Chapter 8. Uniform Convergence and Differentiation.

This chapter continues the study of the consequence of uniform convergence of a series of function. In Chapter 7 we have observed that the uniform limit of a sequence of continuous function is continuous (Theorem 14 Chapter 7). We shall now investigate whether the uniform limit of a differentiable function is differentiable. Convergence is most effectively treated in the setting of metric spaces which allow for generalization to the space of bounded functions, whose codomain is a complete metric space. But we shall not introduce this setting here, preferring to use the equivalent technique not phrased in that setting. Some observation as extension to "complete metric space" is apparent. We shall confine strictly to real valued functions on subset of the real numbers.

8.1 The Weierstrass M Test

The first test for uniform convergence of a series of function is a form of comparison test.

Theorem 1 (Weierstrass M Test). Suppose ($f_k : E \to \mathbf{R}$, k = 1, 2, …) is a sequence of functions. Suppose (M_k) is a sequence of non-negative real numbers for which $\sum_{k=1}^{\infty} M_k$ is convergent and that for each integer $k \geq 1$, the function f_k is bounded by M_k , i.e., $|f_k(x)| \leq M_k$ for all x in E. Then the series $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on E.

Proof. Since for each x in E, $|f_k(x)| \leq M_k$ and since $\sum_{k=1}^{\infty} M_k$ is convergent , $\sum_{k=1}^{\infty}|f_k(x)|$ is convergent for each x in E, by the Comparison Test (Proposition 12 Chapter 2 Series). It follows by Proposition 14 of Chapter 3 that $\sum_{k=1}^{\infty} f_k(x)$ is convergent for each x in E. Hence $\sum_{k=1}^{\infty} f_k(x)$ is pointwise convergent. Uniform convergence of $\sum_{k=1}^{\infty} f_k(x)$ is a consequence of $\sum_{k=1}^{\infty} M_k$ is uniformly convergent (since it is independent of x). Here is how we deduce this.
$\sum_{k=1}^{\infty} M_k$ is a Cauchy series, since it is convergent. Hence given any $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$ and for any p in $\boldsymbol{P}$,

$$\sum_{k=n+1}^{n+p} M_k < \frac{\varepsilon}{2}.$$

Thus, it follows that for all $n \geq N$, for any p in $\boldsymbol{P}$ and for any x in E,

$$\left|\sum_{k=n+1}^{n+p} f_k(x)\right| \leq \sum_{k=n+1}^{n+p}|f_k(x)| \leq \sum_{k=n+1}^{n+p} M_k < \frac{\varepsilon}{2} \quad \text{-------------------} \ (1)$$

Therefore, for all $n \geq N$ and for all x in E,

$$\left|\sum_{k=n+1}^{\infty} f_k(x)\right| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence, for any integer $n \geq N$ and for all x in E,

$$\left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{\infty} f_k(x)\right| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| < \varepsilon.$$

Therefore, this says that $\sum_{k=1}^{n} f_k(x)$ converges uniformly to $f(x) = \sum_{k=1}^{\infty} f_k(x)$.

Remark. Condition (1) above defines a notion which we shall call "uniformly Cauchy". We may formulate a criterion for uniform convergence in terms of inequality (1) or being uniformly Cauchy, but it is the M-test that is more readily applicable, easy to apply.

8.2 A criterion for Uniform Convergence: Uniformly Cauchy

Definition 2. A sequence of functions ($f_k : E \to \mathbf{R}$) is said to be *uniformly Cauchy* if given any ε > 0, there exists an integer N such that for all $n > m \geq N$ and for all $x \in E$.

$$|f_n(x) - f_m(x)| < \varepsilon .$$

Theorem 3. The sequence of functions ($f_k : E \to \mathbf{R}$) converges uniformly if and only if ($f_k : E \to \mathbf{R}$) is uniformly Cauchy.

Proof. If the sequence (f_k) converges uniformly to $\boldsymbol{f}$ **,** then given any ε > 0, there exists an integer N such that for all $n \geq N$ and for all $x \in E$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

Therefore, for all integers n, m such that n > m ≥ N and for all $x \in E$,

$$\begin{aligned}|f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon ,\end{aligned}$$

Thus, by Definition 2, (f_k) is uniformly Cauchy.

Conversely now suppose (f_k) is uniformly Cauchy. Then given any ε > 0, there exists an integer N such that for all $m > n \geq N$ and for all $x \in E$.

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} . \qquad \text{-----------------------} \quad (1)$$

Hence for each x, ($f_k(x)$) is a Cauchy sequence and so (by Cauchy Principle of Convergence), ($f_k(x)$) converges to a function $f(x)$ pointwise. Thus for any x in E,

$$|f_n(x) - f(x)| = \left|f_n(x) - \lim_{k\to\infty} f_k(x)\right| = \lim_{k\to\infty} |f_n(x) - f_k(x)| \quad \text{-------------} (2)$$

Now by (1), for any $k > n \geq N$ and for all $x \in E$.

$$|f_n(x) - f_k(x)| < \frac{\varepsilon}{2}.$$

Therefore, for all x in E, $\lim_{k\to\infty} |f_n(x) - f_k(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$

It follows, then from (2), that for all $n \geq N$ and for all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$. That is to say, $f_n \to f$ uniformly on E.

Example 4.

The following three statements are consequence of the Weierstrass M Test.

(1) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is uniformly convergent on the closed and bounded interval [−1,1].

Since for all x in [−1,1] and for all positive integers n, $\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by Weierstrass M Test (Theorem 1), the series is uniformly convergent.

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ is uniformly convergent on **R** by Weierstrass M Test since for each positive integer n and for all x in **R**, $\left|\frac{1}{n^2+x^2}\right| \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(3) $\sum_{n=1}^{\infty} \frac{x}{n^2+x^2}$ is uniformly convergent on the closed and bounded interval $[-a, a]$, where $a > 0$. Since for each positive integer n and for all x in $[-a, a]$. , $\left|\frac{x}{n^2+x^2}\right| \le \frac{a}{n^2}$ and $\sum_{n=1}^{\infty} \frac{a}{n^2}$ is convergent, by the Weierstrass M Test, the series $\sum_{n=1}^{\infty} \frac{x}{n^2+x^2}$ is uniformly convergent on the $[-a, a]$.

Example 5.
We can have a sequence of functions converging non-uniformly to a constant function.
For example the sequence of functions (f_n) where for each positive integer n, f_n :**R** $\to$ **R** is defined by $f_n(x) = \frac{nx}{1+n^2x^2}$, is such a sequence.
For each x in **R**, $f_n(x) \to 0$. We deduce this as follows.
For each $x \neq 0$ in **R**, $f_n(x) = \frac{\frac{x}{n}}{\frac{1}{n^2}+x^2} \to \frac{0}{0+x^2} = 0$ as $n \to \infty$. For $x = 0$, for each positive integer n, $f_n(0) = 0$. Hence $f_n(0) \to 0$. Thus the pointwise limit of the sequence (f_n) is the zero constant function, i.e., $f_n \to f$ pointwise, where $f(x) = 0$ for all x.
To see that the convergence is not uniform, we examine what it means for a convergence not to be uniform. We start with the negation of the definition of uniform convergence in Definition 11 Chapter 7. The sequence (f_n) does not converge uniformly to f means there exists an $\varepsilon > 0$ such that for any positive integer N, there exists an integer $n \ge N$ and an element x_n in the domain of f_n such that $|f_n(x_n) - f(x_n)| \ge \varepsilon$.
So we shall proceed to find this ε by examining the values that $|f_n(x) - f(x)|$ can take. Recall that $f(x) = 0$ for all x. Hence for all positive integer n and for all x in **R**,

$$|f_n(x) - f(x)| = |f_n(x)| = \left|\frac{nx}{1+n^2x^2}\right| = \frac{n|x|}{1+n^2x^2} \le 1.$$

Thus the set $\{|f_n(x) - f(x)| : x \in \mathbf{R}\}$ is bounded above by 1 for all positive integer n.
Therefore, $\sup\{\, |f_n(x) - f(x)| : x \in \mathbf{R}\}$ exists for each positive integer n. Now by quick inspection of the function rule for f_n, we see that $f_n(\frac{1}{n}) = \frac{1}{2}$ for each positive integer n. Consequently, $\sup\{\, |f_n(x) - f(x)| : x \in \mathbf{R}\} \ge f_n(\frac{1}{n}) = \frac{1}{2}$ for all positive integer n. We can thus take $\varepsilon = 1/2$. Thus for each positive integer N in ***P***, choose $n = N$ and $x_N = \frac{1}{N}$ then we have

$$|f_n(x_n) - f(x_n)| = |f_N(x_N)| = \frac{1}{2} \ge \varepsilon.$$

This means the convergence is not uniform.

We may also show that the sequence (f_n) is not uniformly Cauchy so that by Theorem 3 the convergence is not uniform. Observe that for any positive integers n and m,

$$|f_n(x) - f_m(x)| = \left|\frac{nx}{1+n^2x^2} - \frac{mx}{1+m^2x^2}\right| = \left|\frac{(n-m)x + mn(m-n)x^3}{(1+n^2x^2)(1+m^2x^2)}\right|$$
$$= \left|\frac{(n-m)x + mn(m-n)x^3}{(1+n^2x^2)(1+m^2x^2)}\right|.$$

Thus, $$\left|f_n(\tfrac{1}{n}) - f_m(\tfrac{1}{n})\right| = \left|\frac{(1-\frac{m}{n}) + \frac{m}{n}(\frac{m}{n}-1)}{2\left(1+\frac{m^2}{n^2}\right)}\right|. \quad \text{---------------------------} \quad (1)$$

For each positive integer N, choose any $n \geq N$, choose $m = 3n$ and take $x_N = \frac{1}{n}$. Then we have using (1), $\left|f_n(\frac{1}{n}) - f_m(\frac{1}{n})\right| = \left|\frac{(1-3)+3(2)}{2(10)}\right| = \frac{1}{5}$. So taking $\varepsilon = \frac{1}{5}$, we have shown that, for each positive integer N, we can find integers n and $m \geq N$ and an element x_N in the domain **R**, such that $|f_n(x_N) - f_m(x_N)| \geq \varepsilon = \frac{1}{5}$. Thus, by Definition 2,(f_n) is not uniformly Cauchy.

8.3 Uniform Convergence and Differentiation

Theorem 14 of Chapter 7 says that continuity behaves well under uniform convergence, i.e., the uniform limit of a sequence of continuous functions is continuous. But differentiability is less well behaved and even less well behaved than integrability.

The uniform limit of differentiable functions need not be differentiable. There are various possibilities. Each f_n of the sequence (f_n) may be differentiable but the sequence of the derivatives (f_n') may not be convergent and when (f_n') is convergent, the convergence need not be uniform. Thus, if we were to formulate a result using the uniform convergence of derivatives, the uniform convergence of the sequence of derivatives will have to be assumed. In this way by using the good behaviour of integration under uniform convergence, we use the Fundamental Theorem of Calculus to deduce result about the derivatives of the limiting function of the sequence (f_n) and the uniform convergence of (f_n) if the uniform convergence of the derivatives (f_n') is assumed and that the derivatives f_n' are all continuous.

Example 6. A sequence (f_n) converging uniformly to a function f but (f_n') does not converge to f'.
Let (f_n) be a sequence of function defined on **R** by $f_n(x) = \dfrac{x}{1+nx^2}$ for x in **R**.
Then $f_n \to f$ pointwise, where f is the zero constant function. The convergence is uniform. We deduce this as follows.
$|f_n(x) - f(x)| = \dfrac{|x|}{1+nx^2} = \dfrac{1}{\frac{1}{|x|} + n|x|}$ for $x \neq 0$ and $|f_n(0) - f(0)| = 0$. Now note that $\frac{1}{x} + nx$ achieves its minimum in $(0, \infty)$ at $x = \frac{1}{\sqrt{n}}$. Hence the maximum of the reciprocal is $\sup\{\,|f_n(x) - f(x)| : x \in \mathbf{R}\} = f(\frac{1}{\sqrt{n}}) = \frac{1}{2\sqrt{n}}$. As $\frac{1}{2\sqrt{n}} \to 0$, given any $\varepsilon > 0$, there exists a positive integer N such that $n \geq N \Rightarrow \frac{1}{2\sqrt{n}} < \varepsilon$. Hence for all x in **R**,

$$n \geq N \Rightarrow |f_n(x) - f(x)| \leq \sup\{\,|f_n(y) - f(y)| : y \in \mathbf{R}\} = \frac{1}{2\sqrt{n}} < \varepsilon.$$

This means by Definition 13 Chapter 7, $f_n \to f$ uniformly.

Now for each positive integer n, f_n is differentiable and $f_n'(x) = \dfrac{1-nx^2}{(1+nx^2)^2}$.

For $x \neq 0$, $f_n'(x) = \dfrac{\frac{1}{n^2} - \frac{x^2}{n}}{(\frac{1}{n}+x^2)^2} \to \dfrac{0}{x^4} = 0$ as $n \to \infty$. Plainly $f_n'(0) = 1 \to 1$ as $n \to \infty$.

Therefore, $f_n' \to g$ pointwise, where $g(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Plainly $g \neq f' = 0$.

We note that each f_n is *continuously differentiable*, i.e., f_n' is continuos. Therefore, since g is not continuous at 0, (f_n') does not converge uniformly.

8.4. Uniform Convergence and Integration

It is not unreasonable in the light of Example 6, to make the requirement that (f_n') be uniformly convergent and that each f_n' be continuous. Perhaps then we may be able to deduce $f' = \lim\limits_{n\to\infty} f_n'$. With the condition that each f_n' is continuous and the sequence (f_n') is uniformly convergent, by Theorem 14 Chapter 7, $g = \lim\limits_{n\to\infty} f_n'$ is continuous and hence Riemann integrable on any bounded interval. If we have Riemann integrability how can we then show that $g = f'$? The next question is then when does the following equation

$$\int_a^x g(t)dt = \lim_{n\to\infty} \int_a^x f_n'(t)dt \qquad \text{--------------------------} \quad (*)$$

hold?

That is, dose integrating each f_n' and finding its limit the same as integrating g ? The right hand side of (*) by the Fundamental Theorem of Calculus (Theorem 43 Chapter 5) is just

$$\lim_{n\to\infty} \int_a^x f_n'(t)dt = \lim_{n\to\infty} \left(f_n(x) - f_n(a)\right) = f(x) - f(a)$$

assuming $f_n \to f$ pointwise. So if we assume (*), then we have $\int_a^x g(t)dt = f(x) - f(a)$. It will then follow by the Fundamental Theorem of Calculus (Theorem 45 Chapter 5) that $g(x) = f'(x)$ for each x since g is continuous. Hence $g = f'$. Thus $f_n' \to f'$ uniformly.

In this fashion, information about integration can tell us information about differentiation. What we require is a simple result regarding the convergence of Riemann integrals. So we state the result below.

Theorem 7. Suppose ($g_n : [a, b] \to \mathbf{R}$) is a sequence of continuous function converging uniformly to $g:[a, b] \to \mathbf{R}$. Then g is continuous on $[a, b]$, $\lim\limits_{n\to\infty} \int_a^b |g_n(t) - g(t)|dt = 0$ and $\int_a^b g(t)dt = \lim\limits_{n\to\infty} \int_a^b g_n(t)dt$.

Proof. For each positive integer n, g_n is continuous on $[a, b]$ and so g_n is integrable on $[a, b]$ (see e.g., Theorem 23 Chapter 5). Since $g_n \to g$ uniformly, by Theorem 9 Chapter 7, g is continuous on $[a, b]$ and hence integrable on $[a, b]$. Therefore, $g_n - g$ is Riemann integrable on $[a, b]$.

Now for each positive integer n,

$$\left|\int_a^b g_n(x) - \int_a^b g(x)dx\right| = \left|\int_a^b (g_n(x) - g(x))dx\right| \le \int_a^b |g_n(x) - g(x)|dx \quad \text{---} \quad (1)$$

by Theorem 53 Chapter 5 Integration.

Since $g_n \to g$ uniformly, given any ε > 0 there exists a positive integer N such that for all integer n,

$$n \geq N \Rightarrow |g_n(x) - g(x)| < \frac{\varepsilon}{2(b-a)} \text{ for all } x \text{ in } [a, b].$$

Thus,

$$n \geq N \Rightarrow \int_a^b |g_n(x) - g(x)| dx \leq \int_a^b \frac{\varepsilon}{2(b-a)} dx = \frac{\varepsilon}{2} < \varepsilon \,. \text{ -------------- (2)}$$

Therefore, $\lim_{n\to\infty} \int_a^n |g_n(t) - g(t)| dt = 0$.

Now,

$$n \geq N \Rightarrow \left| \int_a^b g_n(x) - \int_a^b g(x) dx \right| \leq \int_a^b |g_n(x) - g(x)| dx$$

by (1)

$$< \varepsilon$$

by (2).

This means $\int_a^b g_n(x) \to \int_a^b g(x) dx$. This completes the proof.

8.5 Differentiating A Sequence

Now we formulate our theorem about differentiation.

Theorem 8. Let I be a non-empty interval (bounded or unbounded). Suppose we have a sequence of continuously differentiable functions ($f_n : I \to \mathbf{R}$). That is, for each positive integer n, f_n is differentiable and the derived function $f_n{}' : I \to \mathbf{R}$ is continuous.

Suppose the following two conditions are satisfied:

(1) ($f_n : I \to \mathbf{R}$) converges pointwise to a function $f : I \to \mathbf{R}$;

(2) ($f_n{}' : I \to \mathbf{R}$) converges uniformly to a function $g : I \to \mathbf{R}$.

Then $f : I \to \mathbf{R}$ is differentiable, $g : I \to \mathbf{R}$ is continuous $f' = g$ and $f_n \to f$ uniformly on any closed and bounded interval $[a, b] \subseteq I$.

Proof. Fix a point a in I. We shall proceed to use Theorem 7. For each positive integer n, since $f_n : I \to \mathbf{R}$ is continuous, by the Fundamental Theorem of Calculus (Theorem 43 Chapter 5)

$$\int_a^x f_n'(t) dt = f_n(x) - f_n(a). \qquad \text{-------------------- (1)}$$

By Theorem 7, since $f_n' \to g$ uniformly as given by condition (2),

$$\int_a^x f_n'(t) dt \to \int_a^x g(t) dt$$

for each x in I.

Therefore, for each x in I by (1),

$$\int_a^x g(t) dt = \lim_{n\to\infty} \int_a^x f_n'(t) dt = \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a) = f(x) - f(a) \text{ -------- (2)}$$

by condition (1).

Note that g is continuous on I by Theorem 14 Chapter 7. Thus by the Fundamental theorem of Calculus (Theorem 45 Chapter 5), the function G: $I \to \mathbf{R}$ defined by $G(x) = \int_a^x g(t) dt$ is differentiable and $G'(x) = g(x)$ for each x in I. Therefore, it follows from (2) that for each x in I ,

$$g(x) = f'(x)$$

since $G(x) = f(x) - f(a)$. Hence we have proved the first two assertions.

Observe that since $f_n' \to g$ uniformly, the sequence of functions ($F_n : I \to \mathbf{R}$), where for each positive integer n, F_n is defined by $F_n(x) = \int_a^x f_n'(t) dt$ for x in I, also

converges uniformly to $G(x)=\int_a^x g(t)dt$ on any closed interval [a, b] in I. This is easily deduced as follows.
Since $f_n' \to g$ uniformly, for any ε > 0, there exists a positive integer N such that for all integer n,

$$n \geq N \Rightarrow |f_n'(x)-g(x)| < \frac{\varepsilon}{2(b-a)} \text{ for all } x \text{ in } [a, b].$$

Hence $n \geq N \Rightarrow |F_n(x)-G(x)| = \left|\int_a^x f_n'(t)dt - \int_a^x g(t)dt\right|$

$$\leq \int_a^x |f_n'(t)-g(t)|dt \leq \int_a^x \frac{\varepsilon}{2(b-a)} = \frac{(x-a)\varepsilon}{2(b-a)} < \varepsilon$$

for all x in [a, b].
Thus $F_n \to G$ uniformly on [a, b].
Since for each positive integer n, $F_n(x) = f_n(x) - f_n(a)$ for all x in I (by (1)) and since $F_n \to G$ uniformly on [a, b], $f_n(x) - f_n(a)$ converges uniformly to $f(x) - f(a)$ uniformly on [a, b] and so since $f_n(a) \to f(a)$ uniformly, $f_n \to f$ uniformly on [a, b]. This proves the last assertion and thus completes the proof.

Remark.
1. If I is a closed and bounded interval, say [a, b], then the conclusion of Theorem 8 will give uniform convergence of (f_n).
2. Since by Theorem 3, (f_n') converges uniformly is equivalent to (f_n') being uniformly Cauchy, we may replace condition (2) of Theorem 8 by requiring that (f_n') be uniformly Cauchy.
3. Condition (1) of Theorem (8) may be replaced by a simpler looking condition (1)' :
"There exists an element a in I such that the sequence ($f_n(a)$) converges."
Then condition (2) would imply pointwise convergence for (f_n) on I. We deduce this as follows. By (1) in the proof of Theorem 8,

$$f_n(x) = \int_a^x f_n'(t)dt + f_n(a).$$

By Theorem 7, since $f_n' \to g$ uniformly by condition (2),

$$\int_a^x f_n'(t)dt \text{ converges pointwise to } \int_a^x g(t)dt .$$

Therefore, if ($f_n(a)$) is convergent and converges to, say $f(a)$, then f_n converges pointwise to $\int_a^x g(t)dt + f(a)$.
4. A stronger version of Theorem 8 is also true. Under the hypothesis (1) that each f_n is differentiable on I (not necessarily continuously differentiable), (2) there exists an element a in I such that the sequence ($f_n(a)$) converges and (3) ($f_n' : I \to \mathbf{R}$) converges uniformly to a function $g : I \to \mathbf{R}$, we can conclude that the sequence (f_n) converges to a function f such that $f' = g$. The proof is more delicate since f_n' may not be integrable and we shall need to use only the consequence of differentiability. We shall prove this below.

Theorem 8'. Let I be a non-empty interval (bounded or unbounded). Suppose we have a sequence of differentiable functions ($f_n : I \to \mathbf{R}$).
Suppose the following two conditions are satisfied:
(1) There exists a point x_0 such that the sequence ($f_n(x_0)$) is convergent
(2) ($f_n' : I \to \mathbf{R}$) converges uniformly to a function $g : I \to \mathbf{R}$.
Then ($f_n : I \to \mathbf{R}$) converges on I to a differentiable function $f : I \to \mathbf{R}$ such that $f' = g$ and $f_n \to f$ uniformly on any closed and bounded interval [a, b] $\subseteq I$.

Proof. Take a closed and bounded interval $[a, b]$ *containing* x_0 in I. We shall show that $(f_n : I \to \mathbf{R})$ is uniformly convergent on $[a, b]$.
Since $(f_n' : I \to \mathbf{R})$ converges uniformly to a function $g : I \to \mathbf{R}$, $(f_n' : I \to \mathbf{R})$ is uniformly Cauchy on $[a, b]$. This means given $\varepsilon > 0$, there exists an integer N such that *for all* x in I and

$$n, m \geq N \Rightarrow |f_n'(x) - f_m'(x)| < \varepsilon/(2L), \quad \text{------------} \ (1)$$

where $L = b - a$ is the length of a closed and bounded $[a, b]$ in I.
Now for any integer $n, m > 0$,

$$|f_n(x) - f_m(x)| = |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0) - f_m(x_0))|$$
$$= |(f_n'(c) - f_m'(c))(x - x_0) + (f_n(x_0) - f_m(x_0))|$$

for some c between x and x_0 by the Mean Value Theorem

$$\leq |(f_n'(c) - f_m'(c))| \, |x - x_0| + |f_n(x_0) - f_m(x_0)| \quad \text{----------} \ (2)$$

by the triangle inequality.
Since $(f_n(x_0))$ is convergent, it is Cauchy. Hence there exists an integer M, such that,

$$n, m \geq M \Rightarrow |f_n(x_0) - f_m(x_0)| < \varepsilon/2 \quad \text{----------------------------} \ (3)$$

Therefore, it follows from (2) and (3) that for all x in [a, b] ,

$$n, m \geq \max(N, M) \Rightarrow |f_n(x) - f_m(x)| < \varepsilon\, |x - x_0|/(2L) + \varepsilon/2 < \varepsilon .$$

This proves that (f_n) is uniformly Cauchy on $[a, b]$. Therefore, by Theorem 3, (f_n) converges uniformly to a function, say f on $[a, b]$. For any x in I, there exists a closed and bounded interval D containing both x and x_0 . Thus, by we have just proved, f_n converges uniformly to a function, f on D. By uniqueness of limit, the limiting function f is unique. Hence f_n converges pointwise to a function, f on *the* interval I. In particular, by the above proceeding we can conclude that f_n converges uniformly to a function on *any* closed and bounded interval D in I.
We shall now show that the limiting function f is differentiable and that $f' = g$.
Take any c in I. We shall show that $f'(c) = g(c)$.

Define $g_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c}, & x \neq c \\ f_n'(c), & x = c \end{cases}$. Then g_n is continuous on I since f_n is differentiable at c and $g_n(c) = f_n'(c)$. Observe that the sequence (g_n) is pointwise convergence on $I - \{c\}$, since (f_n) is. Because the sequence $((f_n'(c))$ is convergent and converges to $g(c)$, (g_n) is pointwise convergent on I .
We shall show that (g_n) is uniformly convergent on I.
For any $x \neq c$,

$$|g_n(x) - g_m(x)| = \left| \frac{f_n(x) - f_n(c)}{x - c} - \frac{f_m(x) - f_m(c)}{x - c} \right|$$
$$= |f_n'(d) - f_m'(d)|$$

for some d between c and x by the Mean Value Theorem.
Since $(f_n' : I \to \mathbf{R})$ is uniformly Cauchy on I , *for all* x in I, there exists an integer N_0 such that

$$n, m \geq N_0 \Rightarrow |f_n'(x) - f_m'(x)| < \varepsilon.$$

It follows that for any $x \neq c$,

$$n, m \geq N_0 \Rightarrow |g_n(x) - g_m(x)| = |f_n'(d) - f_m'(d)| < \varepsilon.$$

Also, $n, m \geq N_0 \Rightarrow |g_n(c) - g_m(c)| = |f_n'(c) - f_m'(c)| < \varepsilon.$
Hence (g_n) is uniformly convergent on I.

Note that for $x \neq c$, $g_n(x) \to \dfrac{f(x)-f(c)}{x-c}$ and $g_n(c) = f'_n(c) \to g(c)$. Thus

$g_n(x) \to G(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c}, & x \neq c \\ g(c), & x = c \end{cases}$. Since each g_n is continuous, the uniform limit G is continuous on I. Therefore,

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c} G(x) = G(c) = g(c).$$

This shows that f is differentiable at c and $f'(c) = g(c)$.
This completes the proof.

8.6 Differentiating Power Series

We shall apply Theorem 8 to power series. First an example.

Example 9. For each positive integer n, let $f_n(x) = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^{n-1}}{(n-1)!}$ for x in $\mathbf{R}$. (This is the familiar truncated exponential expansion.) $f_n(x)$ is the n-th partial sum of the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Define $f_0(x) = 0$ for all x in $\mathbf{R}$.
By the Ratio Test (see e.g. Theorem 18 Chapter 7) the series converges for all x, as

$$\left| \frac{\frac{1}{n!}}{\frac{1}{(n-1)!}} \right| = \frac{1}{n} \to 0 < 1.$$

Thus, for each x in $\mathbf{R}$, the sequence $(f_n(x))$ converges to a value which we denote by $f(x)$. In this way we define a function $f : \mathbf{R} \to \mathbf{R}$. This is the well known exponential function. Note that $f_n \to f$ pointwise on $\mathbf{R}$. Now fix a positive number K and consider the closed interval $[-K, K]$.
Then we claim that $f_n \to f$ uniformly on $[-K, K]$. We now proceed to prove just this fact.
Note that for each non-negative integer n, and for all x in $[-K, K]$,

$$\left| \frac{x^n}{n!} \right| \leq \frac{K^n}{n!}.$$

Therefore, since $\displaystyle\sum_{n=0}^{\infty} \frac{K^n}{n!}$ is convergent, by the Weierstrass M Test (Theorem 1), $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on $[-K, K]$. That is to say $f_n \to f$ uniformly on $[-K, K]$.

Now for each positive integer n, f_n is a polynomial function and so is continuous on $\mathbf{R}$ and hence on $[-K, K]$. Therefore, by Theorem 14 Chapter 7, f is continuous on $[-K, K]$.
For each positive integer n, the derived function is given by

$$f'_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-2}}{(n-2)!} = f_{n-1}(x).$$

Thus the sequence $(f_n{}') = (f_{n-1})$ converges uniformly on $[-K, K]$ to f. Hence, by Theorem 8, f is differentiable and $f'(x) = \lim_{n\to\infty} f'_n(x) = \lim_{n\to\infty} f_{n-1}(x) = f(x)$ for all x in $[-K, K]$. Since this is true for all $K > 0$, $f'(x) = f(x)$ for all x in $\mathbf{R}$.

Remark. We could have proved the uniform convergence of f on $[-K, K]$ in Example 9 by a Comparison Test just as in the proof of the Weierstrass M Test. It is really also a test for absolute convergence. Hence the test is restricted in this way for application. Let us follow the argument of the proof.
For each non-negative integer k and for all x in $[-K, K]$,
$$\left|\frac{x^k}{k!}\right| \le \frac{K^k}{k!}.$$
Therefore, for any positive integer n,
$$\left|\sum_{k=n+1}^{n+p} \frac{x^k}{k!}\right| \le \sum_{k=n+1}^{n+p}\left|\frac{x^k}{k!}\right| \le \sum_{k=n+1}^{n+p} \frac{K^k}{k!} \qquad \text{------------------ (1)}$$
Since we know $\sum_{k=0}^{\infty} \frac{K^k}{k!}$ is convergent, the series $\sum_{k=0}^{\infty} \frac{K^k}{k!}$ is a Cauchy series. Hence for any $\varepsilon > 0$, there exists a positive integer N such that for all $n \ge N$ and for all p in $\boldsymbol{P}$,
$$\sum_{k=n+1}^{n+p} \frac{K^k}{k!} < \varepsilon.$$
It then follows from (1) that for all $n \ge N$ and for all p in $\boldsymbol{P}$,
$$\left|\sum_{k=n+1}^{n+p} \frac{x^k}{k!}\right| \le \sum_{k=n+1}^{n+p} \frac{K^k}{k!} < \varepsilon$$
for all x in $[-K, K]$. Therefore, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is uniformly Cauchy on $[-K, K]$ and so by Theorem 3 it converges uniformly on $[-K, K]$.

Our next result is about the disk of convergence of the power series.

Lemma 10. The power series $\sum_{n=0}^{\infty} a_n x^n, \sum_{n=1}^{\infty} n a_n x^{n-1}, \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ all have the same radius of convergence.

Proof. It is sufficient to show that $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence. Let r be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and r' the radius of convergence of $\sum_{n=1}^{\infty} n a_n x^{n-1}$. Let x be such that $|x| < r'$. Then $\sum_{n=1}^{\infty} |n a_n x^{n-1}|$ is convergent. Since for each integer $n \ge 1$, $|a_n x^{n-1}| \le |n a_n x^{n-1}|$, by the Comparison Test for Series (Proposition 12 Chapter 6), $\sum_{n=1}^{\infty} |a_n x^{n-1}|$ is convergent (for $|x| < r'$). Therefore, $|x| \sum_{n=1}^{\infty} |a_n x^{n-1}| = \sum_{n=1}^{\infty} |a_n x^n|$ is convergent for $|x| < r'$. Thus, $|x| \le r$. Hence $r' \le r$. (This is because if $r' > r$, then we can choose a x_0 such that $r < |x_0| < r'$. Then by the above argument we can show that $\sum_{n=0}^{\infty} |a_n x_0^n|$ is convergent and consequently contradicting that $\sum_{n=0}^{\infty} |a_n x_0^n|$ is divergent since $|x_0| > r$.)
Now we shall show that $r \le r'$. Suppose $|x| < r$. Choose a real number c such that $|x| < c < r$. Then both series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} a_n c^n$ converge. It follows that $a_n c^n \to 0$ (see Proposition 10 Chapter 6). Therefore, given any $\varepsilon > 0$, there exists a positive integer N such that for all integer $n \ge N$, $|a_n c^n| < \varepsilon$. Now take $\varepsilon = c > 0$. It then follows that for all integer n,

$$n \geq N \;\Rightarrow |\, a_n\, c^{n-1} \,| < 1.$$

Therefore, for all integer $n \geq N$,

$$\sum_{k=N}^{n} |k a_k x^{k-1}| = \sum_{k=N}^{n} |a_k c^{k-1}|\,|k| \left|\frac{x}{c}\right|^{k-1} < \sum_{k=N}^{n} k \left|\frac{x}{c}\right|^{k-1} \quad \text{------------------} \quad (1)$$

Now notice that $\sum_{k=N}^{\infty} k\left|\frac{x}{c}\right|^{k-1}$ is convergent by the Ratio Test (Theorem 21 Chapter 6) because for $x \neq 0$, $\dfrac{(n+1)\left|\frac{x}{c}\right|^{n}}{n\left|\frac{x}{c}\right|^{n-1}} = (1+\frac{1}{n})\left|\frac{x}{c}\right| \to \left|\frac{x}{c}\right| < 1$ and it is plainly convergent for $x = 0$. Therefore, using (1), by the Comparison Test (Proposition 12 Chapter 6), $\sum_{k=N}^{\infty} |k a_k x^{k-1}|$ is convergent. It follows that , $\sum_{k=1}^{\infty} |k a_k x^{k-1}|$ is convergent. Therefore, $|x| \leq r\,'$. It then follows that $r \leq r\,'$. (This is because if $r > r'$ then choose x such that $r > |x| > r\,'$. But we have shown that $|x| \leq r\,'$ and this contradicts $|x| > r\,'$.). Therefore, $r = r\,'$. So $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence. For each positive integer, n let $b_n = (n+1)a_{n+1}$. Then $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} b_n x^n$ and

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} (n-1) b_{n-1} x^{n-2} = \sum_{n=1}^{\infty} n b_n x^{n-1}.$$

Therefore, by what we have just shown, $\sum_{n=0}^{\infty} b_n x^n$ and $\sum_{n=1}^{\infty} n b_n x^{n-1}$ have the same radius of convergence. It follows that $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ have the same radius of convergence. Now if we let $c_0 = 0$ and for each integer $n \geq 1$, let $c_n = \dfrac{a_{n-1}}{n}$. Then

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=0}^{\infty} c_{n+1} x^{n+1} = \sum_{n=0}^{\infty} c_n x^n$$

and

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

Thus again by what we have proved $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=1}^{\infty} n c_n x^{n-1}$have the same radius of convergence and so $\sum_{n=0}^{\infty} \dfrac{a_n}{n+1} x^{n+1}$ and $\sum_{n=0}^{\infty} a_n x^n$ have the same radius of convergence.

We deduce from Lemma 10 that the power series obtained from one by differentiating term by term have the same radius of convergence. We shall now show that we can indeed obtain the derivative of the function represented by the power series by term by term differentiation within the radius of convergence.

Theorem 11. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a real power series with radius of convergence $\boldsymbol{r}$ and $D_r = \{x\colon |x| < r\}$ is the disc of convergence, then the function $f : D_r \to \mathbf{R}$ is differentiable and $f\,'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for each x in D_r. Moreover, both $f(x)$ and $f\,'(x)$ as power series converge uniformly (and absolutely) on any closed interval $[-c, c] \subseteq D_r$.

Proof. We show that for any power series with disc of convergence D_r , the power series converges uniformly on any closed and bounded interval $[-c, c]$ in $D_r = (-r, r)$.

Let $0 < c < r$. Take a fixed real number K such that $c < K < r$. Then $\sum_{n=0}^{\infty} a_n K^n$ converges absolutely (Theorem 4 Chapter 7). Now since $0 < c < K$, for all x in $[-c, c]$, $|x| < K$. Therefore, for any integer $n \geq 0$, and for all x in $[-c, c]$,

$$|a_n x^n| \leq |a_n K^n| .$$

Hence, by the Weierstrass M Test (Theorem 1) $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on the interval $[-c, c]$.

Thus, if we write $f(x)$ for $\sum_{n=0}^{\infty} a_n x^n$ for each x in $[-c, c]$, then the n-th partial sum $s_n(x) = \sum_{k=0}^{n} a_k x^k \to f(x)$ uniformly on $[-c, c]$. Similarly, since by Lemma 10 $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$have the same radius of convergence and hence the same disc of convergence, $s'_n(x) = \sum_{k=1}^{n} k a_k x^{k-1}$converges uniformly on $[-c, c]$. Therefore, by Theorem 8, f is differentiable on $[-c, c]$, s_n' converges uniformly to f' on $[-c, c]$. That is

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

on $[-c, c]$. Since this is true for any c with $0 < c < r$, f is differentiable on $D_r = (-r, r)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for each x in D_r . This completes the proof.

Remark.

1. Theorem 11 says that we can differentiate a power series term by term in its disc of convergence. This is a very important property of power series function.
2. It is Taylor's Theorem that links power series with other theory of functions.
3. Thus a real power series represents an infinitely differentiable function f on its interval of convergence and all the derivatives can be obtained by term wise differentiation (Theorem 11). We can thus express the coefficient a_n in terms of the derivatives of f.
4. We may prove Theorem 11 directly without using Theorem 8 as follows:

   Take x with $|x| < r$ and T, S such that $|x| < T < S < r$. .
   For real x, and h such that $0 < |h| \leq T - |x|$, we have

   $$\frac{1}{h}((x+h)^n - x^n) = (x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x$$

   Hence $$\frac{1}{h}(f(x+h) - f(x)) = \sum_{n=1}^{\infty} g_n(h),$$

   where $g_n(h) = a_n((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x)$ for $h \neq 0$.
   Define $g_n(0) = n a_n x^{n-1}$. Then g_n is continuous for all h in $\mathbf{R}$. In particular

   $$|g_n(h)| \leq n|a_n|T^{n-1} \text{ for } |h| \leq T - |x| \text{ ------------------- } (1)$$

   Now $n(\frac{T}{S})^n \to 0$ because $0 < T/S < 1$ Therefore, for all sufficiently large n we have

   $$n(\tfrac{T}{S})^n \leq T \quad \text{or} \quad nT^{n-1} \leq S^n$$

Since $\sum_{n=0}^{\infty} |a_n|S^n$ is convergent, it follows by the Comparison Test that $\sum_{n=0}^{\infty} n|a_n|T^{n-1}$ is convergent. Then by the Weierstrass M Test (Theorem 1), it follows from (1) that $\sum_{n=1}^{\infty} g_n(h)$ converges uniformly on $\{h: |h| \le T - |x|\}$. Therefore, its sum, i.e., its limiting function is continuous at 0 by Theorem 13 Chapter 7. This means

$$\frac{1}{h}(f(x+h) - f(x)) = \sum_{n=1}^{\infty} g_n(h) \to \sum_{n=1}^{\infty} g_n(0) = \sum_{n=1}^{\infty} na_n x^{n-1} \text{ as } h \to 0.$$

Hence $f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$.

Thus by Lemma 10, both $f(x)$ and $f'(x)$ have the same radius of convergence and so converge uniformly (and absolutely) on any closed interval $[-c, c] \subseteq D_r$.

5. Putting $x = 0$ we can deduce from Theorem 11 that $f'(0) = a_1$ and $f^{(n)}(0) = n!a_n$. This shows that one can deduce the coefficient a_n from the sum function, that is the limiting function f .

8.7 Using Taylor's Theorem

Example 12. Use of Taylor's Theorem.
Suppose we have the following differential equation:

$$f' = f \quad \text{with initial condition } f(0) = 1.$$

Suppose we have proved that sufficiently well behaved differential equations have unique solutions. Then suppose this equation has a solution f on the interval $[-K, K]$. Then f is differentiable and so it is continuous and hence bounded on $[-K, K]$. Thus

$$|f(x)| \le M \text{ for all } x \in [-K, K].$$

Since $f' = f$, $|f'(x)| \le M$ for all $x \in [-K, K]$. Now Apply Taylor's Theorem (Theorem 44 Chapter 4) with expansion around $x_0 = 0$. Then we have for each $n \ge 2$,

$$f(x) = f(0) + x f'(0) + \cdots + \frac{1}{k!}x^k f^{(k)}(0) \cdots + \frac{1}{n!}x^n f^{(n)}(0) + x^{n+1}\frac{f^{(n+1)}(\theta_{x,n})}{(n+1)!},$$

where $\theta_{x,n}$ is some point between 0 and x. Now by the initial condition $f'(0) = f(0) = 1$. It follows that $f^{(n)}(0) = f(0) = 1$ for any positive integer n. Thus the Taylor expansion becomes

$$f(x) = 1 + x + \cdots + \frac{1}{k!}x^k \cdots + \frac{1}{n!}x^n + x^{n+1}\frac{f(\theta_{x,n})}{(n+1)!} .$$

Now we know that the series (see Example 9)

$$1 + x + \cdots + \frac{1}{k!}x^k \cdots + \frac{1}{n!}x^n + \cdots .$$

converges uniformly on $[-K, K]$. In particular, the modulus of the Lagrange form of the remainder $\left| x^{n+1}\frac{f(\theta_{x,n})}{(n+1)!} \right| \le K^{n+1}\frac{M}{(n+1)!}$. Since $K^{n+1}\frac{M}{(n+1)!} \to 0$ as $n \to \infty$, $\left| x^{n+1}\frac{f(\theta_{x,n})}{(n+1)!} \right| \to 0$ as $n \to \infty$ by the Comparison Test for sequences. Hence for each x in $[-K, K]$, $\left| f(x) - \sum_{k=0}^{n} \frac{1}{k!}x^k \right| = \left| x^{n+1}\frac{f(\theta_{x,n})}{(n+1)!} \right| \to 0$ as $n \to \infty$. Therefore, by the

Comparison Test for sequences (Proposition 8 Chapter 2), $f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ for any x in $[-K, K]$. Since this is true for any $K > 0$, $f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ for all x in $\mathbf{R}$. Thus, the vanishing of the Lagrange remainder $R_n(x)=x^{n+1}\frac{f^{(n+1)}(\theta_{x,n})}{(n+1)!}$ plays a critical role in showing that the solution of the differential equation is given by the infinite Taylor series $\sum_{k=0}^{n}\frac{1}{k!}x^k$.

Example 13. A function that does not admit an infinite Taylor series expansion.

Let $f:\mathbf{R}\to\mathbf{R}$ be defined by $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0 \\ 0, & x=0\end{cases}$.

Then since the exponential function is differentiable, f is thus a composition of two differentiable functions on $x \neq 0$ and so is differentiable at $x \neq 0$. Now

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}=\lim_{x\to 0^+}\frac{1/x}{e^{1/x^2}}=\lim_{x\to 0^+}\frac{-1/x^2}{e^{1/x^2}(-\frac{2}{x^3})}=\lim_{x\to 0^+}\frac{x}{2e^{1/x^2}}=0$$

by L'Hôpital's Rule and that $\lim_{x\to 0^+}\frac{1}{e^{1/x^2}}=0$. We can show in exactly the same manner that

$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=0$. Hence, $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0$ and so $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0$.

(We can also use the fact that for $x \neq 0$, $e^{\frac{1}{x^2}}\geq\frac{1}{x^2}$ so that $\frac{1}{e^{1/x^2}}\leq x^2$. Thus for $x\neq 0$, $0<\left|\frac{1/x}{e^{1/x^2}}\right|\leq |x|$ and so by the Comparison Test, $\lim_{x\to 0}\frac{1/x}{e^{1/x^2}}=0$. It follows that $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{1/x}{e^{1/x^2}}=0$.).

For $x\neq 0$, $f'(x)=e^{-\frac{1}{x^2}}\frac{2}{x^3}=\frac{2}{x^3e^{1/x^2}}=p_1(\frac{1}{x})e^{-\frac{1}{x^2}}$, where $p_1(\frac{1}{x})$ is a polynomial in $\frac{1}{x}$, $p_1(y) = 2y^3$.

We now examine the limit of the first derivative.

$$\lim_{x\to 0^+}f'(x)=\lim_{x\to 0^+}p_1(\frac{1}{x})e^{-\frac{1}{x^2}}=\lim_{t\to\infty}p_1(t)e^{-t^2}=\lim_{t\to\infty}\frac{p_1(t)}{e^{t^2}}=\lim_{t\to\infty}\frac{6t^2}{2te^{t^2}}=\lim_{t\to\infty}\frac{3t}{e^{t^2}}=\lim_{t\to\infty}\frac{3}{2te^{t^2}}=0$$

by L'Hôpital's Rule and that $\lim_{t\to\infty}\frac{1}{te^{t^2}}=0$ because $\lim_{t\to\infty}te^{t^2}=\infty$. In exactly the same way we show that $\lim_{x\to 0^-}f'(x)=0$. Hence

$$\lim_{x\to 0}f'(x)=0.$$

Note that f is continuous at $x = 0$, since $\lim_{x\to 0}f(x)=0=f(0)$. Therefore, the existence of $\lim_{x\to 0}f'(x)$ implies that f is differentiable at x = 0 and that $f'(0)=\lim_{x\to 0}f'(x)=0$. This means that f' is continuous at $x = 0$.

We shall show that for each positive integer n, $\lim_{x\to 0}f^{(n)}(x)=0$ and consequently, $f^{(n)}(x)=0$.

First we claim that for each positive integer n,

$$f^{(n)}(x)=p_n(\frac{1}{x})e^{-\frac{1}{x^2}} \quad \text{-----------------------------} \quad (1)$$

where $p_n(\frac{1}{x})$ is a polynomial in $\frac{1}{x}$.

We shall prove this statement by induction. Note that (1) is true for n =1, as we have observed. Now assume that (1) is true for n, i.e., $f^{(n)}(x) = p_n(\frac{1}{x})e^{-\frac{1}{x^2}}$.
Differentiating we have,

$$\begin{aligned} f^{(n+1)}(x) &= p_n'(\frac{1}{x})(-\frac{1}{x^2})e^{-\frac{1}{x^2}} + p_n(\frac{1}{x})(\tfrac{2}{x^3})e^{-\frac{1}{x^2}} \\ &= \left(-\frac{1}{x^2}p_n'(\frac{1}{x}) + \tfrac{2}{x^3}p_n(\frac{1}{x})\right)e^{-\frac{1}{x^2}}. \end{aligned}$$

But $-\frac{1}{x^2}p_n'(\frac{1}{x}) + \frac{2}{x^3}p_n(\frac{1}{x})$ is a polynomial in $\frac{1}{x}$. Therefore, letting $p_{n+1}(\frac{1}{x}) = -\frac{1}{x^2}p_n'(\frac{1}{x}) + \frac{2}{x^3}p_n(\frac{1}{x})$, we see that $f^{(n+1)}(x) = p_{n+1}(\frac{1}{x})e^{-\frac{1}{x^2}}$. Thus (1) is true for n+1 and so by mathematical induction, (1) is true for all positive integers.
We next examine the limit $\lim_{x\to 0} f^{(n)}(x)$. We shall now show that for all positive integer n,

$$\lim_{x\to 0} f^{(n)}(x) = 0.$$

Now note $\lim_{x\to 0^+} f^{(n)}(x) = \lim_{x\to 0^+} p_n(\frac{1}{x})e^{-\frac{1}{x^2}} = \lim_{t\to\infty} p_n(t)e^{-t^2} = \lim_{t\to\infty} \frac{p_n(t)}{e^{t^2}} = 0$ by a repeated use of the L'Hôpital's Rule.
[We can first compute the limit

$$\lim_{t\to\infty} \frac{t^{2k+1}}{e^{t^2}} = \lim_{t\to\infty} \frac{(2k+1)}{2}\frac{t^{2k-1}}{e^{t^2}} = \lim_{t\to\infty} \frac{(2k+1)!!}{2^k}\frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{(2k+1)!!}{2^{k+1}}\frac{1}{te^{t^2}} = 0$$

by a repeated use of the L'Hôpital's Rule.]
Similarly, $\lim_{x\to 0^-} f^{(n)}(x) = \lim_{x\to 0^-} p_n(\frac{1}{x})e^{-\frac{1}{x^2}} = \lim_{t\to -\infty} p_n(t)e^{-t^2} = \lim_{t\to -\infty} \frac{p_n(t)}{e^{t^2}} = 0.$
Therefore, $\lim_{x\to 0} f^{(n)}(x) = 0$. Note that , $\lim_{x\to 0} f^{(n)}(x) = 0$ and $f^{(n-1)}$ is continuous at $x = 0$ implies that $f^{(n)}(0) = \lim_{x\to 0} f^{(n)}(x) = 0$ and consequently $f^{(n)}$ is continuous at 0 since it is differentiable there.
[We can use L'Hôpital's Rule, for this deduction.

$$f^{(n)}(0) = \lim_{x\to 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0} = \lim_{x\to 0} \frac{f^{(n-1)}(x)}{x} = \lim_{x\to 0} \frac{f^{(n)}(x)}{1} = 0$$ by L'Hôpital's Rule, since $\lim_{x\to 0} f^{(n)}(x)$ exists and equals 0.]

We thus have for integer $n \geq 1$,

$$f^{(n)}(x) = \begin{cases} p_n(\frac{1}{x})e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Therefore, the n-th degree Taylor expansion of f about x = 0 gives,

$$\begin{aligned} f(x) &= f(0) + x f'(0) + \cdots + \frac{1}{k!}x^k f^{(k)}(0)\cdots + \frac{1}{n!}x^n f^{(n)}(0) + x^{n+1}\frac{f^{(n+1)}(\theta_{x,n})}{(n+1)!} \\ &= 0 + x\cdot 0 + \frac{0}{2!}\cdot x^2 + \cdots + \frac{1}{k!}x^k\cdot 0\cdots + \frac{1}{n!}x^n\cdot 0 + x^{n+1}\frac{f^{(n+1)}(\theta_{x,n})}{(n+1)!} \\ &= x^{n+1}\frac{f^{(n+1)}(\theta_{x,n})}{(n+1)!} \end{aligned}$$

for some $\theta_{x,n}$ between 0 and x.
Hence the remainder, $x^{n+1}\frac{f^{(n+1)}(\theta_{x,n})}{(n+1)!}$ cannot converge to 0 as n tends to ∞ for otherwise f would be identically the zero constant function and thus giving a contradiction as f is not a constant zero function. Therefore, we cannot write $f(x)$ as an infinite Taylor series. In particular the sequence $(f^{(n+1)}(\theta_{x,n}))$ cannot be bounded.

8.8 Convergence of Taylor Polynomials.

We now state Taylor's Theorem with Lagrange form of the remainder without proof.

Theorem 14. Taylor's Theorem (with Lagrange form of the remainder).
Let I be an open interval containing the point x_0 and n be a non-negative integer. Suppose $f : I \to \mathbf{R}$ has n+1 derivatives. Then for any x in I,

$$f(x) = f(x_0) + \tfrac{1}{1!}(x - x_0)\, f'(x_0) + \cdots + \tfrac{1}{k!}(x - x_0)^k f^{(k)}(x_0) \cdots + \tfrac{1}{n!}(x - x_0)^n f^{(n)}(x_0) + R_n(x)$$
,
where the term $R_n(x)$ is the Lagrange form of the remainder and is given by

$$R_n(x) = \frac{1}{(n+1)!}(x - x_0)^{n+1} f^{(n+1)}(\eta)$$

for some η between x and x_0.

(Reference: Theorem 44 Chapter 4.)

As we have seen in Example 12 and 13, in order to write f as a Taylor series we need to show that the remainder $R_n(x)$ converges to 0 as $n \to \infty$ for all x. One advantage of having the series representation of a function is to consider differentiating the function by simply differentiating the terms of the series within the disk of convergence or to consider integrating the function term by term within the disk of convergence.

Now if f is a function defined on an open interval I having derivatives of all order, i.e., f is a smooth function, then Theorem 14 says that for all integer $n \geq 1$, f has a Taylor polynomial

$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0)$$

about the point x_0 in I and $f(x) = p_n(x) + R_n(x)$. If $p_n(x) \to f(x)$ for x in I, then a *Taylor series* expansion of the function $f : I \to \mathbf{R}$ about the point x_0 is the series

$$\sum_{k=0}^{\infty} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0).$$

In particular, at each point x in I,

$$\lim_{n\to\infty} |p_n(x) - f(x)| = 0$$

which is equivalent to $\lim\limits_{n\to\infty} |R_n(x)| = 0$. Thus f admits a Taylor series expansion if and only if it has derivatives of all order and $\lim\limits_{n\to\infty} R_n(x) = 0$.

We shall now investigate the convergence of $p_n(x)$ to $f(x)$. We do this via the Lagrange form of the remainder $R_n(x)$.

The next result makes use of a criterion of the convergence of $R_n(x)$ to 0.

Theorem 15. Suppose $f : I \to \mathbf{R}$ is a function defined on the open interval I, having derivatives of all order. Let x_0 be a point in I. Suppose there exists a closed interval $[x_0 - r, x_0 + r]$ in I such that for every integer $n \geq 1$ and for all x in $[x_0 - r, x_0 + r]$, there exists $M \geq 0$ such that

$$|f^{(n)}(x)| \leq M^n.$$

Then $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0)$ if $|x - x_0| \leq r$.

Proof. By Theorem 14, for all x in $[x_0 - r, x_0 + r]$,

$$|p_n(x) - f(x)| \le \left| \frac{1}{(n+1)!}(x-x_0)^{n+1} f^{(n+1)}(\eta) \right| \text{ for some } \eta \text{ between } x \text{ and } x_0$$
$$\le \left| \frac{1}{(n+1)!} r^{n+1} M^{n+1} \right|$$
since $|f^{(n+1)}(x)| \le M^{n+1}$ for all x such that $|x - x_0| \le r$,
$$\le \frac{(Mr)^{n+1}}{(n+1)!}.$$

Since $\frac{(Mr)^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$, by the Comparison Test, $p_n \to f$ uniformly on $[x_0 - r, x_0 + r]$. Hence, $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$ if $|x - x_0| \le r$.

Theorem 15 can be applied to functions with easily observed bounded derivatives of all order. Thus sine and cosine are such functions.

Example 16.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$

for all x in **R**.

Let $f(x) = \sin(x)$. We only need to know that $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$. Hence $f'(x) = \cos(x)$, $f^{(2)}(x) = -\sin(x)$, $f^{(3)}(x) = -\cos(x)$, $f^{(4)}(x) = \sin(x)$, $f^{(5)}(x) = -\cos(x)$, and in general, $f^{(2n+1)}(x) = (-1)^n \cos(x)$, $f^{(2n)}(x) = (-1)^n \sin(x)$ for integer $n \ge 0$. Therefore, $f^{(2n+1)}(0) = (-1)^n$ and $f^{(2n)}(0) = 0$. Thus the Taylor polynomial about $x = 0$ has only odd powers of x.

For integer $n \ge 0$, $p_{2n+1}(x) = \sum_{k=0}^{n} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} == \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ and $p_{2n+2}(x) = p_{2n+1}(x)$.

For all integer $n \ge 0$, $| f^{(n)}(x) | \le 1$, assuming that $|\sin(x)|$, $|\cos(x)| \le 1$. We denote here $f(x)$ by $f^{(0)}(x)$. Therefore, by Theorem 15, $p_n \to f$ uniformly on $[-K, K]$, for any $K > 0$. Hence $p_n(x) \to f(x) = \sin(x)$ for all x in **R**.

The following is an application of Taylor's series.

Proposition 17. The Euler constant e is irrational.

Proof. By Taylor's Theorem (Theorem 14), for each integer $n \ge 0$,

$$e^x = 1 + x + \cdots + \frac{1}{k!}x^k \cdots + \frac{1}{n!}x^n + \frac{x^{n+1}}{(n+1)!}e^{\theta_n}$$

for some θ_n between 0 and x. Hence for x in [0, 1],

$$e^x - \left(1 + x + \cdots + \frac{1}{k!}x^k \cdots + \frac{1}{n!}x^n\right) = \frac{x^{n+1}}{(n+1)!}e^{\theta_n} \quad .$$

Hence taking x =1,

$$0 < e - \left(1 + 1 + \frac{1}{2!} \cdots + \frac{1}{k!} \cdots + \frac{1}{n!}\right) = \frac{1}{(n+1)!}e^{\theta_n} \le \frac{1}{(n+1)!}e.$$

This means for any integer $n \ge 0$,

$$0 < e - \left(1 + 1 + \frac{1}{2!} \cdots + \frac{1}{k!} \cdots + \frac{1}{n!}\right) \le \frac{1}{(n+1)!}e \text{ ------------- } (1)$$

We have shown in Chapter 6 in the section on Euler constant γ , that

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \int_1^n \frac{1}{t}dt = \ln(n) < \sum_{k=1}^{n-1} \frac{1}{k}.$$

Hence we have $\ln(4) > \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$ and consequently taking exponentiation we get $e < 4$. Thus from (1) we get

$$0 < \; e - \left(1 + 1 + \frac{1}{2!} \cdots + \frac{1}{k!} \cdots + \frac{1}{n!}\right) \le \frac{4}{(n+1)!} \quad \text{-------------- (2)}$$

for any integer $n \ge 0$. Thus if e is rational, say $e = \dfrac{p}{q}$ in its lowest terms, then from (2) we get for any inter $n \ge 0$

$$0 < \; \frac{p}{q} - \left(1 + 1 + \frac{1}{2!} \cdots + \frac{1}{k!} \cdots + \frac{1}{n!}\right) \le \frac{4}{(n+1)!} \quad \text{-------------- (3)}$$

Let $n = \max(4, q)$. Multiply (3) by $n!$ we get

$$0 < \; \frac{n!p}{q} - \left(2n! + \frac{n!}{2!} + \cdots + \frac{n!}{k!} \cdots + 1\right) \le \frac{4}{n+1} \le \frac{4}{5}. \quad \text{--------- (4)}$$

But the term $\dfrac{n!p}{q} - \left(2n! + \dfrac{n!}{2!} + \cdots + \dfrac{n!}{k!} \cdots + 1\right)$ is an integer since every term in the expression is an integer. (4) then says it is an integer in (0, 4/5], contradicting that there is no integer in (0, 4/5]. Hence e is irrational.

8.9 Continuity of Power Series, Abel's Theorem

Now we go back to the question of continuity of a power series function at the boundary of the disc of convergence, if the power series is convergent there. For real power series, if the series is convergent at the boundary of the disc of convergence, then it is also continuous there, a result attributed to Abel. Even if we do not have convergence at the boundary, for instance if R is the radius of convergence and if $\lim_{x\to R^-} \sum_{n=0}^{\infty} a_n x^n$ exists, though $\sum_{n=0}^{\infty} a_n R^n$ is divergent, then one has a definition of "sum" for the divergent series to take on this limit. This means that it is possible to define the sum of a series in entirely new ways that give finite sum to series that are divergent in Cauchy's sense. For series that are convergent in Cauchy's sense and if it is also convergent in these new ways of summing the series, then we call this a regularity or consistency result. The notions of Abel summability and Cesaro summability are regular ones. The results called Tauberian theorems that give condition so that given the summability in whatever new way of a series, it will also be convergent in Cauchy's sense. For example, Alfred Tauber (1886-1942) proved that if $\sum_{n=0}^{\infty} a_n$ is Abel summable to the value A and if $na_n \to 0$ as $n \to \infty$, then $\sum_{n=0}^{\infty} a_n$ converges to A in the sense of Cauchy. We shall prove Abel's regularity theorem.

Theorem 18 (Abel's Theorem, Abel, Niels Henrik, 1802-29)

Suppose the real power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. If it converges at $x = R$, then it converges uniformly in $[0, R]$. Similarly, if it converges at $-R$, then it converges uniformly in $[-R, 0]$.

Proof. We may assume that the radius of convergence is 1. This makes the proof easier and more elegant. We may use the change of variable $x = Ry$ to change the power series if need be to one with radius of convergence 1. With this change of variable,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n y^n = \sum_{n=0}^{\infty} b_n y^n,$$

where $b_n = a_n R^n$. Thus $\sum_{n=0}^{\infty} b_n y^n$ converges absolutely for $|y| < 1$ and diverges when $|y| > 1$.

Now we assume the radius of convergence is 1. Suppose at the boundary $x = 1$, $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n$ is convergent. We may assume that $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = 0$.

(If need be, we may redefine the new a_0 to be the old $a_0(\text{old}) - \lim_{n\to\infty} \sum_{k=0}^{n} a_k$. Suppose $\lim_{n\to\infty} \sum_{k=0}^{n} a_k = L$. Then we let $c_k = a_k$ for integer $k > 0$, $c_0 = a_0 - L$. Then $\sum_{n=0}^{\infty} c_n = 0$ and $\sum_{n=0}^{\infty} a_n$ is convergent if and only if $\sum_{n=0}^{\infty} c_n$ is convergent. Plainly $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on [0, 1] if and only if $\sum_{n=0}^{\infty} c_n x^n$ is uniformly convergent on [0, 1] because the constant term a_0 and c_0 do not affect the Cauchy condition.)

Thus we may assume that

(i) the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 1;

(ii) $\sum_{n=0}^{\infty} a_n$ is convergent and

(iii) $\sum_{n=0}^{\infty} a_n = 0$.

For each integer $n \geq 0$, let $s_n = \sum_{k=0}^{n} a_k$. Then (iii) says $s_n \to 0$. Plainly, $a_n = s_n - s_{n-1}$ for integer $n \geq 1$ and $a_0 = s_0$. We shall rewrite the partial sums of $\sum_{n=0}^{\infty} a_n x^n$ in a more useful form. For each integer $n \geq 0$,

$$\sum_{k=0}^{n} a_k x^k = a_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k = s_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k. \quad \text{--------------------} \quad (1)$$

We shall show that $\sum_{n=0}^{\infty} a_n x^n$ is uniformly Cauchy on [0, 1].

(We can actually use (1) to deduce that the power series is continuous at $x = 1$. We shall pursue this later.)

For any integer $N \geq 1$ and for any integer $p \geq 1$,

$$\begin{aligned}
\sum_{k=N+1}^{N+p} a_k x^k &= \sum_{k=N+1}^{N+p} (s_k - s_{k-1}) x^k = \sum_{k=N+1}^{N+p} s_k x^k - \sum_{k=N+1}^{N+p} s_{k-1} x^k \\
&= \sum_{k=N+1}^{N+p} s_k x^k - \sum_{k=N}^{N+p-1} s_k x^{k+1} = \sum_{k=N+1}^{N+p} s_k x^k - x \sum_{k=N}^{N+p-1} s_k x^k \\
&= (1-x) \sum_{k=N+1}^{N+p-1} s_k x^k + s_{N+p} x^{N+p} - s_N x^{N+1} \\
&= (1-x) \sum_{k=N}^{N+p-1} s_k x^k + s_{N+p} x^{N+p} - s_N x^N . \quad \text{------------------------} \quad (2)
\end{aligned}$$

Therefore, it follows from (2) and triangle inequality that for any integer $N \geq 1$, any integer $p \geq 1$ and for $x \in [0, 1]$,

$$\left| \sum_{k=N+1}^{N+p} a_k x^k \right| \leq (1-x) \sum_{k=N}^{N+p-1} |s_k| x^k + \left| s_{N+p} \right| x^{N+p} + |s_N| x^N$$

$$\leq (1-x) \sum_{k=N}^{N+p-1} |s_k| x^k + \left| s_{N+p} \right| + |s_N|. \quad \text{------------------} \quad (3)$$

Now since $s_n \to 0$, $|s_n| \to 0$. For each integer $n \geq 0$, let

$$M_n = \sup\{|s_n|, |s_{n+1}|, \ldots\} = \sup\{|s_j| : j \text{ is an integer and } j \geq n\}.$$

Then for each integer $n \geq 0$, $M_n \geq 0$ and $\lim_{n\to\infty} M_n = \limsup_{n\to\infty} |s_n| = 0$ since $\lim_{n\to\infty} |s_n| = 0$. From (3) we have that for any integer $N \geq 1$, any integer $p \geq 1$ and for $x \in [0, 1]$,

$$\begin{aligned}
\left|\sum_{k=N+1}^{N+p} a_k x^k\right| &\leq (1-x)\sum_{k=N}^{N+p-1} M_N x^k + 2M_N = M_N \sum_{k=N}^{N+p-1} x^k(1-x) + 2M_N \\
&\leq M_N\left(\sum_{k=N}^{N+p-1} x^k - \sum_{k=N}^{N+p-1} x^{k+1}\right) + 2M_N \\
&\leq M_N(x^N - x^{N+p}) + 2M_N = M_N x^N(1-x^p) + 2M_N \\
&\leq M_N + 2M_N = 3M_N. \qquad \text{-------------------------------------} \ (4)
\end{aligned}$$

Now since $M_n \to 0$ as $n \to \infty$, given any $\varepsilon > 0$, there exists a positive integer N_0 such that for any integer n, $n \geq N_0 \Rightarrow M_n < \varepsilon/3$. Thus it follows from (4) that for any integer $N \geq N_0$, any integer $p \geq 1$ and for *any* $x \in [0, 1]$,

$$\left|\sum_{k=N+1}^{N+p} a_k x^k\right| \leq 3M_N < \varepsilon.$$

Therefore, $\sum_{k=0}^{\infty} a_k x^k$ is uniformly Cauchy on [0, 1]. Thus, by Theorem 3, $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on [0,1].

The case that $\sum_{k=0}^{\infty} a_k x^k$ is convergent at the other end point -1 and $\sum_{k=0}^{\infty} a_k(-1)^k = 0$ is similar. Just note that for any integer $N \geq 1$, for any integer $p \geq 1$ and for x in $[-1,0]$,

$$\sum_{k=N+1}^{N+p} a_k x^k = \sum_{k=N+1}^{N+p} (-1)^k a_k |x|^k \sum_{k=N+1}^{N+p} (s_k - s_{k-1})|x|^k = \sum_{k=N+1}^{N+p} s_k x^k - \sum_{k=N+1}^{N+p} s_{k-1} x^k,$$

where $s_n = \sum_{k=0}^{n} (-1)^k a_k$. We can then deduce (3) and (4) with the same notation but with $|x|$ in place of x and deduce in like manner the uniform convergence of $\sum_{k=0}^{\infty} a_k x^k$ on $[-1,0]$.

Corollary 19. Suppose the real power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. If it converges at $x = R$ to a value L, then $\lim_{x\to R^-} \sum_{n=0}^{\infty} a_n x^n = L$. That is to say the power series function $\sum_{n=0}^{\infty} a_n x^n$ is continuous at $x = R$. If the series converges at $x = -R$ to a value L', then $\lim_{x\to -R^+} \sum_{n=0}^{\infty} a_n x^n = L'$, hence the power series function $\sum_{n=0}^{\infty} a_n x^n$ is continuous at $x = -R$.

Proof. By Theorem 18, if $\sum_{n=0}^{\infty} a_n x^n$ is convergent at R, then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, R]$. Therefore, by Theorem 14 Chapter 7, $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $[0, R]$ because for each integer $n \geq 1$, the n-th partial sum $s_n(x) = \sum_{k=0}^{n-1} a_k x^k$ is a continuous polynomial function and (s_n) converges uniformly on $[0, R]$. Thus $\lim_{x\to R^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n = L$. Similarly if $\sum_{n=0}^{\infty} a_n x^n$ is convergent at $-R$, then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-R, 0]$ by Theorem 18. Again by Theorem 14 of Chapter 7, the consequence of uniform convergence is continuity at $x = -R$. The conclusion about the right limit at $-R$ then follows.

Example 20.
We shall illustrate the technique of using Abel's formula in the proof of Theorem 18 to deduce continuity at an end point of the interval of convergence.
Suppose the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 1 and $\sum_{n=0}^{\infty} a_n = 0$. Then $\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = 0$.
Proof.
For each integer $n \geq 0$,

$$\sum_{k=0}^{n} a_k x^k = a_0 + \sum_{k=1}^{n}(s_k - s_{k-1})x^k = s_0 + \sum_{k=1}^{n}(s_k - s_{k-1})x^k, \quad \text{------------} \quad (1)$$

where $s_n = \sum_{k=0}^{n} a_k$.
Then following (1) for all x in $[0, 1]$,

$$\begin{aligned}\sum_{k=0}^{n} a_k x^k &= s_0 + \sum_{k=1}^{n} s_k x^k - \sum_{k=1}^{n} s_{k-1} x^k = \sum_{k=1}^{n} s_k x^k - \sum_{k=0}^{n-1} s_k x^{k+1} + s_0 \\ &= \sum_{k=1}^{n} s_k x^k - x\sum_{k=0}^{n-1} s_k x^k + s_0 = (1-x)\sum_{k=1}^{n-1} s_k x^k + s_n x^n - s_0 x + s_0 \\ &= (1-x)\sum_{k=0}^{n-1} s_k x^k + s_n x^n.\end{aligned}$$

Suppose N is a positive integer. Then for $n > N+1$, for all x in $[0, 1]$,

$$\sum_{k=0}^{n} a_k x^k = (1-x)\sum_{k=0}^{N} s_k x^k + (1-x)\sum_{k=N+1}^{n-1} s_k x^k + s_n x^n.$$

Thus, for $n > N+1$, for all x in $[0, 1]$,

$$\left|\sum_{k=0}^{n} a_k x^k\right| \leq (1-x)\sum_{k=0}^{N} |s_k| x^k + (1-x)\sum_{k=N+1}^{n-1} |s_k| x^k + |s_n| x^n \qquad \text{by triangle inequality}$$

$$\leq (1-x)\sum_{k=0}^{N} |s_k| x^k + (1-x)\sum_{k=N+1}^{n-1} |s_k| x^k + |s_n|, \qquad \text{----------------} \quad (2)$$

since $0 \leq x \leq 1$.
Note that $s_n \to 0$, $|s_n| \to 0$. For each integer $n \geq 0$, let

$M_n = \sup \{|s_n|, |s_{n+1}|, \dots\} = \sup \{|s_j| : j \text{ is an integer and } j \geq n\}$.

Then $M_n \geq 0$ for all positive integer n and $\lim_{n\to\infty} M_n = \limsup_{n\to\infty} |s_n| = 0$ since $\lim_{n\to\infty} |s_n| = 0$.
So if $n \geq N+1$, $|s_n| \leq M_N$. It then follows from (2) that for $n > N+1$ and for all x in $[0, 1]$,

$$\begin{aligned}\left|\sum_{k=0}^{n} a_k x^k\right| &\leq (1-x)\sum_{k=0}^{N} |s_k| x^k + (1-x)\sum_{k=N+1}^{n-1} M_N x^k + M_N \\ &\leq (1-x)\sum_{k=0}^{N} |s_k| x^k + M_N + M_N\end{aligned}$$

since $(1-x)\sum_{k=N+1}^{n-1} x^k \leq 1$

$$\le (1-x)\sum_{k=0}^{N}|s_k| + 2M_N. \qquad \text{------------------------} \quad (3)$$

Since $M_n \to 0$, there exists a positive integer L such that for all *integer n*,

$$n \ge L \Rightarrow M_n < \frac{\varepsilon}{4}.$$

Thus, from (3), for all $n \ge L+2$ and all x in [0,1],

$$\left|\sum_{k=0}^{n} a_k x^k\right| \le (1-x)\sum_{k=0}^{L}|s_k| + 2M_L \,.$$

Therefore, $\left|\sum_{k=0}^{\infty} a_k x^k\right| \le (1-x)\sum_{k=0}^{L}|s_k| + 2M_L < (1-x)\sum_{k=0}^{L}|s_k| + \frac{\varepsilon}{2}.$

If we let $\delta = \dfrac{\varepsilon/2}{1+\sum_{k=0}^{L}|s_k|} > 0$, then we have

$$1-\delta < x < 1 \Rightarrow \left|\sum_{k=0}^{\infty} a_k x^k\right| < (1-x)\sum_{k=0}^{L}|s_k| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This means $\lim\limits_{x\to 1^-}\sum_{n=0}^{\infty} a_n x^n = 0 = \sum_{n=0}^{\infty} a_n$. Hence $\sum_{n=0}^{\infty} a_n x^n$ is continuous at $x = 1$.

Example 21.

ln (1+ *x*) has the following power series expansion for $-1 < x \le 1$.

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \quad \text{for } -1 < x \le 1.$$

We shall start with the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{for } |x| < 1.$$

This is a power series expansion for $\frac{1}{1+x}$. The radius of convergence plainly is 1.

Take any real number K such that $0 < K < 1$. Then $\sum_{n=0}^{\infty}(-1)^n x^n$ converges uniformly on $[-K, K]$ by Theorem 11. It follows from Theorem 7 that we can integrate the function term by term in $[-K, K]$. Thus

$$\int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} \text{ for all } x \text{ in } [-K, K].$$

But the left hand side is *ln*(1+*x*). Hence for any real number K such that $0 < K < 1$,

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} \text{ for all } x \text{ in } [-K, K].$$

Therefore, $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ for all x in $(-1, 1)$. Now for $x = 1$, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent by Leibniz's Alternating series test. Therefore, by Abel's Theorem (Corollary 19), $\lim\limits_{x\to 1^-}\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = \lim\limits_{x\to 1^-}\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. By the continuity of ln(1+*x*) at $x = 1$, we then have $\ln(2) = \ln(1+1) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. Thus $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ for $-1 < x \le 1$.

Exercises 22.

1. Determine whether each of the following sequences (of functions) converge uniformly on the given domain.

   (i) $\frac{\sin(nx)}{n}$ on [0, 1] ; (ii) $\frac{1}{3n-x}$ on [0, 1] ; (iii) $\frac{1}{nx+2}$ on [0, 1] l

   (iv) $(x-\frac{1}{n})^2$ on [0, 1] ; (v) $x - x^n$ on [0, 1] ; (vi) $\frac{2n+x}{n+3}$ on $[a, b]$, $a < b$.

2. Use the Weierstrass M-Test to prove that each of the following series is uniformly convergent on the given domain.

   (i) $.\sum_{n=1}^{\infty}\frac{\sin(nx)}{2^n}$ on **R ;** (ii) $.\sum_{n=1}^{\infty}\frac{x^2+n}{x^2+n^4}$ on $[-a, a]$, $a > 0$;

   (iii) $.\sum_{n=1}^{\infty}(n+1)x^n$ on $[-a, a]$, $0 < a < 1$; (iv) $\sum_{n=1}^{\infty}\frac{x^n(1-x)}{n}$ on [0, 1]

   (Hint: Find maximum value of $x^n(1-x)$ in [0, 1]) .

3. Let $f(x)=\sum_{n=1}^{\infty}\frac{x^{n/2}}{n(n!)^2.}$. Discuss how you might prove that f is continuous on [0, 1].

4. (Realizing function as a power series.)

   (i) Prove that $\frac{1}{1+x}=1-x+x^2+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$.

   Discuss how you might prove that

   (ii) $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}$ for $|x| < 1$,

   and (iii) $\frac{-1}{(1+x)^2}=-1+2x-3x^2+\cdots=\sum_{n=1}^{\infty}(-1)^n nx^{n-1}$ for $|x| < 1$.

5. Use the power series expansion of $\frac{1}{1-x}=1+x+x^2+\cdots=\sum_{n=0}^{\infty}x^n$ for $|x| < 1$ to prove that

   (a) $.\sum_{n=1}^{\infty}nx^n=\frac{x}{(1-x)^2}$ if $|x| < 1$;

   (b) $.\sum_{n=0}^{\infty}\frac{n}{n+1}x^n=\begin{cases}\dfrac{x+(1-x)\ln(1-x)}{(1-x)x} & \text{if } 0<|x|<1\\ 0, & \text{if x = 0}\end{cases}$;

   (c) $.\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}=\begin{cases}(\frac{1}{x}-1)\ln(1-x)+1 & \text{if } 0<|x|<1\\ 0, & \text{if x = 0}\end{cases}$.

   Give reasons for the steps you take.

   (This question is an example of power series manipulation.)

6. (Optional) Determine the radius of convergence of the Bessel function of the first kind of order zero $J_0(x)$ given by $J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$. Write out the first 4 terms of $J_0(x)$.

   Show that $J_0(x)$ satisfies the differential equation $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$

   (Bessel's differential equation of order zero).

7. Find all those x for which the following series converge.

(i) $\sum_{n=1}^{\infty}\frac{n^2(n+2)}{(n+5)3^n}x^n$; (ii) $\sum_{n=1}^{\infty}\frac{3^{\sqrt{n}}}{n}x^n$; (iii) $\sum_{n=1}^{\infty}\frac{2^n+3^n}{n^2}(2x+1)^n$.

(Hint: Use ratio test.)

8. Use trigonometric formula to prove that $4\sin^3(x) = 3\sin(x) - \sin(3x)$. Use this and the power series expansion for $\sin(x)$ to show that

(i) $\sin^3(x)=\frac{3}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{2n}-1}{(2n+1)!}x^{2n+1}$ for all real x;

(ii) Use partial fraction and obvious series expansion of the resulting rational functions, or otherwise, show that $\frac{x}{1+x-2x^2}=\frac{1}{3}\sum_{n=1}^{\infty}[1-(-2)^n]x^n$ for $|x| < ½$.

9. Assuming that $y'' + y = 0$, $y(0) = 0$, $y'(0) =1$ has a solution given by a power series. Find the power series and determine its radius of convergence.
(Hint: Use the three conditions to obtain relation among the coefficients of the power series and solving the relation.)

10. Find the radius of convergence of $y(x)=a_0(1-x^2)-a_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)(2n-1)}$, where a_0 and a_1 are arbitrary real numbers.
Show that $y(x)$ satisfies the differential equation $(1-x^2)y'' = -2y$ on its interval of convergence.

11. Show that $f_n(x) = (1 - x^2)x^n$ converges uniformly on $[-1, 1]$ and find its limiting function g. Hence conclude that $\int_0^1 f_n(x)dx \to 0$.

12. Explain what results you would use to show that $\sum_{n=1}^{\infty}\frac{e^{-nx^2}}{n^2}$ is continuous on **R**.

13. Show that any power series is the Taylor's series of its sum.

14. For the following functions determine their Taylor series centred at the points indicated and determine the radius of convergence in each case.
(a) $\sin^{-1}(x)$, 0 (b) $\cos(x)$, $\pi / 2$ (c) $\tan^{-1}(x)$, 0 (d) $\cosh(x)$, 0
(e) $\ln\left(\frac{1+x}{1-x}\right)$, 0 (f) $\tan^{-1}(x)$, 0 (g) $\ln(1+x)$, 0 (h) a^x , -1 ($a > 0$)
(i) $\int_0^x e^{-t^2}dt$, 0 (j) $\int_0^x\frac{\sin^2(x)}{t^2}dt$, 0 [Hint: $\cos(2x) = 1 - 2\sin^2(x)$.]

15. Prove that for $|x| < 1/2$.
$$\frac{9x}{(1+2x)(1-x)^2}=\sum_{n=1}^{\infty}\{3n+2+(-1)^{n+1}2^{n+1}\}x^n .$$

16. Prove that

(i) $\ln((1+x)^{(1+x)})+\ln((1-x)^{(1-x)})=x^2+\frac{x^4}{2\cdot 3}+\frac{x^6}{3\cdot 5}+\frac{x^8}{4\cdot 7}+\cdots$ for $|x| < 1$,

(ii) $2\ln(x)-\ln(x+1)-\ln(x-1)=\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\cdots$ for $|x| > 1$,

(iii) $\frac{1}{2}\ln(x) = \frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots$ for $x > 0$.

17. For each $\beta > 1$, prove that the series $\sum_{n=1}^{\infty} \sin\left(\left(\frac{x}{n}\right)^{\beta}\right)$ converges pointwise on the interval $[0, \infty)$, to a continuous function, but the convergence is not uniform on $[0, \infty)$, (Hint for non-uniform convergence: use the inequality for any $x \geq 0$, $\sin(x) \geq x - x^3/6$.)

18. For each β such that $0 \leq \beta \leq 1$, $\sum_{n=1}^{\infty} \sin\left(\left(\frac{x}{n}\right)^{\beta}\right)$ diverges for all $x > 0$. Show that whenever x^{β} is defined and not zero, $\sum_{n=1}^{\infty} \sin\left(\left(\frac{x}{n}\right)^{\beta}\right)$ is divergent. (Hint: use the hint for question 17.)

19. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x}{\sqrt{n}}\right)$ converges uniformly to a differentiable function on $[-K, K]$ for any constant $K > 0$. Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x}{\sqrt{n}}\right)$ converges pointwise to a differentiable function f on **R** such that

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, n} \cos\left(\frac{x}{\sqrt{n}}\right).$$

However, prove that $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{x}{\sqrt{n}}\right)$ is not uniformly convergent on **R**. (Hint: see the hint for question 17.)