$$

Then since f is Riemann integrable on $[x_n, b]$, f_n is Riemann integrable for each n in $\boldsymbol{P}$. Therefore, by Theorem 32, f_n is measurable and Lebesgue integrable for each n in $\boldsymbol{P}$. Obviously, (f_n) is an increasing sequence of non-negative functions converging pointwise to f. Therefore, by the Lebesgue Monotone Convergence Theorem, f is measurable and

$$\int_{[a,b]} f_n d\mu \nearrow \int_{[a,b]} f\, d\mu.$$

Now the Lebesgue integral $\int_{[a,b]} f_n d\mu = \int_a^b f_n(x)dx = \int_{x_n}^b f_n(x)dx$. Hence

$$\int_{x_n}^b f_n(x)dx \nearrow \int_{[a,b]} f\, d\mu.$$

If f is Lebesgue Integrable, then the limit $\lim_{n\to\infty} \int_{x_n}^b f_n(x)dx$ exists for any sequence (x_n) in (a, b) such that $x_n \to a^+$. That is to say, the improper Cauchy Riemann integral $\int_a^b f(x)dx$ exists. On the other hand, if the improper Riemann integral exists, then for any sequence $x_n \to a$, $\int_{x_n}^b f_n(x)dx \nearrow \int_{[a,b]} f\, d\mu$ and so $\int_{[a,b]} f\, d\mu < \infty$ and f is Lebesgue integrable.

Theorem 45. Suppose $f:[a, b] \to \mathbf{R}$ is such that f is Riemann integrable on any subinterval in $(a, b]$. Suppose there is a strictly decreasing sequence (x_n) in (a, b) such that $x_n \to a$ but $|f(x_n)| \to \infty$, that is, $|f|$ is unbounded in any neighbourhood containing a. Then f is Lebesgue integrable on $[a, b]$ if and only if the improper Cauchy Riemann integral $\int_a^b |f(x)|dx$ exists. Moreover, when f is Lebesgue integrable, the Lebesgue integral $\int_{[a,b]} f\, d\mu$ is equal to the improper Cauchy Riemann integral, $\int_a^b f x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$.

Proof. By Definition 30, f is Lebesgue integrable if and only if $|f|$ is Lebesgue integrable if and only if both f^+ and f^- are Lebesgue integrable. By Theorem 44, the Lebesgue integral $\int_{[a,\, b]} f^+ d\mu = \lim_{t\to a^+} \int_t^b f^+(x)dx$ = the improper Cauchy Riemann integral $\int_a^b f^+(x)dx$. Similarly the Lebesgue integral $\int_{[a,\,b]} f^- d\mu = \lim_{t\to a^+}\int_t^b f^-(x)dx = \int_a^b f^-(x)dx$. In particular, if the improper Riemann integral $\int_a^b |f(x)|dx$ exists, then the improper integrals $\int_a^b f^+(x)dx$ and $\int_a^b f^-(x)dx$ exist (by the Comparison Test) and the Lebesgue integral

$$\begin{aligned}\int_{[a,\,b]} |f|\,d\mu &= \int_{[a,\,b]} f^+ d\mu + \int_{[a,\,b]} f^+ d\mu \\ &= \int_a^b f^+(x)dx + \int_a^b f^-(x)dx = \int_a^b |f(x)|\,dx\end{aligned}$$

the sum of the improper Cauchy Riemann integrals. Moreover the Lebesgue integral $\int_{[a,\,b]} f\,d\mu$ too exists and

$$\int_{[a,\,b]} f\,d\mu = \int_{[a,\,b]} f^+ d\mu - \int_{[a,\,b]} f^+ d\mu = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx\,.$$

Conversely, suppose that f is Lebesgue integrable, then $|f|$ is Lebesgue integrable and so both f^+ and f^- are Lebesgue integrable. Therefore, by Theorem 44, both improper Cauchy Riemann integrals $\int_a^b f^+(x)dx$ and $\int_a^b f^-(x)dx$ exist and are finite. It follows that

$$\int_a^b |f(x)|\,dx = \int_a^b f^+(x)dx + \int_a^b f^-(x)dx$$

exists. Moreover, $\displaystyle \int_a^b f(x)dx = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx$

$$= \int_{[a,\,b]} f^+ d\mu - \int_{[a,\,b]} f^- d\mu = \int_{[a,\,b]} f\,d\mu.$$

We have a similar result when the function is unbounded in any neighbourhood containing b.

Theorem 46. Suppose $f:[a,\,b] \to \mathbf{R}$ is such that f is Riemann integrable on any subinterval in $[a,\,b)$. Suppose there is a strictly increasing sequence (x_n) in $(a,\; b)$ such that $x_n \to b$ but $|f(x_n)| \to \infty$, that is, $|f|$ is unbounded in any neighbourhood containing a. Then f is Lebesgue integrable on $[a,\,b]$ if and only if the improper Cauchy Riemann integral $\int_a^b |f(x)|dx$ exists. Moreover, when f is Lebesgue integrable, the Lebesgue integral $\int_{[a,b]} f\,d\mu$ is equal to the improper Cauchy Riemann integral, $\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$.

The proof of Theorem 46 is exactly the same as for Theorem 45.

Definition 47.
Suppose one of the following is satisfied.
(1) There is a sequence of points (x_n) in $[a,\,b]$ such that $|f(x_n)| \to \infty$ as $x_n \to a$;
(2) There is a sequence of points (y_n) in $[a,\,b]$ such that $|f(y_n)| \to \infty$ as $y_n \to b$;
(3) There is a sequence of points (x_n) in $[a,\,b]$ such that $|f(x_n)| \to \infty$ as $x_n \to a$ and a sequence of points (y_n) in $[a,\,b]$ such that $|f(y_n)| \to \infty$ as $y_n \to b$.
The improper Cauchy Riemann integral $\int_a^b f(x)dx$ *converges absolutely* if $\int_a^b |f(x)|dx$ exists (or converges) or equivalently if both $\int_a^b f^+(x)dx$ and $\int_a^b f^-(x)dx$ exist or in view of Theorems 45 and 46 if f is Lebesgue integrable on $[a,\,b]$.

The improper Cauchy Riemann integral $\int_a^b f(x)dx$ *converges conditionally* if $\int_a^b f(x)dx$ converges and $\int_a^b |f(x)|dx$ diverges (does not exist) or equivalently f is not Lebesgue integrable on $[a, b]$.

If for a point b, there is a sequence of points (x_n) such that $|f(x_n)| \to \infty$ as $x_n \to b$, i.e., f is not bounded in any neighbourhood containing b, then we say f has a *singularity* at b. Thus condition (1) is the same as saying that f has a singularity at a and condition (2) a singularity at b while condition (3) a singularity at both a and b.

14.6 Convergence Tests

Theorem 48. Cauchy Criterion.
Suppose $f:[a, b] \to \mathbf{R}$ is such that f is Riemann integrable on any subinterval in $[a, b)$. Then the improper Cauchy Riemann integral $\int_a^b f(x)dx$ exists if and only if for every $\varepsilon > 0$, there exists a number $M < b$ (depending on ε) such that for all $M < s < t < b$,

$$\left|\int_s^t f(x)dx\right| < \varepsilon.$$

The proof is similar to Theorem 2.

Theorem 49.
Let b be a finite point or ∞ and f a non-negative function on $[a, b)$. Suppose f is Riemann integrable on $[a, c]$ for any c, $a < c < b$ with a singularity at b.
Then the improper Cauchy Riemann integral $\int_a^b f(x)dx$ exists if and only if the function defined by $F(x) = \int_a^x f(t)dt$ is bounded on $[a, b)$.

Proof. Plainly since $F(x)$ is non-decreasing, $\underset{x\to b^-}{Lim} F(x)$ exists if and only if F(x) is bounded by the completeness property of **R,** i.e., the improper Cauchy Riemann integral $\int_a^b f(x)dx$ is convergent if and only if $F(x)$ is bounded.

Theorem 50. Comparison Test.
Suppose f and g are non-negative functions defined on $I = [a, b)$, integrable on $[a, c]$ for any c, $a < c < b$. Suppose f and g each has a singularity at b and that $f(x) \le g(x)$ for all x in I.
Then

(1) If the improper Cauchy Riemann integral $\int_a^b g(x)dx$ converges, then so does $\int_a^b f(x)dx$.

(2) If the improper Cauchy Riemann integral $\int_a^b f(x)dx$ diverges, then so does $\int_a^b g(x)dx$.

Proof. This a consequence of Theorem 49. If $\int_a^b g(x)dx$ converges, then $\int_a^x g(t)dt$ is bounded on I. Since $\int_a^x f(t)dt \le \int_a^x g(t)dt$, $\int_a^x f(t)dt$ is bounded consequently $\int_a^b f(x)dx$ is convergent.

$\int_a^b f(x)dx$ diverges implies that $\int_a^x f(t)dt$ is unbounded and so $\int_a^x g(t)dt$ is unbounded and $\int_a^b g(x)dx$ diverges.

Theorem 51. Limit Comparison Test
Suppose f and g are positive functions defined on $I = [a, b)$, integrable on $[a, c]$ for any c, $a < c < b$. Suppose that $0 < \lim_{x\to b^-} \frac{f(x)}{g(x)} < \infty$.
Then either both improper Cauchy Riemann integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or they both diverge.

Proof. Suppose $\lim_{x\to b^-} \frac{f(x)}{g(x)} = q > 0$. Then taking $\varepsilon = q/2 > 0$, there exists $\delta > 0$ such that

$$b-\delta < x < b \Rightarrow \left|\frac{f(x)}{g(x)} - q\right| < \varepsilon = \frac{q}{2}.$$

Therefore, taking $X = b - \delta$, for all x such that $X < x < b$, *we have*

$$\frac{q}{2} < \frac{f(x)}{g(x)} < \frac{3q}{2},$$

i.e.,
$$\frac{q}{2}g(x) < f(x) < \frac{3q}{2}g(x). \quad \text{-------------------------} \quad (1)$$

By *hypothesis* $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge if and only if $\int_X^b f(x)dx$ and $\int_X^b g(x)dx$ converge. It follows from (1) and Theorem 50 *that either both* $\int_X^b f(x)dx$ and $\int_X^b g(x)dx$ converge or they both diverge. Consequently, either both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or they both diverge.

14.7 Anti-derivative and Improper Integral

We can make use of primitive or more specifically limit of primitive to determine Lebesgue integral. For example Theorem 45 tells us that the Lebesgue integral is an improper Cauchy Riemann integral. Then we can make use of the Fundamental Theorem of Calculus to calculate the improper integral and hence the Lebesgue integral.

Theorem 52.
(1) Suppose $f : [a, \infty) \to \mathbf{R}$ is continuous and non-negative on $[a, \infty)$. Suppose F is an antiderivative of f. Then f is Lebesgue integrable on $[a, \infty)$ if and only if the limit $\lim_{x\to\infty} F(x)$ exists (and is finite).
In this case, the Lebesgue integral

$$\int_{[a,\infty)} f\,d\mu = \int_a^\infty f(x)dx = \lim_{x\to\infty} F(x) - F(a).$$

(2) Suppose $f : (a, b] \to \mathbf{R}$ is continuous and non-negative on $(a, b]$ and has a singularity at a. Suppose F is an antiderivative of f. Then f is Lebesgue integrable on $(a, b]$ if and only if the limit $\lim_{x\to a^+} F(x)$ exists (and is finite).
In this case, the Lebesgue integral

$$\int_{(a,b])} f\,d\mu = \int_a^b f(x)dx = F(b) - \lim_{x\to a^+} F(x).$$

Proof. Part (1) is a consequence of Theorem 34 and the Fundamental Theorem of Calculus, as $\int_a^\infty f(x)dx = \lim_{x\to\infty}\int_a^x f(t)dt = \lim_{x\to\infty} F(x) - F(a)$. Part (2) is a consequence of Theorem 44 and the Fundamental Theorem of Calculus.

Example 53.

For $x > 0$ and $k > 0$, an anti-derivative for $x^{-k} = \frac{1}{x^k}$ is $\frac{1}{-k+1}x^{-k+1}$ for $k \neq 1$ and $\ln(x)$ for $k = 1$.

Therefore, for $a > 0$,

$$\int_a^x \frac{1}{t^k}dt = \begin{cases} \frac{1}{-k+1}x^{-k+1} - \frac{1}{-k+1}a^{-k+1}, \text{ for } k > 0 \text{ and } k \neq 1 \\ \ln(x) - \ln(a), \text{ for } k = 1 \end{cases}.$$

Now $(-k+1) < 0$ if and only if $k > 1$ and so $\int_a^\infty \frac{1}{t^k}dt$ converges when $k > 1$, since $\frac{1}{-k+1}x^{-k+1} \to 0$ as $x \to \infty$. Thus $\int_a^\infty \frac{1}{t^k}dt = \frac{1}{k-1}a^{-k+1} = \frac{1}{k-1}\frac{1}{a^{k-1}}$ for $k > 1$ and $\int_a^\infty \frac{1}{t^k}dt$ diverges when $0 < k < 1$. Likewise, $\int_a^\infty \frac{1}{t}dt$ diverges because $\ln(x) \to \infty$ as $x \to \infty$. Plainly $\int_a^\infty \frac{1}{t^k}dt$ diverges when $k \leq 0$.

Thus we have the following summary

(1) $\int_a^\infty \frac{1}{x^k}dx$ for $a > 0$ converges if and only if $k > 1$.

We can similarly deduce the following

(2) $\int_0^a \frac{1}{x^k}dx$ for $a > 0$ converges if and only if $k < 1$.

It then follows that

(3) $\int_a^\infty \frac{1}{(x-x_0)^k}dx$ for $a > x_0$ converges if and only if $k > 1$

and

(4) $\int_{x_0}^a \frac{1}{(x-x_0)^k}dx$ for $a > x_0$ converges if and only if $k < 1$.

Example 54.

(1) $\int_1^\infty e^{-x}x^p dx$ converges for all p.

Use a simple comparison test.

$e^{-x}x^{p+2} \to 0$ as $x \to \infty$.

Therefore, there exist a number M such that $x \geq M \Rightarrow e^{-x}x^{p+2} < 1 \Rightarrow e^{-x}x^p < \frac{1}{x^2}$.

Since $\int_1^\infty \frac{1}{x^2}dx$ is convergent, by the Comparison Test (Theorem 50) $\int_1^\infty e^{-x}x^p dx$ is convergent.

Hence the Lebesgue integral $\int_{[1,\infty)} e^{-x}x^p d\mu = \int_1^\infty e^{-x}x^p dx$.

(2) $\int_0^a e^{\frac{1}{x}}x^p dx$, $a > 0$ diverges for all p.

Note that $e^{\frac{1}{x}}x^{p+1} \to \infty$ as $x \to 0^+$ because $e^{\frac{1}{x}}x^{p+1} > \dfrac{1}{n!}\dfrac{1}{x^n}x^{p+1}$ for $n > p+1$ and $\lim\limits_{x\to 0^+} \dfrac{1}{n!}\dfrac{1}{x^n}x^{p+1} = \infty$. Therefore, there exists ε > 0 such that

$$0 < x < \varepsilon \Rightarrow e^{\frac{1}{x}}x^{p+1} > 1 \Rightarrow e^{\frac{1}{x}}x^{p} > \frac{1}{x}.$$

Since $\int_0^{\varepsilon} \frac{1}{x}dx$ is divergent, consequently by the Comparison Test (Theorem 50), $\int_0^{\varepsilon} e^{\frac{1}{x}}x^p dx$ is divergent for any p. It follows that $\int_0^{a} e^{\frac{1}{x}}x^p dx$ is divergent for all p. Therefore, by Theorem 44 $e^{\frac{1}{x}}x^p$ is not Lebesgue integrable on $[0, a]$.

(3) $\int_0^a \ln(x)dx$ converges for any $a > 0$.

For any $0 < t \le a$, by integration by parts,

$$\int_t^a \ln(x)dx = [x\ln(x)]_t^a - \int_t^a 1dx = a\ln(a) - a - (t\ln(t) - t).$$

Now, it can be easily deduced by using L'Hôpital's Rule that $\lim\limits_{t\to 0^+} t\ln(t) = 0$.
Therefore,

$$\int_0^a \ln(x)dx = \lim_{t\to 0^+}\int_t^a \ln(x)dx = a\ln(a) - a.$$

Hence, by Theorem 45, $\ln(x)$ is Lebesgue integrable on $[0, a]$ for any $a > 0$ and the Lebesgue integral is $a\ln(a)-a$.

(4) $\int_1^{\infty} \frac{1}{\ln(x)}dx$ diverges.

By using L'Hôpital's Rule, we can show that $\lim\limits_{x\to\infty} \dfrac{x}{\ln(x)} = \infty$.

Therefore, there exists $K > 0$ such that $x \ge K \Rightarrow \dfrac{x}{\ln(x)} > 1 \Rightarrow \dfrac{1}{\ln(x)} > \dfrac{1}{x}$.

Since $\int_K^{\infty} \frac{1}{x}dx$ diverges, it follows that $\int_K^{\infty} \frac{1}{\ln(x)}dx$ diverges by Theorem 50 (Comparison Test).

Therefore, $\int_1^{\infty} \frac{1}{\ln(x)}dx$ diverges. Hence $\dfrac{1}{\ln(x)}$ is not Lebesgue integrable on $[1, \infty)$.

For a function, not necessarily non-negative, we can, with sufficient condition satisfied by the function, formulate a convergence criterion along the line of the Leibnitz alternating series test.

Theorem 55.
Suppose $f:[a, \infty) \to \mathbf{R}$ is such that f is Riemann integrable on $[a, b]$ for every $b > a$. Suppose there is a strictly increasing sequence (a_n) in $[a, \infty)$ with $a_0 = a$ and $a_n \to\infty$ satisfying :
(1) $f(x)$ has constant sign in each interval (a_n , a_{n+1}),
(2) $f(x)$ changes sign from (a_{n-1} , a_n) to (a_n , a_{n+1}),
(3) $\left|\int_{a_{n-1}}^{a_n} f(x)dx\right| \ge \left|\int_{a_n}^{a_{n+1}} f(x)dx\right|$,
(4) $\int_{a_{n-1}}^{a_n} f(x)dx \to 0$ as $n \to \infty$.
Then the improper Riemann integral $\int_a^{\infty} f(x)dx$ converges.

Proof. Let $c_n = \int_{a_{n-1}}^{a_n} f(x)dx$. Then by condition (4) $c_n \to 0$. By condition (3) ($|c_n|$) is a decreasing sequence. By condition (2) $\sum\limits_{n=1}^{\infty} c_n$ is an alternating series. Therefore, by Leibnitz's Alternating Series Test, (see Chapter 6 Series, Theorem 20), $\sum_{n=1}^{\infty} c_n$ is convergent.

We shall show next that $\sum_{n=1}^{\infty} c_n$ is the improper integral.

Now take any $b > a$. Since $a_n \to \infty$, there exists an integer n_0 with $a_{n_0} \geq b$. Let n_0 be the least integer such that $a_{n_0} \geq b$ so that $a_{n_0-1} < b \leq a_{n_0}$.

Thus, $\int_a^b f(x)dx = \sum_{k=1}^{n_0-1} c_k + \int_{a_{n_0-1}}^b f(x)dx.$

Since $a_{n_0-1} < b \leq a_{n_0}$ and $f(x)$ is of constant sign on (a_{n_0-1}, a_{n_0}).

$$\left|\int_{a_{n_0-1}}^{b} f(x)dx\right| \leq \left|\int_{a_{n_0-1}}^{a_{n_0}} f(x)dx\right| = |c_{n_0}| \text{ ---------------------- (1)}$$

Plainly, as $b \to \infty$, $n_0 \to \infty$.

Now as $c_n \to 0$, given $\varepsilon > 0$, there exists an integer N_1 such that

$$n \geq N_1 \Rightarrow |c_n| < \varepsilon/2 \qquad \text{-------------------- (2)}$$

Since $\sum_{n=1}^{\infty} c_n$ is convergent, there exists an integer N_2 such that

$$k \geq N_2 \Rightarrow \left|\sum_{n=k}^{\infty} c_n\right| < \varepsilon/2. \quad \text{----------------------- (3)}$$

Let $N = \max(N_1, N_2)$. Then since $n_0 \to \infty$ as $b \to \infty$, there exists a number $K > 0$ such that

$$b > K \Rightarrow n_0 > N.$$

Therefore, for $b > K$,

$$\begin{aligned}\left|\int_a^b f(x)dx - \sum_{k=1}^{\infty} c_k\right| &\leq \left|\int_a^b f(x)dx - \sum_{k=1}^{k=n_0-1} c_k\right| + \left|\sum_{k=n_0}^{\infty} c_k\right| \\ &\leq |c_{n_0}| + \left|\sum_{k=n_0}^{\infty} c_k\right| \qquad \text{by (1)} \\ &< \varepsilon/2 + \left|\sum_{k=n_0}^{\infty} c_k\right| \quad \text{by (2) since } n_0 > N \geq N_1 \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{by (3) since } n_0 > N \geq N_2.\end{aligned}$$

Hence $\lim_{b\to\infty} \int_a^b f(x)dx = \sum_{k=1}^{\infty} c_k$. Thus the improper integral $\int_a^{\infty} f(x)dx$ is convergent.

Example 56. Use of the Lebesgue Dominated Convergence Theorem.

$$\lim_{n\to\infty} \int_0^1 \frac{1-e^{-nx}}{\sqrt{x}} dx = 2.$$

Define $f_n : (0, 1] \to \mathbf{R}$ by $f_n(x) = \frac{1-e^{-nx}}{\sqrt{x}}$, Then f_n converges pointwise to $\frac{1}{\sqrt{x}}$ on (0, 1].

Now $0 < e^{-nx} \leq 1$ for $x \geq 0$ so that $0 \leq 1 - e^{-nx} \leq 1$. Hence

$$0 \leq f_n(x) \leq \frac{1}{\sqrt{x}} \quad \text{for } x \text{ in } (0, 1].$$

Note that each f_n is continuous on (0, 1] and so is measurable on (0, 1].

Now the improper integral $\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t\to 0^+} \left[2\sqrt{x}\,\right]_t^1 = 2$. Therefore, by Theorem 52(2), $\frac{1}{\sqrt{x}}$ is Lebesgue integrable on (0, 1] and its Lebesgue integral,

$$\int_{(0,1]} \frac{1}{\sqrt{x}} d\mu = \int_0^1 \frac{1}{\sqrt{x}} dx = 2.$$

Therefore, by the Lebesgue Dominated Convergence Theorem,

$$\int_{(0,1]} f_n d\mu = \int_0^1 \frac{1-e^{-nx}}{\sqrt{x}} dx \to \int_{(0,1]} \frac{1}{\sqrt{x}} d\mu = 2.$$

Note that by the Comparison Test (Theorem 50), the improper integral $\int_0^1 \frac{1-e^{-nx}}{\sqrt{x}} dx = \int_0^1 f_n(x)dx$ is convergent and equals the Lebesgue integral of $\int_{(0,1]} f_n d\mu$.

14. 8 Differentiation Under the Integral Sign and Lebesgue Integral

Since differentiation is a process of limit, we shall investigate the processes of limits before and after a Lebesgue integral.

Here is a result that makes use of the Lebesgue Dominated Convergence Theorem.

Theorem 57. Suppose E is a non-empty (Lebesgue) measurable subset of $\mathbf{R}$ and I is an interval. Suppose $f : E \times I \to \mathbf{R}$ is a function satisfying
(1) for each t in I, the function

$$f_t : E \to \mathbf{R} \text{ defined by } f_t(x) = f(x, t)$$

is measurable;
(2) there exists an integrable function g: $E \to \mathbf{R}$ such that for almost all x in E and all t in I,

$$|f_t(x)| = |f(x, t)| \leq g(x);$$

(3) for some point t_0 (t_0 may be $\pm\infty$,) in the closure of I, $\bar{I}$, there exists a function $h : E \to \mathbf{R}$ such that, for almost all x in E, $\lim_{t\to t_0} f_t(x) = \lim_{t\to t_0} f(x,t) = h(x)$.
Then
(1) h is Lebesgue integrable and
(2) $\lim_{t\to t_0} \int_E f_t(x)d\mu(x) = \int_E \lim_{t\to t_0} f_t(x)d\mu(x) = \int_E h(x)d\mu(x)$.

Proof. Since $t_0 \in \bar{I}$, there exists a sequence (t_n) in I such that $t_n \to t_0$. For each n in P define the function, $g_n : E \to \mathbf{R}$ by

$$g_n(x) = f_{t_n}(x) = f(x, t_n), \ x \text{ in } E.$$

By Condition (2), f_t is Lebesgue integrable for each t in I and so g_n is Lebsgue integrable on E. (Note that f_t is measurable implies that $|f_t|$ is measurable. Then by Lemma 27, there is a sequence of increasing simple non-negative function tending pointwise to $|f_t|$, then by condition (2), the integral of these non-negative simple functions is bounded above by the Lebesgue integral of g and so by the Lebesgue Monotone Convergence Theorem (Theorem 28), $|f_t|$ is Lebesgue integrable. Consequently, f_t is Lebesgue integrable.) By Condition (3) g_n converges pointwise to $h(x)$ almost everywhere. Therefore, by the Lebesgue Dominated Convergence Theorem, since $|g_n(x)| \leq g(x)$ for almost all x in E and g is Lebesgue integrable,

$$\int_E g_n(x)d\mu \to \int_E h(x)d\mu.$$

Note that since $g_n = f_{t_n}$, for any sequence (t_n) in I such that $t_n \to t_0$, we conclude that

$$\lim_{t\to t_0} \int_E f_t(x)d\mu = \lim_{n\to\infty} \int_E g_n(x)d\mu = \int_E h(x)d\mu = \int_E \lim_{t\to t_0} f_t(x)d\mu.$$

Now we specialize Theorem 57 to differentiation under an integration sign.

Theorem 58. Suppose E is a non-empty (Lebesgue) measurable subset of **R** and $f : E \times (a, b) \to \mathbf{R}$ is a function satisfying the following three condition.

(1) For each t in I, the function

$$f_t : E \to \mathbf{R} \text{ defined by } f_t(x) = f(x, t)$$

is Lebesgue integrable.

(2) For some t_0 in (a, b), the partial derivative $\frac{\partial f}{\partial t}(x, t_0)$ exists for almost all x in E.

(3) There exists a neighbourhood V of t_0 and a Lebesgue integrable function g: $E \to \mathbf{R}$ such that for almost all x in E and all t in V,

$$|D_{t_0}(x,t)| = \left|\frac{f(x,t)-f(x,t_0)}{t-t_0}\right| \leq g(x),$$

where $|D_{t_0}(x,t)| = \begin{cases} \left|\dfrac{f(x,t)-f(x,t_0)}{t-t_0}\right|, & t \neq t_0 \\ 0, \text{ if } t = t_0 \end{cases}$.

Then,

(i) the function F: $E \to \mathbf{R}$ defined by $F(x) = \frac{\partial f}{\partial t}(x, t_0)$ is a (Lebesgue) integrable function

and

(ii) the function H: $(a, b) \to \mathbf{R}$, defined by $H(t) = \int_E f_t d\mu = \int_E f(x,t) d\mu$ is differentiable *at* t_0 and

$$H'(t_0) = \int_E F(x) d\mu = \int_E \frac{\partial f}{\partial t}(x, t_0) d\mu.$$

Proof. Let $G(t) = \dfrac{H(t)-H(t_0)}{t-t_0} = \displaystyle\int_E \frac{f_t - f_{t_0}}{t-t_0} d\mu = \int_E \frac{f(x,t) -- f(x,t_0)}{t-t_0} d\mu$ for t in V,
$= \int_E D_{t_0}(x,t) d\mu$.

Note that for each t in V, the function $x \mapsto D_{t_0}(x,t)$ is measurable since f_t and f_{t_0} are integrable and so measurable. Therefore, $D_{t_0}(x,t)$ satisfies the condition of Theorem 57 and

$$\lim_{t \to t_0} D_{t_0}(x,t) = \frac{\partial f}{\partial t}(x, t_0) \text{ for almost all } x \text{ in } E.$$

Hence, by Theorem 57,

$$H'(t_0) = \lim_{t \to t_0} \frac{H(t)-H(t_0)}{t-t_0} = \lim_{t \to t_0} \int_E \frac{f_t - f_{t_0}}{t-t_0} d\mu = \lim_{t \to t_0} \int_E D_{t_0}(x,t) d\mu \text{ for } t \text{ in } V,$$
$$= \int_E \lim_{t \to t_0} D_{t_0}(x,t) d\mu = \int_E \frac{\partial f}{\partial t}(x, t_0) d\mu.$$

This shows that H is differentiable at t_0 and the derivative $H'(t_0) = \int_E \frac{\partial f}{\partial t}(x, t_0) d\mu$.

Remark.

(1) Conditions (2) and (3) are satisfied by any function f : $[c, d] \times (a, b) \to \mathbf{R}$ whose partial derivative $\frac{\partial f}{\partial t}(x,t)$ is continuous on $[c, d] \times (a, b) \to \mathbf{R}$ so that it is uniformly continuous on $[c, d] \times V$, where V is a closed interval in (a, b) containing t_0. In this case, by continuity and compactness of $[c, d] \times V$, $\left|\frac{\partial f}{\partial t}(x,t)\right|$ is bounded on $[c, d] \times V$, say by M. By the Mean Value Theorem, for any (x, t) in $[c, d] \times V$,

$$|D_{t_0}(x,t)| = \left|\frac{f(x,t) - f(x,t_0)}{t - t_0}\right| = \left|\frac{\partial f}{\partial t}(x,t_1)\right|$$

for some t_1 in V. Therefore, $|D_{t_0}(x,t)| \le M$. We can just take g in Theorem 58 to be the constant function M.

More generally, if $\frac{\partial f}{\partial t}(x,t)$ exists for all t in a neighbourhood V of t_0 and for almost all x in E and if there exists an integrable function g(x) such that $\left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x)$for all t in V and for almost all x in E, then by using the Mean Value Theorem, condition (3) of Theorem 58 is satisfied.

(2) We can replace the interval (a, b) in Theorem 58 by $[a, b)$, or $(a, b]$ or $[a, b]$. Taking the derivative at any end point to be the appropriate one sided limit, the conclusion of Theorem 58 is still valid.

The next Theorem is a special case.

Theorem 59. Suppose $f : [c, d] \times [a, b] \to \mathbf{R}$ is a continuous function such the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists for all (x, t) in $[c, d] \times [a, b]$ and is continuous on $[c, d] \times [a, b]$. Let F: $[a, b] \to \mathbf{R}$ be defined by $F(t) = \int_c^d f(x,t)dx$.
Then, F is differentiable and $F'(t) = \int_c^d \frac{\partial f}{\partial t}(x,t)dx$.

Proof. For any t_0 in $[a, b]$, there exists a closed interval V containing t_0 such that $\frac{\partial f}{\partial t}(x,t)$ is continuous on $[c, d]\times V$. Therefore, by the remark preceding the theorem, conditions (2) and (3) of Theorem 58 are satisfied. Condition (1) of Theorem 58 is of course satisfied since for each t, $f_t : [a, b] \to \mathbf{R}$ defined by $f_t(x) = f(x, t)$, is continuous and so Riemann integrable, hence Lebesgue integrable and the Lebesgue integral $\int_{[c,d]} f(x,t)d\mu(x)$ is equal to the Riemann integral $\int_c^d f(x,t)dx$. Similarly the Lebesgue integral $\int_{[c,d]} \frac{\partial f}{\partial t}(x,t)d\mu(x)$ is the same as the Riemann integral $\int_c^d \frac{\partial f}{\partial t}(x,t)dx$. Therefore, by Theorem 58,

$$F'(t) = \int_c^d \frac{\partial f}{\partial t}(x,t)dx.$$

What happens if the integral is given by improper integral? If the improper integral is of the type with singularity at d, or $d = \infty$, then even if $\frac{\partial f}{\partial t}(x,t)$ exists for all (x, t), $\frac{\partial f}{\partial t}(x,t)$ may not be continuous on $[c, d] \times [a, b]$ or that $[c, d]$ is unbounded when ($d = \infty$). $\frac{\partial f}{\partial t}(x,t)$ is not generally uniformly bounded. Hence we may need to introduce the notion of uniform convergence for the improper Riemann Cauchy integral $\int_c^d \frac{\partial f}{\partial t}(x,t)dx$.

Theorem 60. Suppose $f : [c, \infty) \times [a, b] \to \mathbf{R}$ is a continuous function such that the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists for all (x, t) in $[c, \infty) \times [a, b]$ and is continuous on $[c, \infty) \times [a, b]$.
Suppose that the improper Cauchy Riemann integral $\int_c^\infty f(x,t)dx$ converges for each t in $[a, b]$. Suppose furthermore the improper integral $\int_c^\infty \frac{\partial f}{\partial t}(x,t)dx$ converges uniformly for t in $[a, b]$. Let F: $[a,b] \to \mathbf{R}$ be defined by $F(t) = \int_c^\infty f(x,t)dx$.
Then, F is differentiable and $F'(t) = \int_c^\infty \frac{\partial f}{\partial t}(x,t)dx$.

Proof. $\int_c^\infty \frac{\partial f}{\partial t}(x,t)dx$ converge uniformly in $[a, b]$ means that

$$\int_c^d \frac{\partial f}{\partial t}(x,t)dx \to G(t) \text{ as } d \to \infty \text{ for each } t \text{ in } [a, b],$$

where $G(t) = \int_c^\infty \frac{\partial f}{\partial t}(x,t)dx$, and that given $\varepsilon > 0$, there exists a real number $K > 0$ such that for all $d > K$ and for all t in $[a, b]$,

$$\left| \int_c^d \frac{\partial f}{\partial t}(x,t)dx - G(t) \right| < \varepsilon \qquad \text{----------------------------- (1)}$$

For each $d > c$, define $F_d(t) = \int_c^d f(x,t)dx$. Then F_d converges pointwise to F on $[a, b]$, where $F(t) = \int_c^\infty f(x,t)dx$. By Theorem 59, F_d is differentiable and

$$F'_d(t) = \int_c^d \frac{\partial f}{\partial t}(x,t)dx .$$

Then condition (1) says that $F_d\,'(t)$ converges uniformly to $G(t)$ on $[a, b]$ as $d \to \infty$. By taking any sequence (d_n) in (c, ∞) such that $d_n \to \infty$ and letting $H_n = F_{d_n}$, by the above argument, we have that

H_n converges pointwise to F on $[a, b]$,
$H_d\,'$ converges uniformly to G on $[a, b]$.

Then by Theorem 8, Chapter 8, F is differentiable and $F' = G$.

Remark. As remark in Chapter 8, we only need require that there exists a t_0 in $[a, b]$ such that $H_n(t_0)$ converges to $F(t_0)$. This is equivalent to $\int_c^\infty f(x,t_0)dx$ is convergent. In Theorem 60, instead of requiring $\int_c^\infty f(x,t)dx$ to be convergent for each t in $[a, b]$, we need only require convergence at some point t_0 in $[a, b]$. That it is convergent for all t is a consequence of the uniform convergence of $F_d\,'$ as d tends to infinity.

As an application of the theorems in this section, we shall give an indirect method of computing the probability integral, more specifically $\int_0^\infty e^{-x^2}dx$.

Example 61. $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

By Theorem 10 (a comparison test), since $e^{-x^2} \le e^{-x}$ for $x \ge 1$ and $\int_1^\infty e^{-x}dx$ is convergent, $\int_1^\infty e^{-x^2}dx$ is convergent. Consequently, the improper Riemann integral $\int_0^\infty e^{-x^2}dx$ is convergent. Since e^{-x^2} is non-negative, by Theorem 34, $\int_0^\infty e^{-x^2}dx$ is a Lebesgue integral on $[0, \infty)$. Consider the following two functions:

$$f(t) = \left(\int_0^t e^{-x^2}dx\right)^2 \quad \text{and} \quad g(t) = \int_0^1 \frac{e^{-t^2(x^2+1)}}{x^2+1}dx.$$

We shall take the limit of these two functions to determine $\int_0^\infty e^{-x^2}dx$.
Observe that by the Chain Rule and the Fundamental Theorem of Calculus,

$$f'(t) = 2e^{-t^2}\left(\int_0^t e^{-x^2}dx\right) \text{ ---------------------- } (1)$$

Let $h(x,t) = \dfrac{e^{-t^2(x^2+1)}}{x^2+1}$. Then h is continuous on $[0,1]\times[0,\infty)$ and so $h_t(x) = h(x,t)$ is continuous on $[0,1]$ and so is Riemann integrable on $[0, 1]$ and hence Lebesgue integrable on $[0, 1]$. The partial derivative $\dfrac{\partial h}{\partial t}(x,t) = -2te^{-t^2(x^2+1)}$ exists for all x in $[0, 1]$ and for all $t > 0$. In particular, $\left|\dfrac{\partial h}{\partial t}(x,t)\right| = \left|-2te^{-t^2(x^2+1)}\right| \le |2t| \le M$ for some constant number M (depending on $t_0 > 0$) for some neighbourhood of any t_0. Therefore, by Theorem 58,

$$g'(t) = \int_0^1 \frac{\partial h}{\partial t}(x,t)dx = \int_0^1 -2te^{-t^2(x^2+1)}dx = -2te^{-t^2}\int_0^1 e^{-t^2x^2}dx.$$

For $t > 0$, using a change of variable $y = xt$,

$$g'(t) = -2e^{-t^2}\int_0^t e^{-y^2}dy = -2e^{-t^2}\int_0^t e^{-x^2}dx \text{ -------------------- } (2)$$

Thus from (1) and (2), we see that $f'(t) + g'(t) = 0$ for all $t > 0$. Therefore, $f(t) + g(t) = c$ for some constant c, for all $t > 0$. Note that f is obviously continuous at 0. Since $|h(x,t)| = \left|\dfrac{e^{-t^2(x^2+1)}}{x^2+1}\right| \le \dfrac{1}{x^2+1}$ and $\dfrac{1}{x^2+1}$ is Lebesgue integrable on $[0,1]$, by Theorem 57, $g(t)$ is continuous at $t = 0$. Hence $f(t) + g(t)$ is continuous at $t = 0$. Therefore, $f(t) + g(t) = c$ for some constant c, for all $t \ge 0$. Thus

$$c = f(0) + g(0) = 0 + \int_0^1 \frac{1}{x^2+1}dx = \tan^{-1}(1) = \frac{\pi}{4}.$$

So we have the equation,

$$f(t) + g(t) = \frac{\pi}{4} \text{ ------------------------------- } (3).$$

By Theorem 57,

$$\lim_{t\to\infty} g(t) = \int_0^1 \lim_{t\to\infty} \frac{e^{-t^2(x^2+1)}}{x^2+1}dx = 0 \text{ since } \lim_{t\to\infty}\frac{e^{-t^2(x^2+1)}}{x^2+1} = 0 \text{ for all } x \text{ in } [0, 1].$$

Thus, passing to the limit (3) becomes

$$\lim_{t\to\infty} f(t) + \lim_{t\to\infty} g(t) = \frac{\pi}{4}$$

and so $\lim\limits_{t\to\infty} f(t) = \dfrac{\pi}{4}$. This means $\left(\int_0^\infty e^{-x^2}dx\right)^2 = \dfrac{\pi}{4}$ and so $\int_0^\infty e^{-x^2}dx = \dfrac{\sqrt{\pi}}{2}$.

Exercises 62.

1. Determine for each of the following integrals whether it is convergent, absolute convergent, or conditionally convergent. Evaluate the convergent ones if possible.

   (a) $\int_0^1 \frac{1}{x^n}dx$, $n \ge 1$, (b) $\int_1^\infty \frac{1}{x^n}dx$, $n > 1$, (c) $\int_1^\infty \frac{1}{x^n}dx$, $n \le 1$,

   (d) $\int_0^{\pi/2} \tan(x)dx$, (e) $\int_1^\infty \frac{\sin(x)}{x^n}dx$, $n > 1$ (f) $\int_0^1 (\ln(x))^2 dx$,

   (g) $\int_0^\infty e^{-x}\cos(2x)dx$, (h) $\int_0^{\pi/2} \frac{1}{1-\cos(x)}dx$, (i) $\int_0^\infty x^n e^{-x}dx$, $n > 0$,

   (j) $\int_0^\infty \sin^2(2x)\cos^2(2x)dx$ (k) $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$, (l) $\int_0^\infty \frac{1}{1+e^x}dx$

   (m) $\int_0^\infty x^2 e^{-x^2}dx$, (n) $\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}dx$, (o) $\int_0^\infty x^{2n+1}e^{-x^2}dx$, $n \ge 0$.

2. Suppose $\int_a^b f(x)dx$ is an absolutely convergent improper integral and g is a bounded integrable function on $[a, b]$. Prove that $\int_a^b f(x)g(x)dx$ converges absolutely.

3. Prove that $\int_0^\infty \sin(x^2)dx$ and $\int_0^\infty \cos(x^2)dx$ are convergent.

4. Show that $\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \int_0^\infty \frac{\sin(x)}{x}dx$.

5. Determine $\int_0^{\pi/2} \ln(\sin(x))dx$.

6. Test the following improper integrals for convergence.

(a) $\int_0^\infty \sin(x^\alpha)dx$, $\alpha \geq 1$, (b) $\int_0^\infty \frac{x^2}{(1+x^2)^2}dx$, (c) $\int_0^\infty \frac{x^3}{(1+x^2)^2}dx$,
(d) $\int_1^\infty \frac{1}{x^{1+1/x}}dx$, (e) $\int_1^4 \frac{\sqrt{x}}{\ln(x)}$, (f) $\int_0^\infty \frac{1}{\sqrt{x}}\cos\left(\frac{1}{x}\right)dx$,
(g) $\int_0^\pi \frac{x}{\sin(x)}dx$, (h) $\int_0^\infty x\sin(e^x)dx$ (i) $\int_0^\infty \frac{e^{\cos(x)}}{x}dx$.

7. Suppose f is a positive continuous function defined on $[a, \infty)$. Prove that $\int_a^\infty f(x)dx$ is convergent if $\lim_{x\to\infty}\frac{f(x+1)}{f(x)} < 1$.

8. Prove that $\int_0^\infty e^{-t}t^\beta \sin(t)dt$ converges absolutely if $\beta > -2$. [Hint: consider the integral from 0 to 1 and from 1 to ∞ separately.]

9. Test the integral $\int_0^{\pi/2} \frac{1}{\sqrt{1-\sin(x)}}dx$ for convergence.

10. Prove that for $t \geq 0$, $\int_0^\infty e^{-tx}\frac{\sin(x)}{x}dx = \frac{\pi}{2} - \tan^{-1}(t)$. Hence deduce that

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

11. Show that $\int_0^\infty e^{-tx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$ for $t > 0$.

12. Show that the function $f(x) = \ln(x)/x^2$ is Lebesgue integrable over $[1, \infty)$ and that the Lebesgue integral $\int_{[1,\ \infty)} f\,d\mu = 1$.

13. For any t in **R**, prove that $\int_0^\infty e^{-x^2}\cos(2tx)dx = \frac{\sqrt{\pi}}{2}e^{-t^2}$.