Denjoy Saks Young Theorem for Arbitrary Function

By

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This article is the third of a series of articles towards the proof of the Denjoy Saks Young Theorem. The first two articles are “Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem” and “Functions of Bounded Variation and de La Vallée Poussin's Theorem”. The definition of Dini derivates is given in “Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem” and we shall use the notation given there. The total variation function of a function of bounded variation is defined in “Functions of Bounded Variation and de La Vallée Poussin's Theorem” and we shall adopt the notation therein.

Suppose $A$ is an arbitrary subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ is a finite valued function defined on $A$. We shall begin with the case when the right upper Dini derivate of $f$ is less than $\infty$.

Let $B = \{ x \in A : \quad \text{D}^+ f(x) < \infty \}$. For simplicity, we may assume that every point of $A$ is a two-sided limit point of $A$, for non-limit points or only one-sided limit points constitute at most a denumerable set.

We assume that $m^*(B) > 0$.

The function $f$ need not be bounded. We shall limit the domain to where $f$ is bounded and pass to the whole space by extending the domain. To begin this approach, take a strictly increasing positive sequence of real numbers, $(M_n)$, such that $M_n \nearrow \infty$. For each positive integer $n$, let $E_n = \{ x \in A : |f(x)| < M_n \}$.

Then plainly, $E_n \subseteq E_{n+1}$ and $A = \bigcup_{n=1}^{\infty} E_n$. Thus, $m^*(E_n) \to m^*(A)$, by the continuity from below property of Lebesgue outer measure. For each integer $k \geq 1$, let

$$E_{n,k} = \left\{ x \in B : \frac{f(x+h) - f(x)}{h} < M_n, x+h \in A \cap \left( x, x + \frac{1}{k} \right) \right\}$$

$$= \left\{ x \in B : \frac{f(x+h) - f(x)}{h} < M_n, 0 < h < \frac{1}{k}, x+h \in A \right\}$$
= \left\{ x \in B : \frac{f(t) - f(x)}{t - x} < M_n, t \in A \cap \left( x, x + \frac{1}{k} \right) \right\}.

Since \( M_n < M_{n+1} \), \( E_{n,k} \subseteq E_{n+1,k} \) and \( E_{n,k} \subseteq E_{n,k+1} \). Thus,

\[ E_{n,k} \subseteq E_{n,k+1} \subseteq E_{n+1,k+1} \subseteq \cdots \quad \text{------------------ (*)} \]

We claim that \( B = \bigcup_{n,k} E_{n,k} \).

Suppose \( x \in B \). Then \( D^* f(x) < M_n \) for some integer \( n > 0 \). That is to say,

\[ \limsup_{t \to x^+, t \in A} \frac{f(t) - f(x)}{t - x} = \limsup_{h \to 0^+} \frac{f(x + h) - f(x)}{h} : x + h \in A < M_n. \]

This implies that there exists \( \delta_x > 0 \) such that

\[ \sup \left\{ \frac{f(x + h) - f(x)}{h} : 0 < h < \delta_x, x + h \in A \right\} < M_n. \]

Therefore, \( \frac{f(x + h) - f(x)}{h} < M_n \), for all \( x + h \in A \) and \( 0 < h < \delta_x \). Let \( k \) be a positive integer such that \( \frac{1}{k} < \delta_x \). Then we have

\[ \frac{f(x + h) - f(x)}{h} < M_n, \text{ for all } x + h \in A \text{ and } 0 < h < \frac{1}{k}. \]

Consequently, \( x \in E_{n,k} \). Hence, we conclude that \( B \subseteq \bigcup_{n,k} E_{n,k} \).

Conversely, suppose \( x \in E_{n,k} \) for some positive \( n \) and \( k \). Then

\[ \frac{f(x + h) - f(x)}{h} < M_n, \text{ for all } x + h \in A \text{ and } 0 < h < \frac{1}{k}. \]

Hence,

\[ \sup \left\{ \frac{f(x + h) - f(x)}{h} : 0 < h < \frac{1}{k}, x + h \in A \right\} \leq M_n \]

and so,
\[
\limsup_{k \to \infty} \left\{ \frac{f(x+h) - f(x)}{h} : 0 < h < \frac{1}{k}, x + h \in A \right\} \leq M_a.
\]

Therefore, \( A^+ f(x) \leq M_a < \infty \) and so \( x \in B \). Thus, we can conclude that
\[
B = \bigcup_{n,k} E_{n,k}.
\]

Observe that \( \bigcup_{n,k} E_{n,k} = \bigcup_{n} E_{n,n} \). This is because if \( k < n \), then \( E_{n,k} \subseteq E_{n,n} \) and if \( k > n \), then \( E_{n,k} \subseteq E_{k,k} \). Thus, \( \bigcup_{n,k} E_{n,k} \subseteq \bigcup_{n} E_{n,n} \) and so equality follows.

Thus, by the continuity from below property of Lebesgue outer measure,
\[
m^*(B) = \lim_{n \to \infty} m^*(E_{n,n}).
\]

Note that \( B = B \cap A = B \left( \bigcap_{n=1}^{\infty} E_n \right) = \bigcup_{n=1}^{\infty} (B \cap E_n) \).

We observe that \( B = \bigcup_{n=1}^{\infty} (E_{n,n} \cap E_n) \). We deduce this as follows.

Take any \( x \) in \( B \). Then \( x \in E_{k,k} \) for some integer \( k \) and \( x \in E_L \). If \( L \leq k \), then \( x \in E_k \), so that \( x \in E_{k,k} \cap E_k \) and if \( L > k \), then \( x \in E_{L,L} \), so that \( x \in E_{L,L} \cap E_L \). It follows that \( B \subseteq \bigcup_{n=1}^{\infty} (E_{n,n} \cap E_n) \). Consequently, \( B = \bigcup_{n=1}^{\infty} (E_{n,n} \cap E_n) \).

Note that \( E_{n,n} \cap E_n \subseteq E_{n+1,n+1} \cap E_{n+1} \subseteq \cdots \) and so we have also that
\[
m^*(B) = \lim_{n \to \infty} m^*(E_{n,n} \cap E_n).
\]

Suppose that \( m^*(B) < \infty \) or that the set \( B \) is bounded. Hence, for an \( \varepsilon > 0 \) such that \( m^*(B) - \varepsilon > 0 \) there exists a positive integer \( N \) such that
\[
n \geq N \Rightarrow 0 < m^*(B) - \varepsilon < m^*(E_{n,n} \cap E_n) \leq m^*(B).\]

Thus, taking \( n = N \), we have that
\[
0 < m^*(B) - \varepsilon < m^*(E_{N,N} \cap E_N) \leq m^*(B).
\]

Suppose \( m^*(B) = \infty \), then for any \( K > 0 \), there exists integer \( N \) such that
\[
n \geq N \Rightarrow m^*(E_{n,n} \cap E_n) \geq K.
\]
Let \( A_i = E_{N,N} \cap E_N \). Then \( A_i \subseteq B \). Moreover, for \( x \in A_i \), we have for any \( 0 < \delta < \frac{1}{N} \),

\[
|f(x)| < M_N, \quad \frac{f(x+h) - f(x)}{h} < M_N \quad \text{for } 0 < h < \delta \text{ and } x+h \in A. \quad \text{--------- (1)}
\]

We have the following technical lemma.

**Lemma 1.** The function \( f \) is locally of bounded variation on \( A_i \). That is to say, for any \( c < d \), with \( c, d \) in \( A_i \) and \( d - c < \frac{1}{N} \). The restriction of \( f \) on \( A_i \cap [c,d] \) is of bounded variation on \( A_i \cap [c,d] \). More generally, for any positive integer, \( n \), \( f \) is locally of bounded variation on \( E_{n,n} \cap E_n \).

**Proof.**

Take any \( x \in A_i \cap [c,d] \) with \( c < x \leq d \) and \( d - c < \frac{1}{N} \). For any partition

\[
Q : c = x_0 < x_1 < \cdots < x_n = x \quad \text{by points in } A_i \cap [c,d],
\]

The positive variation with respect to the partition \( Q \), by (1), satisfies

\[
P(Q) = \sum_{i=1}^{n} \max \left( f(x_i) - f(x_{i-1}), 0 \right) \leq \sum_{i=1}^{n} (x_i - x_{i-1})M_N = (x - x_0)M_N < \frac{1}{N}M_N.
\]

The negative variation with respect to the partition \( Q \) is

\[
N(Q) = \sum_{i=1}^{n} \min \left( f(x_i) - f(x_{i-1}), 0 \right)
\]

\[
\geq -2 \sum_{i=1}^{n} (x_i - x_{i-1})M_N = -2(x-c)M_N.
\]

Thus, the negative variation is bounded below. It follows that the negative variation function of the restriction of \( f \) to \( A_i \cap [c,d] \),

\[
n(x) = \inf \{ N(Q) : Q \text{ any partition of } [c,x] \text{ by points in } A_i \cap [c,d] \} \geq -2(x-c)M_N.
\]

Therefore, with \( c \) as the anchor point, the positive variation function, \( p(x) \) of the restriction of \( f \) to \( A_i \cap [c,d] \) satisfies that \( p(x) \leq \delta M \) with \( 0 < \delta < \frac{1}{N} \) and \( n(x) \geq -2(d-c)M_N \). (For the definition of positive variation, negative
variation and total variation, please refer to the definition in the proof of Theorem 6 in *Functions of Bounded Variation and de La Vallée Poussin’s Theorem.* Hence, the total variation function,

\[ v_f(x) = p(x) - n(x) \leq \delta M_N + 2(d - c)M_N < 3\delta M_N < \frac{3M_N}{N}. \]

Thus, the restriction of \( f \) to \( A_i \cap [c,d] \) is of bounded variation on \( A_i \cap [c,d] \).

The proof for \( E_{n,n} \cap E_n \) is exactly the same.

**Remark.** In Lemma 1, we need not choose the closed interval to have end points in \( A \). However, we assume that \( A_i \cap [c,d] \neq \emptyset \). We may just pick any point, \( a \), in \( A_i \cap [c,d] \) as anchor point for the definition of total variation function of \( f \).

Now for \( x > a \) with \( x \in A_i \cap [c,d] \), \( v_{f,a}(x) = p(x) - n(x) \leq (x - a)M_N + 2(x - a)M_N \) and for \( y < a \) with \( y \in A_i \cap [c,d] \), \( |v_{f,a}(y)| = |v_{f,a}(a)| \leq (a - y)M_N + 2(a - y)M_N \). Hence, the total absolute variation for \( f \) on \((x,y) \cap A_i \) is less than

\[ (x - y)M_N + 2(x - y)M_N < 3\delta M_N < \frac{3M_N}{N}. \]

It follows that \( f \) is of bounded variation on \( A_i \cap [c,d] \) with bounds \( 3M_N \).

It is customary to proceed from finite Dini derivates to possibly infinite ones. So, our next result will be on set with finite Dini derivates. We shall make similar definitions of \( E_{n,k} \) and \( E_n \) with \( B \) replaced by \( \{x \in A : -\infty < \_A D^+ f(x) < \infty\} \) and \( E_{n,k} = \{x \in B : -M_n < \frac{f(x+h)-f(x)}{h} < M_n, x+h \in A \cap (x, x+\frac{1}{k})\} \). Analogous properties that have been stated for \( E_{n,k} \) and \( E_n \) hold true and that Lemma 1 holds when \( B = \{x \in A : -\infty < \_A D^+ f(x) < \infty\} \) with the corresponding set \( E_{n,n} \cap E_n \). Note that the definition of \( E_n \) remains unchanged, \( E_n = \{x \in A : |f(x)| < M_n\} \).

We verify these facts below:

Suppose \( B = \{x \in A : -\infty < \_A D^+ f(x) < \infty\} \).
Let \((M_n)\) be a strictly increasing positive sequence of real numbers, \((M_n)\), such that \(M_n \nrightarrow \infty\). As before let \(E_n = \{x \in A: |f(x)| < M_n\}\). We shall use the same notation as before. Define

\[
E_{n,k} = \left\{ x \in B: -M_n < \frac{f(x+h) - f(x)}{h} < M_n, x + h \in A \cap \left( x, x + \frac{1}{k} \right) \right\}
\]

\[
= \left\{ x \in B: -M_n < \frac{f(x+h) - f(x)}{h} < M_n, 0 < h < \frac{1}{k}, x + h \in A \right\}
\]

\[
= \left\{ x \in B: -M_n < \frac{f(t) - f(x)}{t-x} < M_n, t \in A \cap \left( x, x + \frac{1}{k} \right) \right\}.
\]

Since \(M_n < M_{n+1}\), \(E_{n,k} \subseteq E_{n+1,k}\) and \(E_{n,k} \subseteq E_{n,k+1}\). It follows that \(E_{n,n} \subseteq E_{n+1,n+1} \subseteq \cdots\).

We can show as before that \(B = \bigcup_{n=1}^{\infty} E_{n,n}\).

Suppose \(x \in B\). Then \(-M_n < \int_A^x f(x) < M_n\) for some integer \(n > 0\). That is to say,

\[-M_n < \limsup_{t \to x^+, x \in A} \frac{f(t) - f(x)}{t-x} = \limsup_{h \to 0^+} \left\{ \frac{f(x+h) - f(x)}{h} : x + h \in A \right\} < M_n.\]

This implies that \(\sup_{\delta, h > 0} \left\{ \frac{f(x+h) - f(x)}{h} : x + h \in A \right\} > -M_n\) and for any \(\delta > 0\) there exists \(\delta_x > 0\) such that

\[\sup_{\delta, h > 0} \left\{ \frac{f(x+h) - f(x)}{h} : 0 < h < \delta_x, x + h \in A \right\} < M_n.\]

Therefore, \(-M_n < \frac{f(x+h) - f(x)}{h} < M_n\), for all \(x + h \in A\) and \(0 < h < \delta_x\). Let \(k\) be a positive integer such that \(\frac{1}{k} < \delta_x\). Then we have

\[-M_n < \frac{f(x+h) - f(x)}{h} < M_n, \text{ for all } x + h \in A \text{ and } 0 < h < \frac{1}{k}.\]

Consequently, \(x \in E_{n,k}\). Hence, we conclude that \(B \subseteq \bigcup_{n,k} E_{n,k}\).

Conversely, suppose \(x \in E_{n,k}\) for some positive \(n\) and \(k\). Then
\[-M_n < \frac{f(x+h) - f(x)}{h} < M_n, \text{ for all } x + h \in A \text{ and } 0 < h < \frac{1}{k}.\]

Hence,
\[-M_n < \sup \left\{ \frac{f(x+h) - f(x)}{h} < M_n : 0 < h < \frac{1}{k}, x + h \in A \right\} \leq M_n\]

and so,
\[-M_n \leq \limsup_{k \to \infty} \left\{ \frac{f(x+h) - f(x)}{h} < M_n : 0 < h < \frac{1}{k}, x + h \in A \right\} \leq M_n.\]

Therefore, \(-\infty < A^+ D^+ f(x) < \infty\) and so \(x \in B\). Thus, we can conclude that
\[B = \bigcup_{n,k}^{\infty} E_{n,k} = \bigcup_{n=1}^{\infty} E_{n,n}.\]

Similarly, we deduce that
\[B = \bigcup_{n=1}^{\infty} (E_{n,n} \cap E_n).\]

We make the tacit understanding that when \(B = \{x \in A : -\infty < A^+ D^+ f(x) < \infty\}\),
\[E_{n,k} = \left\{ x \in B : -M_n < \frac{f(x+h) - f(x)}{h} < M_n, x + h \in A \cap \left( x, x + \frac{1}{k} \right) \right\}\]
and when
\[B = \{x \in A : A^+ D^+ f(x) < \infty\}, \ E_{n,k} = \left\{ x \in B : \frac{f(x+h) - f(x)}{h} < M_n, x + h \in A \cap \left( x, x + \frac{1}{k} \right) \right\}.\]

Moreover, inequality (1) still holds, i.e., for \(x \in E_{n,n}\) and for any \(0 < \delta < \frac{1}{n}\), we have
\[|f(x)| < M_n, \ \frac{f(x+h) - f(x)}{h} < M_n \text{ for } 0 < h < \delta \text{ and } x + h \in A. \quad \text{--------- (1)*}\]

We have that \(f\) is locally of bounded variation on \(E_{n,n} \cap E_n\). This is a consequence of Lemma 1.

**Lemma 2.** Suppose \(f : A \to \mathbb{R}\) is a finite-valued function. Let
\[B = \{x \in A : -\infty < A^+ D^+ f(x) < \infty\}, \ E_{n,n}\} \text{ and } \{E_n\}\} \text{ be defined as above. For a fixed
positive integer $N$, let $A_i = E_{N,N} \cap E_N$. Then there exists a set, $\widetilde{N} \subseteq A_i$ such that $m^*(\widetilde{N}) = m^*(f(\widetilde{N})) = 0$ and for all $x \in A_i - \widetilde{N}$, $\mathcal{A}D^+ f(x) = \mathcal{A}Df(x) = \mathcal{A}D_f(x)$ and is finite. Moreover, $f$ is a Lusin function on $A_i = E_{N,N} \cap E_N$.

**Proof.**

By Lemma 1, for any $c < d$, with $c, d$ in $A_i$ and $d - c < \frac{1}{N}$, $f$ is of bounded variations on $A_i \cap [c, d]$. Therefore, by Theorem 15 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, there is a subset $N_i \subseteq A_i \cap [c, d]$ of $A$ such that $m(f(N_i)) = m(N_i) = 0$,

and for each $x \in A_i \cap [c, d] - N_i$, $\mathcal{A}Df(x)$ exist and is finite since $\mathcal{A}D^+ f(x)$ is finite. We can cover $A_i$ by countable number of such closed interval of length strictly less than $\frac{1}{N}$. In the proof of Lemma 1, we need not use precisely the closed interval $[c, d]$ with $c, d$ in $A_i$ because the total absolute variation is independent of the anchor point for the definition of the total variation. (See remark after Lemma 1.) For instance, if $a$ is the anchor point and $x < a < y$ with $x, y \in A_i \cap [c, d]$, the total absolute variation function is given by $|\nu_{f,a}(x) + \nu_{f,a}(y)|$ and so it is bounded and so the function is of bounded variation on $A_i \cap [c, d]$.

Hence, we can conclude that there is a subset $N_2 \subseteq A_i$ such that $m(f(N_2)) = m(N_2) = 0$ and that $\mathcal{A}Df(x)$ exist and is finite for all $x$ in $A_i - N_2$.

Moreover, $f$ is a Lusin function on $A_i$. This is because by Theorem 18, *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, $f$ is a Lusin function on $A_i - N_2$ and as $m(f(N_2)) = 0$, $f$ is a Lusin function on $A_i$.

Let $A_2 \subseteq A_i$ be the points of density of $A_i$ at which $\mathcal{A}Df(x)$ exist and is finite.

\[ \lim_{m(I) \to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A)}{m(I)} = 1. \] \quad (1)

Now \( x \in A_2 \) implies that \( \mathcal{A} \) \( Df(x) \) is finite. Suppose \( x \in A_2 \) and \( \mathcal{A} D^+ f(x) > \mathcal{A} Df(x) + \eta \) for some \( \eta > 0 \). Then, by definition of \( \mathcal{A} D^+ f(x) \), there exists a sequence \((\xi_n)\) such that \( \xi_n > x \), \( \xi_n \to x \) and

\[ \frac{f(\xi_n) - f(x)}{\xi_n - x} > \mathcal{A} Df(x) + \eta. \] \quad (2)

We claim that for a fixed integer \( n \) in (2) and with \( \xi_n \) sufficiently close to \( x \), for any \( \xi \) in \((x, \xi_n) \cap A\),

\[ \frac{f(\xi) - f(x)}{\xi - x} < \mathcal{A} Df(x) + \eta', \] \quad (3)

for \( 0 < \eta' < \eta \).

**Proof of this claim.**

As \( x \in A_2 \) is a point of density of \( A \),

\[ \lim \sup_{m(I) \to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A)}{m(I)} = \lim \inf_{m(I) \to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A)}{m(I)} = 1. \]

Therefore, given any \( \tilde{\xi} > 0 \), there exists \( \tilde{\delta} > 0 \) such that

\[ 1 - \tilde{\xi} < \frac{m^*(I_{\delta} \cap A)}{m(I_{\delta})} \leq 1, \]

for all closed interval, \( I_{\delta} \) containing \( x \) with length, \( m(I_{\delta}) = \delta < \tilde{\delta} \). Take \( \tilde{\xi} \) to be sufficiently small so that we have,

\[ m^*(A_1 \cap I_{\delta}) > m(I_{\delta}) - \tilde{\xi} m(I_{\delta}) = (1 - \tilde{\xi}) m(I_{\delta}) > 0. \] \quad (4)

By definition of \( \mathcal{A} Df(x) \), for any \( 0 < \eta' < \eta \), there exists \( \zeta > 0 \) such that

\[ \frac{f(\xi) - f(x)}{\xi - x} < \mathcal{A} Df(x) + \eta', \] \quad (5)

for all \( \xi \in A_1 \cap (x, \zeta) \). Thus, if we choose \( \xi_n < \zeta \) and \( \xi_n - x < \tilde{\delta} \), then we get
\[
\frac{f(\xi) - f(x)}{\xi - x} < \lambda Df(x) + \eta',
\]
for all \( \xi \in A_i \cap (x, \xi_n) \). This proves our claim (3).

We note that we can choose \( \xi \in A_i \cap (x, \zeta) \) sufficiently close to \( \xi_n \) so that
\[
\epsilon = \frac{\xi_n - \xi}{\xi_n - x}
\]
can be made as small as we wish. We deduce this below.

In general, take \( \zeta \) closed to \( \xi_n \), with \( \zeta < \xi_n \) such that \( 1 - \epsilon < \frac{\zeta - x}{\xi_n - x} < 1 - \theta \), with \( \theta < \epsilon \) and \( \tilde{\zeta} < \theta \). If \( (\zeta, \xi_n) \cap A_i = \emptyset \) or \( m^*((\zeta, \xi_n) \cap A_i) = 0 \), then
\[
1 - \tilde{\zeta} < \frac{m^*((x, \xi_n) \cap A_i)}{m((x, \xi_n))} = \frac{m^*((x, \zeta) \cap A_i)}{m((x, \xi_n))} \leq \frac{\zeta - x}{\xi_n - x} < 1 - \theta
\]
and so \( \tilde{\zeta} > \theta \) and we have a contradiction. This means that \( (\zeta, \xi_n) \cap A_i \neq \emptyset \).

Thus, there exists a point \( \xi \in (\zeta, \xi_n) \cap A_i \). This means there exists a point \( \xi \in (\zeta, \xi_n) \cap A_i \) such that
\[
f(\xi) - f(x) < \lambda Df(x) + \eta'.
\]
Now we show that
\[
\frac{\xi_n - x}{\xi_n - \xi} > \frac{1}{\epsilon}.
\]

Observe that
\[
\frac{\xi_n - x}{\xi_n - \xi} > \frac{\xi_n - x}{\xi_n - \zeta}.
\]
We have
\[
\frac{\xi_n - \zeta}{\xi_n - x} = \frac{\xi_n - x - (\zeta - x)}{\xi_n - x} = \frac{\xi_n - x}{\xi_n - x} < 1 - (1 - \epsilon) = \epsilon.
\]
Hence,
\[
\frac{\xi_n - x}{\xi_n - \xi} > \frac{\xi_n - x}{\xi_n - \zeta} > \frac{1}{\epsilon}.
\]  \hspace{1cm} \text{------------------------ (6)}

Indeed, we have that \( \frac{\xi_n - x}{\xi_n - \xi} < \epsilon \).

From inequality (2) we get,
\[
f(\xi_n) - f(x) > (\xi_n - x) \lambda Df(x) + \eta(\xi_n - x) \hspace{1cm} \text{------------------------ (7)}
\]
From inequality (3), we get, for any \( \xi \in (x, \xi_n) \cap A \),
\[
f(\xi) - f(x) < (\xi - x) A Df(x) + \eta'(\xi - x). \quad \text{-------------- (8)}
\]
Hence, from (7) and (8) we obtain,
\[
f(\xi_n) - f(\xi) > (\xi_n - \xi) A Df(x) + \eta(\xi_n - x) - \eta'(\xi - x)
= (\xi_n - \xi) A Df(x) + (\xi_n - x) \left[ \eta - \eta' \left( \frac{\xi - x}{\xi_n - x} \right) \right].
\]
Therefore, for \( \xi \in (x, \xi_n) \cap A \),
\[
\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > A Df(x) + \left( \frac{\xi_n - x}{\xi_n - \xi} \right) \left[ \eta - \eta' \left( \frac{\xi - x}{\xi_n - x} \right) \right]. \quad \text{------------- (9)}
\]
Note that \( \xi_n \in A \) but \( \xi \in A \). It follows that for all \( \xi_n \) with \( |\xi_n - \xi| < \frac{1}{N} \),
\[
\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} < M_N. \quad \text{------------------------ (10)}
\]
From (9) we deduce that for our choice of \( \xi \in (x, \xi_n) \cap A \),
\[
\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > A Df(x) + \frac{1}{\ell} (\eta - \eta').
\]
Since \( \eta - \eta' > 0 \), we can choose \( \hat{x} > 0 \) as small as we like and we can choose \( \theta \) and \( \epsilon \) to be as smaller as we like so that
\[
\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > A Df(x) + \frac{1}{\ell} (\eta - \eta') > M_N.
\]
This contradicts inequality (10) and so at points of \( A_2 \), \( A \, D^+ f(x) \leq A \, Df(x) \). Since \( A \, D^+ f(x) \geq A \, Df(x) \), we conclude that \( A \, D^+ f(x) = A \, Df(x) \).

Now we consider \( A \, D^- f(x) \) at point \( x \in A_2 \).

Suppose \( x \in A_2 \) and \( A \, D^- f(x) < A \, Df(x) - \eta \) for some \( \eta > 0 \). Then, by definition of \( A \, D^- f(x) \), there exists a sequence \( (\xi_n) \) such that \( \xi_n < x \), \( \xi_n \to x \) and
\[ \frac{f(\xi_n) - f(x)}{\xi_n - x} \leq \alpha \, Df(x) - \eta. \] \text{------------------------ (11)}

By definition of \( \alpha \, Df(x) = \alpha \, Df(x) \), for any \( 0 < \eta' < \eta \), there exists \( \zeta > 0 \) such that
\[ \frac{f(\xi) - f(x)}{\xi - x} > \alpha \, Df(x) - \eta', \] \text{------------------------ (12)}

for all \( \xi \in A_1 \cap (\zeta, x) \).

We claim that for a fixed integer \( n \) in (2) and with \( \xi_n \) sufficiently close to \( x \), there exists a \( \xi \) in \( (\zeta, \xi_n) \cap A \) such that
\[ \frac{f(\xi) - f(x)}{\xi - x} > \alpha \, Df(x) - \eta', \] \text{------------------------ (13)}

for \( 0 < \eta' < \eta \).

**Proof of this claim.**

As \( x \in A_2 \) is a point of density of \( A_i \),
\[ \limsup_{m(I) \to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A_i)}{m(I)} = \liminf_{m(I) \to 0, I \text{ an interval containing } x} \frac{m^*(I \cap A_i)}{m(I)} = 1. \]

Therefore, given any \( \hat{\zeta} > 0 \), there exists \( \hat{\delta} > 0 \) such that
\[ 1 - \hat{\zeta} < \frac{m^*(I_{\hat{\delta}} \cap A_i)}{m(I_{\hat{\delta}})} \leq 1, \]

for all closed interval, \( I_{\hat{\delta}} \) containing \( x \) with length, \( m(I_{\hat{\delta}}) = \delta < \hat{\delta} \). Take \( 0 < \hat{\zeta} < \frac{1}{2} \)
to be sufficiently small so that we have,
\[ m^*(A_i \cap I_{\hat{\delta}}) > m(I_{\hat{\delta}}) - \hat{\zeta} m(I_{\hat{\delta}}) = \left(1 - \hat{\zeta}\right) m(I_{\hat{\delta}}) > 0. \] \text{------------------------ (14)}

Thus, if we take \( \zeta < x \) and \( x - \zeta < \hat{\delta} \), and choose \( \zeta < \xi_n < x \), then
\[ \frac{f(\xi) - f(x)}{\xi - x} > \alpha \, Df(x) - \eta', \]

for all \( \xi \in A_i \cap (\zeta, \xi_n) \).
We note that we can choose \( \xi \in A \cap (\zeta, x) \) sufficiently close to \( \xi_n \) so that
\[
\ell = \frac{\xi_n - \xi}{\xi_n - x}
\]
can be made as small as we wish. We deduce this below.

Choose \( n \) sufficiently large that \( x - \xi_n < \frac{1}{2} (x - \zeta) \). There must exist a point in
\( \xi \in (\zeta, \xi_n) \cap A \), i.e., \( (\zeta, \xi_n) \cap A \neq \emptyset \). If it were empty, then
\[
1 - \frac{\xi_n - \xi}{m((\xi_n, x) \cap A)} = \frac{m^*(((\xi_n, x) \cap A))}{x - \zeta} < \frac{x - \xi_n}{x - \zeta} < \frac{1}{2}.
\]
But \( 1 - \frac{\xi_n - \xi}{x - \zeta} > \frac{1}{2} \). Thus \( (\zeta, \xi_n) \cap A \neq \emptyset \). So, take a point \( \xi \in (\zeta, \xi_n) \cap A \).

In general, take \( \zeta \) closed to \( \xi_n \) such that \( 1 - \xi < \frac{x - \xi_n}{x - \zeta} < 1 - \theta \), with \( \theta < \epsilon \) and \( \xi < \theta \).

As shown above if \( (\zeta, \xi_n) \cap A = \emptyset \), then
\[
1 - \frac{\xi_n - \xi}{m((\xi_n, x) \cap A)} = \frac{m^*((\xi_n, x) \cap A)}{x - \zeta} < \frac{x - \xi_n}{x - \zeta} < 1 - \theta
\]
and so \( \xi > \theta \) and we have a contradiction. Note that as \( \xi \) gets smaller, we can choose \( \theta \) and \( \epsilon \) to get smaller, for instance that they may be chosen to be less than \( 2\xi \).

Note that \( \frac{x - \xi_n}{\xi_n - \xi} > \frac{x - \xi_n}{\xi_n - \zeta} \) and
\[
\frac{\xi_n - \zeta}{x - \xi_n} = \frac{x - \zeta}{x - \xi_n} - \frac{x - \zeta}{x - \xi_n} = \frac{x - \zeta}{x - \xi_n} - 1 < \frac{1}{1 - \epsilon} - 1 = \frac{\epsilon}{1 - \epsilon} < 2\epsilon \quad \text{if} \quad 0 < \epsilon < \frac{1}{2}.
\]
Thus, \( \frac{\xi_n - \zeta}{x - \xi_n} < \frac{\xi_n - \zeta}{x - \xi_n} < 2\epsilon \) and so \( \frac{x - \xi_n}{\xi_n - \zeta} > \frac{1}{2\epsilon} \).

Since \( \xi_n - x \) and \( \xi - x \) are negative, we have,
\[
f(\xi_n) - f(x) > (\xi_n - x) \cdot Df(x) - \eta(\xi_n - x)
\]
and \( f(\xi) - f(x) < (\xi - x) \cdot Df(x) - \eta'(\xi - x) \). Subtracting these two inequalities we get
\[
f(\xi_n) - f(\xi) > (\xi_n - \xi) \cdot Df(x) + (\xi_n - \xi) \left[ \eta - \eta' \left( \frac{x - \xi}{x - \xi_n} \right) \right] \quad \text{and so}
\]
\[
\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > \mathcal{A}Df(x) + \frac{x - \xi_n}{\xi_n - \xi} \left[ \eta - \eta' \left( \frac{x - \xi}{x - \xi_n} \right) \right].
\]

Observe that \((x - \xi_n) > 1\) and so \(\eta - \eta' \left( \frac{x - \xi}{x - \xi_n} \right) \geq \frac{x - \xi_n}{\xi_n - \xi} (\eta - \eta') > \frac{1}{2} (\eta - \eta').\)

Note that \(\eta - \eta' > 0\). We can choose \(\hat{\xi}\) to be arbitrary small along with arbitrary small \(\theta\) and \(\varepsilon\) with \(\hat{\xi} < \theta < \varepsilon\). It follows that there exist \(\xi \in A\) and \(\xi_n\) in \(A\) such that \(\xi < \xi_n\) such that \(\frac{f(\xi_n) - f(\xi)}{\xi_n - \xi} > M_N\) and we have a contradiction. Hence, we must have \(\mathcal{A}Df(x) \geq \mathcal{A}Df(x)\). As \(\mathcal{A}Df(x) \geq D_f(x)\), we have \(\mathcal{A}Df(x) = \mathcal{A}Df(x)\).

Suppose \(A_2 = (A_1 - N_2) - N_3\), where \(m(N_3) = 0\). If we now let \(\tilde{N} = N_2 \cup N_3\), then \(m^*(\tilde{N}) = m^*(f(\tilde{N})) = 0\) and for all \(x \in A_1 - \tilde{N}\), \(\mathcal{A}D^f(x) = \mathcal{A}Df(x)\) and is finite. This completes the proof of Lemma 2.

**Corollary 3.** Suppose \(f : A \to \mathbb{R}\) is a finite valued function. Let

\[
B = \{x \in A : \mathcal{A}D^f(x) < \infty\}, \quad \{E_{n,n}\} \text{ and } \{E_n\} \text{ be defined as above. For a fixed positive integer } N, \text{ let } A_i = E_{n,N} \cap E_N. \text{ Then there exists a set, } \tilde{N} \subseteq A_i \text{ such that}
\]

\[m^*(\tilde{N}) = m^*(f(\tilde{N})) = 0\] and for all \(x \in A_1 - \tilde{N}\), \(\mathcal{A}D^f(x) = \mathcal{A}Df(x)\) is finite or \(-\infty\).

Moreover, the set \(K_N = \{x \in A : \mathcal{A}D^f(x) = -\infty\}\) has measure zero and \(f\) is a Lusin function on \(A_1 - \tilde{N} - K_N\).

**Proof.**

The proof proceeds exactly as for the proof for Lemma 2. This time we have that except for a null set \(N_2\), \(\mathcal{A}Df(x)\) exists finitely or infinitely for \(x \in A_1 - N_2\) and \(m(N_2) = m(f(N_2)) = 0\). Since \(\mathcal{A}D^f(x) < \infty\), \(\mathcal{A}Df(x)\) can take on the value \(-\infty\), when \(\mathcal{A}D^f(x) = -\infty\).

In this case we have \(\mathcal{A}D^f(x) = \mathcal{A}Df(x) = -\infty\). By Theorem 9, Functions of Bounded Variation and de La Vallée Poussin's Theorem, the set \(K_N = \{x \in A_1 : \mathcal{A}Df(x) = -\infty\} = \{x \in A_1 : \mathcal{A}D^f(x) = -\infty\}\) has measure zero. Let
$A_2 \subseteq A_1 - N_2 - K_N$ be the points of density of $A_1$ at which $D_f(x)$ exist and is finite. Suppose $A_2 = (A_1 - N_2 - K_N) - N_3$, where $N_3 \subseteq A_1 - N_2 - K_N$ and $m(N_3) = 0$.

Let $\tilde{N} = N_2 \cup N_3$.

Since $f$ is a Lusin function on $A_1 - N_2 - K_N$, by Theorem 18, *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, $f(N_3) = 0$, as $m(N_3) = 0$. Hence, $m^*(\tilde{N}) = m^*\left(f(\tilde{N})\right) = 0$ and $D^+ f(x) = D^- f(x)$ finitely or infinitely.

**Theorem 4.** Suppose $f : A \rightarrow \mathbb{R}$ is a finite-valued function. Suppose $B$ is a subset of $A$ such that at each point $x$ of $B$, $D^+ f(x) < \infty$. Then, except for a null set, $\tilde{N} \subseteq B$, $D^+ f(x) = D^- f(x)$, its opposite derivate, and is either finite or $-\infty$. Moreover, $m^*(\tilde{N}) = m^*\left(f(\tilde{N})\right) = 0$. The set $K = \{x \in B : D^+ f(x) = -\infty\}$ is a null set and $f$ is a Lusin function on $B - K$.

**Proof.**

We shall use the notation used in Corollary 3.

We note that $B = \bigcup_{n=1}^{\infty} (E_{n,n} \cap E_n)$, $E_{n,n} \cap E_n \subseteq E_{n+1,n+1} \cap E_{n+1} \subseteq \cdots$ and $m^*(B) = \lim_{n \to \infty} m^*(E_{n,n} \cap E_n)$.

For each positive integer, $n$, let $A_n = E_{n,n} \cup E_n$. Then by Corollary 3, there exists a set, $\tilde{N}_n \subseteq A_n$ such that $m^*(\tilde{N}_n) = m^*\left(f(\tilde{N}_n)\right) = 0$ and for all $x \in A_n - \tilde{N}_n$,

$D^+ f(x) = D^- f(x)$ and is finite or $-\infty$. Let $\tilde{N} = \bigcup_{n=1}^{\infty} \tilde{N}_n$. Then $m(\tilde{N}) = 0$. Since $m^*\left(f(\tilde{N}_n)\right) = 0$, for each positive integer, $n$, $m^*\left(f(\tilde{N})\right) = m^*\left(f\left(\bigcup_{n=1}^{\infty} \tilde{N}_n\right)\right) = 0$. Let $B^* = \bigcup_n A_n - \bigcup_n \tilde{N}_n = B - \tilde{N}$. Take any $x \in B^*$. Then $x \in A_k - \tilde{N}_k$ and so $D^+ f(x) = D^- f(x)$ and is finite or $-\infty$. Observe that

$m^*(B^*) = m^*(B) = \lim_{n \to \infty} m^*(A_n)$.

Note that for each positive integer, $n$, $\tilde{N}_n$ contains points, $x$, where $x$ is not a point of density of $A_n$ or $D_f(x)$ does not exist finitely or infinitely.
Note that if \( A D^+ f(x) = -\infty \), there exists an integer \( N \), such that \( x \in E_{n,n} \) for all \( n > N \). This is because if \( A D^+ f(x) = -\infty \), then

\[
\limsup_{h \to 0^+} \left\{ \frac{f(x+h) - f(x)}{h}, x+h \in A \right\} = -\infty
\]

implies that there exists \( \delta > 0 \) such that

\[
\sup_{0 < h < \delta} \left\{ \frac{f(x+h) - f(x)}{h}, x+h \in A, 0 < h < \delta \right\} < 0.
\]

It follows that \( \frac{f(x+h) - f(x)}{h} < 0 \) for \( x+h \in A, 0 < h < \delta \). Thus, \( x \in E_{n,n} \) for all \( n \) such that \( \frac{1}{n} < \delta \). Take any integer \( N \) with \( \frac{1}{N} < \delta \) and we have \( x \in E_{n,n} \) for all \( n > N \).

Let \( K = \{ x \in B : A D^+ f(x) = -\infty \} \). Then we claim that \( K = \bigcup K_n \), where \( K_n \) is as given in Corollary 3. Note that \( x \in K \Rightarrow x \in E_{n,n} \Rightarrow x \in K_n \) for some positive integer \( n \). Therefore, \( K \subseteq \bigcup K_n \). By definition, \( K_n \subseteq K \) for each positive integer \( n \). Hence, \( \bigcup K_n \subseteq K \) and so \( K = \bigcup K_n \). Therefore, \( m^*(K) = m^* \left( \bigcup K_n \right) = 0 \).

Now, \( f \) is a Lusin function on \( B - K - \tilde{N} \) by Theorem 5 below. Since \( m(f(\tilde{N})) = 0 \), \( f \) is a Lusin function on \( B - K \). This completes the proof of the theorem.

**Theorem 5.** Suppose \( f : A \to \mathbb{R} \) is a finite valued function. Suppose \( B \) is a subset of \( A \) such that at each point \( x \) of \( B \), \( A D^+ f(x) \) is finite, i.e.,

\( -\infty < A D^+ f(x) < \infty \). Then, except for a null set, \( \tilde{N} \subseteq B, A D^+ f(x) = A D_- f(x) \), its opposite derivate, and is finite. Moreover, \( m^*(\tilde{N}) = m^* \left( f(\tilde{N}) \right) = 0 \) and \( f \) is a Lusin function on \( B \).

**Proof.**

Now we can apply Lemma 1 and Lemma 2.

For each positive integer, \( n \), let \( A_n = E_{n,n} \cap E_n \). Then by Lemma 2, there exists a set, \( \tilde{N}_n \subseteq A_n \) such that \( m^*(\tilde{N}_n) = m^* \left( f(\tilde{N}_n) \right) = 0 \) and for all \( x \in A_n - \tilde{N}_n \),
$\mathcal{A}D^+ f(x) = \mathcal{A}D_- f(x)$ and is finite. Let $\widetilde{N} = \bigcup_{n=1}^{\infty} \widetilde{N}_n$. Then $m(\widetilde{N}) = 0$. Since

$$m^*(f(\widetilde{N}_n)) = 0,$$

for each positive integer, $n$, $m^*(f(\widetilde{N})) = m^*(\bigcup_{n=1}^{\infty} \widetilde{N}_n) = 0$. Let

$$B^* = \bigcup_{n} A_n - \bigcup_{n} \widetilde{N}_n = B - \widetilde{N}.$$  Take any $x \in B^*$. Then $x \in A_k - \widetilde{N}_k$ and so

$\mathcal{A}D^+ f(x) = \mathcal{A}D_- f(x)$ and is finite.

By Lemma 2, $f$ is a Lusin function on $A_n = E_{n,n} \cap E_n$. Suppose $E \subseteq B$ is a null set in $B$. Then $m^*(E \cap A_n) = 0$. Therefore, $m^*(f(E \cap A_n)) = 0$. As

$$m^*(f(E)) = m^*(f(E \cap \bigcup_{n} A_n)) \leq m^*(\bigcup_{n=1}^{\infty} f(E \cap A_n)),$$

it follows that $m^*(f(E)) = 0$.

Thus, $f$ is a Lusin function on $B$. This completes the proof of Theorem 5.

**Corollary 6.** Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose $B$ is a subset of $A$ such that at each point $x$ of $B$, $\mathcal{A}D^+ f(x) < \infty$. Let

$$K = \{x \in B : \mathcal{A}D^+ f(x) = -\infty\}.$$  Suppose $m(f(K)) = 0$. Then $f$ is a Lusin function on $B$.

**Proof.**

By Theorem 18, *Functions of Bounded Variation and de La Vallée Poussin's Theorem* and Theorem 4, $f$ is a Lusin function on $B - \widetilde{N} - K$, where $\widetilde{N}$ is given in Theorem 4. Since $m^*(f(\widetilde{N} \cup K)) = m^*(f(\widetilde{N}) \cup f(K)) = 0$ as $m(f(K)) = m(f(\widetilde{N})) = 0$, $f$ is a Lusin function on $B$.

We have corresponding results for similar condition on the Dini derivatives.

We shall state these results without proof for the proof involves similar technique.

**Theorem 7.** Suppose $f : A \to \mathbb{R}$ is a finite valued function. Suppose $B$ is a subset of $A$ such that at each point $x$ of $B$, $\mathcal{A}D_+ f(x) > -\infty$. Then, except for a null set, $\widetilde{N} \subseteq B$, $\mathcal{A}D_+ f(x) = \mathcal{A}D^+ f(x)$, its opposite derivate, and is either finite or $+\infty$. Moreover, $m^*(\widetilde{N}) = m^*(f(\widetilde{N})) = 0$. The set $K = \{x \in B : \mathcal{A}D_-(x) = \infty\}$ is a null set and $f$ is a Lusin function on $B - K$.  

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Theorem 8. Suppose \( f : A \to \mathbb{R} \) is a finite valued function. Suppose \( B \) is a subset of \( A \) such that at each point \( x \) of \( B \), \( _A D^+ f(x) < \infty \). Then, except for a null set, \( \widetilde{N} \subseteq B, _A D^+ f(x) = _A D_+ f(x) \), its opposite derivate, and is either finite or \(-\infty\). Moreover, \( m^*(\widetilde{N}) = m^*(f(\widetilde{N})) = 0 \), the set \( K = \{ x \in B : _A D^+ f(x) = -\infty \} \) is a null set and \( f \) is a Lusin function on \( B - K \).

Theorem 9. Suppose \( f : A \to \mathbb{R} \) is a finite valued function. Suppose \( B \) is a subset of \( A \) such that at each point \( x \) of \( B \), \( _A D^- f(x) > -\infty \). Then, except for a null set, \( \widetilde{N} \subseteq B, _A D^- f(x) = _A D_- f(x) \), its opposite derivate, and is either finite or \( +\infty \). Moreover, \( m^*(\widetilde{N}) = m^*(f(\widetilde{N})) = 0 \), the set \( K = \{ x \in B : _A D^- f(x) = \infty \} \) is a null set and \( f \) is a Lusin function on \( B - K \).

As a result of the above theorems we have,

Corollary 10. Suppose \( f : A \to \mathbb{R} \) is a finite valued function. Then for almost all \( x \) in \( A \), \( _A D^+ f(x) > -\infty, _A D^- f(x) > -\infty, _A D_+ f(x) < \infty \) and \( _A D_- f(x) < \infty \).

Theorem 11. Suppose \( f : A \to \mathbb{R} \) is a finite valued function. Suppose \( B \) is a subset of \( A \) such that at each point \( x \) of \( B \), \( f \) has either both finite Dini derivates on the same side or finite bilateral derivates \( \overline{D}f(x) \) or \( \underline{D}f(x) \). Then, \( f \) is differentiable almost everywhere on \( B \), i.e., for almost all \( x \) in \( B \), \( _A Df(x) \) exists and is finite. Moreover, for the subset \( E \) of \( B \), where \( _A Df(x) \) does not exists, \( m^*(E) = m^*(f(E)) = 0 \). \( f \) is a Lusin function on \( B \).

Proof.

Suppose \( _A D^+ f(x) \) and \( _A D_- f(x) \) are finite for all \( x \) in \( B \). Then by Theorem 5, since \( _A D^+ f(x) \) is finite for all \( x \) in \( B \), except for a null set, \( \widetilde{N}_1 \subseteq B, _A D^+ f(x) = _A D_+ f(x) \), its opposite derivate, and is finite. Moreover, \( m^*(\widetilde{N}_1) = m^*(f(\widetilde{N}_1)) = 0 \) and \( f \) is a
Lusin function on $B$. As $\mathcal{D}_x f(x)$ is finite on $B$, by an analogue theorem to Theorem 5, except for a null set, $\widetilde{N}_2 \subseteq B$, $\mathcal{D}_x f(x) = \mathcal{D}_x^+ f(x)$, its opposite derivate, and is finite. We also have that $m^*(\widetilde{N}_2) = m^*(f(\widetilde{N}_2)) = 0$. Therefore, for $x \in B - (\widetilde{N}_1 \cup \widetilde{N}_2)$, $\mathcal{D}_x^+ f(x) = \mathcal{D}_x f(x) = \mathcal{D}_x^- f(x) = \mathcal{D}_x f(x)$ and as $\mathcal{D}_x f(x) \leq \mathcal{D}_x^+ f(x)$, $\mathcal{D}_x f(x) = \mathcal{D}_x^+ f(x) = \mathcal{D}_x^- f(x) = \mathcal{D}_x f(x)$. This means $\mathcal{D}_x f(x)$ exists and is finite. Let $E = \widetilde{N}_1 \cup \widetilde{N}_2$. Then $m(E) = m(\widetilde{N}_1 \cup \widetilde{N}_2) = 0$. This means $f$ is differentiable almost everywhere on $B$. We already knew that $f$ is a Lusin function on $B$.

Suppose $\mathcal{D}_x^+ f(x)$ and $\mathcal{D}_x^- f(x)$ are finite for all $x$ in $B$. Then we can show similarly that for almost all $x$ in $B$, $\mathcal{D}_x f(x)$ exists and is finite and that the set where $\mathcal{D}_x f(x)$ does not exists is a null set and that $f$ is Lusin function on $B$.

Suppose $-\infty < \mathcal{D}_x^- f(x) < \infty$ for all $x$ in $B$. Then $\mathcal{D}_x^+ f(x) < \infty$ and $\mathcal{D}_x^- f(x) < \infty$ for all $x$ in $B$. Then, except for a null set, $\widetilde{N}_1 \subseteq B$, $\mathcal{D}_x^+ f(x) = \mathcal{D}_x f(x)$, its opposite derivate, and is either finite or $-\infty$. Moreover, $m^*(\widetilde{N}_1) = m^*(f(\widetilde{N}_1)) = 0$.

Similarly, except for a null set, $\widetilde{N}_2 \subseteq B$, $\mathcal{D}_x^- f(x) = \mathcal{D}_x f(x)$, its opposite derivate, and is either finite or $\infty$. Moreover, $m^*(\widetilde{N}_2) = m^*(f(\widetilde{N}_2)) = 0$.

Therefore, for $x \in B - (\widetilde{N}_1 \cup \widetilde{N}_2)$, $\mathcal{D}_x^+ f(x) = \mathcal{D}_x f(x)$ and $\mathcal{D}_x^- f(x) = \mathcal{D}_x f(x)$.

Suppose $\mathcal{D}_x^+ f(x) = -\infty$. Then $\mathcal{D}_x^+ f(x) = \mathcal{D}_x f(x) = \mathcal{D}_x^- f(x) = \mathcal{D}_x f(x)$. But this contradicts that $-\infty < \mathcal{D}_x f(x) = \max(\mathcal{D}_x^+ f(x), \mathcal{D}_x^- f(x)) < \infty$. Similarly, $\mathcal{D}_x^- f(x) = -\infty$ leads to a contradiction and so for all $x \in B - (\widetilde{N}_1 \cup \widetilde{N}_2)$

$\mathcal{D}_x^+ f(x) = \mathcal{D}_x f(x) = \mathcal{D}_x^- f(x) = \mathcal{D}_x f(x)$ and is finite. That is to say, $f$ is differentiable on $B - (\widetilde{N}_1 \cup \widetilde{N}_2)$ (with finite derivative). Moreover, denoting $E = (\widetilde{N}_1 \cup \widetilde{N}_2)$, $m^*(E) = m^*(f(E)) = 0$. By Theorem 5, $f$ is a Lusin function on $B - E$. Therefore, $f$ is a Lusin function on $B$.

Suppose $-\infty < \mathcal{D}_x f(x) < \infty$ for all $x$ in $B$. Then $\mathcal{D}_x f(x) > -\infty$ and $\mathcal{D}_x f(x) > -\infty$ for all $x$ in $B$. We can show similarly that except for a null set $E$ in $B$, $f$ is differentiable with finite derivative. Moreover, $m^*(E) = m^*(f(E)) = 0$ and $f$ is a Lusin function on $B$.  

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Theorem 12. Suppose \( f : A \to \mathbb{R} \) is a finite valued function. Suppose \( B \) is a subset of \( A \) such that at each point \( x \) of \( B \), \( |\mathcal{D}^+ f(x)| \leq M \) for some non-negative number \( M \). Then \( m^*(f(B)) \leq Mm^*(B) \).

Proof.

We have \( -M \leq \mathcal{D}^+ f(x) \leq M \) for all \( x \) in \( B \). By Theorem 5, \( \mathcal{D}^+ f(x) = \mathcal{D}^- f(x) \) and is finite for all \( x \) in \( B - \bar{N} \), where \( \bar{N} \) is a subset of \( B \) such that \( m^*(\bar{N}) = m^*(f(\bar{N})) = 0 \). Therefore, for \( x \) in \( B - \bar{N} \), as \( |\mathcal{D}^+ f(x)| \leq M \), \(-M \leq \mathcal{D}_- f(x) = \mathcal{D}^+ f(x) \). It follows by Theorem 10 of Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem, that \( m^*(f(B - \bar{N})) \leq Mm^*(B - \bar{N}) \). Hence,

\[
m^*(f(B)) \leq m^*(f(B - \bar{N})) + m^*(f(\bar{N})) = m^*(f(B - \bar{N})) \leq Mm^*(B - \bar{N}) \leq Mm^*(B).
\]

Denjoy Saks Young Theorem

We are now ready to state the Denjoy Saks Young Theorem for arbitrary function on any subset of \( \mathbb{R} \).

Theorem 13. Suppose \( f : A \to \mathbb{R} \) is a finite valued function. Let

\[
N = \{ x \in A : \mathcal{D}^+ f(x) = -\infty \text{ or } \mathcal{D}^- f(x) = -\infty \text{ or } \mathcal{D}_+ f(x) = \infty \text{ or } \mathcal{D}_- f(x) = \infty \},
\]

\[
S = \{ x \in A : \mathcal{D} f(x) \text{ exists and is finite} \},
\]

\[
T = \{ x \in A : \mathcal{D}^+ f(x) \text{ and } \mathcal{D}^- f(x) \text{ are finite and equal}, \mathcal{D}_+ f(x) = -\infty \text{ and } \mathcal{D}_- f(x) = \infty \},
\]

\[
U = \{ x \in A : \mathcal{D}^- f(x) \text{ and } \mathcal{D}_+ f(x) \text{ are finite and equal}, \mathcal{D}_+ f(x) = \infty \text{ and } \mathcal{D}_- f(x) = -\infty \},
\]

and \( V = \{ x \in A : \mathcal{D}^+ f(x) = \mathcal{D}^- f(x) = \infty \text{ and } \mathcal{D}_+ f(x) = \mathcal{D}_- f(x) = -\infty \} \).

Then \( A = N \cup S \cup T \cup U \cup V \cup E \), where \( E \) is a null set and \( m(f(E)) = 0 \).

Moreover, \( m(N) = 0 \) and \( f \) is a Lusin function on \( S \cup T \cup U \).

Proof.
By Corollary 10, \( N \) is a null set. Consider now the set \( A - N \). Thus, for all \( x \) in \( A - N \), \( _A D^+ f(x) \neq -\infty \), \( _A D^- f(x) \neq -\infty \), \( _A D_f(x) \neq \infty \) and \( _A D_\ast f(x) \neq \infty \).

Consider the set \( B_i = \{ x \in A - N : _A D^+ f(x) < \infty \} \). Then by Theorem 5, except for a null set, \( \bar{N}_i \subseteq B_i, _A D^+ f(x) = _A D_f(x) \), its opposite derivate, and is finite. Moreover, \( m^*(\bar{N}_i) = m^*(f(\bar{N}_i)) = 0 \). Let \( B_2 = \{ x \in A - N : _A D^- f(x) > -\infty \} \). By Theorem 7, except for a null set, \( \bar{N}_2 \subseteq B_2, _A D^- f(x) = _A D_f(x) \), its opposite derivate, and is finite. We also have \( m^*(\bar{N}_2) = m^*(f(\bar{N}_2)) = 0 \). Let \( B_3 = \{ x \in A - N : _A D^- f(x) > -\infty \} \). By Theorem 7, except for a null set, \( \bar{N}_3 \subseteq B_3, _A D^- f(x) = _A D_f(x) \), its opposite derivate, and is finite. We also have \( m^*(\bar{N}_3) = m^*(f(\bar{N}_3)) = 0 \). Let \( B_4 = \{ x \in A - N : _A D^- f(x) < \infty \} \). By Theorem 7, except for a null set, \( \bar{N}_4 \subseteq B_4, _A D^- f(x) = _A D_f(x) \), its opposite derivate, and is finite. Moreover, \( m^*(\bar{N}_4) = m^*(f(\bar{N}_4)) = 0 \).

Let \( C_i \) be the complement of \( B_i \) in \( A - N \), for \( i = 1,2,3 \) and 4. Let

\[
C_1 = \{ x \in A - N : _A D^+ f(x) = \infty \}, \quad C_2 = \{ x \in A - N : _A D^- f(x) = -\infty \}, \\
C_3 = \{ x \in A - N : _A D_f(x) = -\infty \} \quad \text{and} \quad C_4 = \{ x \in A - N : _A D^- f(x) = \infty \}.
\]

The complement of \( C_1 \cup C_2 \cup C_3 \cup C_4 \) is \( B_1 \cap B_2 \cap B_3 \cap B_4 \). By Theorem 11, \( f \) is differentiable on \( B_1 \cap B_2 \cap B_3 \cap B_4 \) except for a null subset \( E_i \) in \( B_2 \cap B_3 \), where \( m(f(E_i)) = 0 \).

First of all,

\[
C_1 \cap C_2 \cap C_3 \cap C_4 = \{ x \in A - N : _A D^+ f(x) = _A D^- f(x) = \infty \text{ and } _A D_f(x) = _A D_\ast f(x) = -\infty \}.
\]

Any intersection of three of the \( C_i \)'s with the complement of the remaining \( C_i \)'s gives a null set, \( E \), such that \( m(f(E)) = 0 \). This is because by Theorem 5 and its analogue, since the complement of the \( C_i \) is \( B_i \), we will have a pair of equal finite opposite derivates and its intersection with the opposite Dini derivate being non-finite will result in a null set, whose image under \( f \) is also a null set. For instance, \( B_1 \cap C_3, \ B_2 \cap C_4, \ B_3 \cap C_1 \) and \( B_4 \cap C_2 \) are null sets whose images under \( f \) are also null sets. Similarly, any intersection of three of the \( B_i \)’s with
the complement of the remaining $B_i$’s results in a null set, whose image under $f$ is also a null set.

Note that $C_1 \cap C_4 \cap B_2 \cap B_3$ and $C_2 \cap C_3 \cap B_i \cap B_4$ are null sets with null images under $f$.

\[
C_1 \cap C_3 \cap B_2 \cap B_4
\]

= \{x \in A - N : A^{+}D_{f}(x) = \infty, A^{-}D_{f}(x) = -\infty, A^{+}D_{f}(x) = A^{-}D_{f}(x) \text{ and is finite} \}

and

\[
C_2 \cap C_4 \cap B_i \cap B_3
\]

= \{x \in A - N : A^{+}D_{f}(x) = \infty, A^{-}D_{f}(x) = -\infty, A^{+}D_{f}(x) = A^{-}D_{f}(x) \text{ and is finite} \}.

Let $S = B_i \cap B_2 \cap B_3 \cap B_4$, $T = C_2 \cap C_4 \cap B_i \cap B_3$, $U = C_1 \cap C_3 \cap B_2 \cap B_4$ and $V = C_1 \cap C_2 \cap C_3 \cap C_4$. Thus, $A = N \cup S \cup T \cup U \cup V \cup E$, where $E$ is a null subset with $m(f(E)) = 0$.

By Theorem 5 and its analogue, $f$ is a Lusin function on $S \cup T \cup U$. This completes the proof of the theorem.