Integration Using Differentiation Under The Integral Sign

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We give examples of how differentiation under the integral sign can be used to evaluate improper integrals.

We are going to use two versions of the differentiation under the integral sign, a proper Riemann integral version and an improper Riemann integral version, which are Theorem 59 and Theorem 60 of [1] Chapter 14, *Mathematical Analysis, An Introduction* in My Calculus Web.

Theorem 1. Suppose one of the following two conditions (i) and (ii) is satisfied.

(i) $f:[c,d]\times[a,b]\to\mathbb{R}$ is a continuous function such the partial derivative $\dfrac{\partial f}{\partial t}(x,t)$ exists for all (*x*, *t*) in [*c*, *d*] × [*a*, *b*] and is continuous on [*c*, *d*] × [*a*, *b*].

(ii) $f:(c,d)\times[a,b]\to\mathbb{R}$ is a continuous function such that the partial derivative $\dfrac{\partial f}{\partial t}(x,t)$ exists for all (*x*, *t*) in (*c*, *d*) × [*a*, *b*] and is continuous on (*c*, *d*) × [*a*, *b*] and that for each *t* in [*a*, *b*], the function $f_t(x)=f(x,t)$ is Lebesgue integrable on (*c*, *d*). Suppose there exists a Lebesgue integrable function *g* on (*c*, *d*) such that $\left|\dfrac{\partial f}{\partial t}(x,t)\right|\le g(x)$ for all (*x*, *t*) in $(c,d)\times[a,b]$.

Let $F:[a,b]\to\mathbb{R}$ be defined by $F(t)=\int_c^d f(x,t)dx$.

Then, *F* is differentiable and $F'(t)=\int_c^d \dfrac{\partial f}{\partial t}(x,t)dx$ for each *t* in [*a*, *b*].

Theorem 2. Suppose $f:[c,\infty)\times[a,b]\to\mathbb{R}$ is a continuous function such that the partial derivative $\dfrac{\partial f}{\partial t}(x,t)$ exists for all (*x*, *t*) in [*c*, ∞) × [*a*, *b*] and is continuous on [*c*, ∞) × [*a*, *b*]. Suppose that the improper Cauchy Riemann integral $\int_c^\infty f(x,t)dx$ converges absolutely for each *t* in [*a*, *b*]. Suppose that the Cauchy Riemann integral $\int_c^\infty \dfrac{\partial f}{\partial t}(x,t)dx$ converges absolutely for each *t* in [*a*, *b*]. Suppose furthermore the improper integral $\int_c^\infty \dfrac{\partial f}{\partial t}(x,t)dx$ converges uniformly for *t* in [*a*, *b*].

Let $F:[a,b]\to\mathbb{R}$ be defined by $F(t)=\int_c^\infty f(x,t)dx$. Then *F* is differentiable and $F'(t)=\int_c^\infty \dfrac{\partial f}{\partial t}(x,t)dx$.

Example 1.

For $t\ge 0$, $\int_0^\infty e^{-tx}\dfrac{\sin(x)}{x}dx=\dfrac{\pi}{2}-\tan^{-1}(t)$. $\int_0^\infty \dfrac{\sin(x)}{x}dx=\dfrac{\pi}{2}$.

We shall first consider the case $t > 0$.

For $x \geq 0$, $t \geq 0$, let $f(x,t) = \begin{cases} e^{-tx}\dfrac{\sin(x)}{x}, & x > 0, \\ 1, & x = 0 \end{cases}$. Then f is a continuous function on $[0,\infty)\times[0,\infty)$.

Case $t > 0$.

For $x \geq 0$, $t > 0$, $|f(x,t)| \leq e^{-tx}$. Fix a real number $a > 0$.

Let $f_t(x) = f(x,t)$.

Then for $t > 0$, since $\int_0^\infty e^{-tx}dx = \lim_{s\to\infty}\left[-\frac{1}{t}e^{-tx}\right]_{x=0}^{x=s} = \frac{1}{t} - \lim_{s\to\infty}\frac{1}{t}e^{-ts} = \frac{1}{t} - 0 = \frac{1}{t}$, the improper Riemann integral $\int_0^\infty f_t(x)dx = \int_0^\infty f(x,t)dx$ is absolutely convergent for t in $(0, a]$

For $t = 0$, $\int_0^\infty f_0(x)dx = \int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$. Since $\int_0^\infty \frac{\sin(x)}{x}dx$ is conditionally convergent, it is not a Lebesgue integral.

Define $F(t) = \int_0^\infty f_t(x)dx = \int_0^\infty f(x,t)dx = \int_0^\infty e^{-tx}\frac{\sin(x)}{x}dx$.

Since $\int_0^\infty |f(x,t)|dx \leq \int_0^\infty e^{-tx}dx = \frac{1}{t}$ for $t > 0$ and $\lim_{t\to\infty}\frac{1}{t} = 0$, $\lim_{t\to\infty}F(t) = 0$.

It is easily seen that for $x > 0$, the partial derivative, $\frac{\partial f}{\partial t}(x,t) = -e^{-xt}\sin(x)$. We also have, $\frac{\partial f}{\partial t}(0,t) = 0$, $\frac{\partial f}{\partial t}(0,0) = 0$, $\frac{\partial f}{\partial t}(x,0) = -\sin(x)$.

Hence, we can write for $x \geq 0$, $t \geq 0$, $\frac{\partial f}{\partial t}(x,t) = -e^{-xt}\sin(x)$. For $x \geq 0$, $t > 0$, $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq e^{-xt}$ and since e^{-xt} is Lebesgue integrable for $t > 0$ on $[0, \infty)$, $\frac{\partial f}{\partial t}(x,t)$ is Lebesgue integrable for each $t > 0$ on $[0, \infty)$. Therefore, $\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ is absolutely convergent for t in $[k, K]$, for any $k > 0$ and any $K > k$.

Note that, for $t > 0$,

$$\int_0^s \frac{\partial f}{\partial t}(x,t)dx = -\int_0^s e^{-xt}\sin(x)dx = \left[-\cos(x)e^{-xt}\right]_{x=0}^{x=s} - t\int_0^s e^{-xt}\cos(x)dx$$

$$= 1 - \cos(s)e^{-st} - t\left\{\left[\sin(x)e^{-xt}\right]_{x=0}^{x=s} + \int_0^s \sin(x)te^{-xt}dx\right\}$$

$$= 1 - \cos(s)e^{-st} - t\sin(s)e^{-st} - t^2\int_0^s \sin(x)e^{-xt}dx.$$

Therefore, $\int_0^s e^{-xt}\sin(x)dx = \frac{1}{1+t^2}\left(1-\cos(s)e^{-st} - t\sin(s)e^{-st}\right)$. Hence,

$$\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx = \int_0^\infty e^{-xt}\sin(x)dx = \lim_{s\to\infty}\int_0^s e^{-xt}\sin(x)dx = \lim_{s\to\infty}\frac{1}{1+t^2}\left(1-\cos(s)e^{-st} - t\sin(s)e^{-st}\right)$$

$$= \frac{1}{1+t^2}.$$

For $t = 0$, $\int_0^\infty \frac{\partial f}{\partial t}(x,0)dx = \int_0^\infty \sin(x)dx$ and is not convergent.

Note that $\left|\frac{\partial f}{\partial t}(x,t)\right| = \left|-e^{-xt}\sin(x)\right| \le e^{-xt} \le e^{-xk}$ for $t \ge k$ and since $\int_0^\infty e^{-xk}dx$ is convergent and independent of t, we can conclude that the improper Riemann integrals, $\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ converges absolutely and uniformly with respect to t in $[k, \infty)$ and hence in $[k,K]$.

By Theorem 2, by considering $f(x, t)$ with domain $[0,\infty)\times[k,K]$, F is differentiable at t in $[k,K]$ and

$$F'(t) = \int_0^\infty \frac{\partial f}{\partial t}(x,t_0)dx = -\int_0^\infty e^{-xt}\sin(x)dx = -\frac{1}{1+t^2} \text{ for all } t\in[k,K].$$

By taking arbitrary $k > 0$ and any $K > k$, we conclude that F is differentiable for all $t > 0$ and $F'(t) = -\frac{1}{1+t^2}$ for all $t > 0$. Hence, for any $c > t$,

$$F(t) - F(c) = -\int_c^t \frac{1}{1+u^2}du = -\left[\tan^{-1}(u)\right]_c^t = \tan^{-1}(c) - \tan^{-1}(t).$$

This means $F(t) = \tan^{-1}(c) - \tan^{-1}(t) + F(c)$ for all $0 < t < c$. But $\lim_{c\to\infty} F(c) = 0$ and so

$$F(t) = \lim_{c\to\infty}\tan^{-1}(c) - \tan^{-1}(t) + \lim_{c\to\infty}F(c) = \frac{\pi}{2} - \tan^{-1}(t).$$

Hence, $\int_0^\infty e^{-tx}\frac{\sin(x)}{x}dx = \frac{\pi}{2} - \tan^{-1}(t)$.

<u>Case *t* = 0.</u>

Firstly, we check that $\int_0^\infty \frac{\sin(x)}{x}dx$ is convergent.

For $t > s > 0$,

$$\int_s^t \frac{\sin(x)}{x}dx = \left[-\frac{\cos(x)}{x}\right]_s^t - \int_s^t \frac{\cos(x)}{x^2}dx = \frac{\cos(s)}{s} - \frac{\cos(t)}{t} - \int_s^t \frac{\cos(x)}{x^2}dx.$$

Therefore,

$$\left|\int_s^t \frac{\sin(x)}{x}dx\right| \le \frac{1}{s} + \frac{1}{t} + \int_s^t \frac{1}{x^2}dx = \frac{2}{s}\ . \text{ -------------------------- (1)}$$

Note that $\lim_{x\to 0^+}\frac{\sin(x)}{x}=1$ and if we let $g(x)=\begin{cases}\frac{\sin(x)}{x}\ , & x>0,\\ 1\ , & x=0\end{cases}$ then

$\int_0^\infty \frac{\sin(x)}{x}dx=\int_0^\infty g(x)dx$.

Given $\varepsilon > 0$, let N be an integer so that $\frac{1}{N}<\frac{\varepsilon}{2}$. Therefore, for any $t > s \geq N$, by (1),

$$\left|\int_s^t \frac{\sin(x)}{x}dx\right|\leq \frac{2}{s}\leq \frac{2}{N}<\varepsilon .$$

It follows by Theorem 2 Chapter 14 that $\int_0^\infty g(x)dx$ exists and so $\int_0^\infty \frac{\sin(x)}{x}dx$ is convergent.

We also have that

$$\lim_{n\to\infty}\int_n^{n+1}\frac{\sin(x)}{x}dx=0. \quad \text{--------------------------------------- (2)}$$

Now we define for $n > 0$,

$$H_n(t)=\int_0^n f_t(x)dx=\int_0^n f(x,t)dx=\int_0^n e^{-tx}\frac{\sin(x)}{x}dx .$$

Plainly,

$$|H_n(n)|\leq \int_0^n e^{-nx}dx=\left[-\frac{1}{n}e^{-nx}\right]_0^n=\frac{1}{n}\left(1-e^{-n^2}\right)<\frac{1}{n} .$$

Since $\frac{1}{n}\to 0$, it follows by the Comparison Test that,

$$\lim_{n\to\infty}H_n(n)=0 \ . \quad \text{-- (3)}$$

By Theorem 1 part(i), taking the domain of $f(x,t)$ as $[0,n]\times[0,K]$ for any $K > t \geq 0$,

$$H_n{}'(t)=\int_0^n \frac{\partial f}{\partial t}(x,t)dx=-\int_0^n e^{-tx}\sin(x)dx .$$

Now,

$$\begin{aligned}\int_0^n e^{-tx}\sin(x)dx &= \left[-\cos(x)e^{-tx}\right]_{x=0}^{x=n}-t\int_0^n e^{-tx}\cos(x)dx\\ &=1-\cos(n)e^{-tn}-t\left\{\left[\sin(x)e^{-tx}\right]_0^n+t\int_0^n e^{-tx}\sin(x)dx\right\}\\ &=1-\cos(n)e^{-tn}-t\sin(n)e^{-tn}-t^2\int_0^n e^{-tx}\sin(x)dx .\end{aligned}$$

Hence, for $t \geq 0$, $\left(1+t^2\right)\int_0^n e^{-tx}\sin(x)dx=1-\cos(n)e^{-tn}-t\sin(n)e^{-tn}$ so that

$$\int_0^n e^{-tx}\sin(x)dx=\frac{1-e^{-tn}\left(\cos(n)+t\sin(n)\right)}{\left(1+t^2\right)} .$$

Thus, for $t \geq 0$, $H_n'(t) = \dfrac{e^{-tn}\left(\cos(n) + t\sin(n)\right) - 1}{\left(1+t^2\right)}$.

Taking limit as n tends to infinity, for all $t > 0$,

$$\lim_{n\to\infty} H_n'(t) = -\frac{1}{\left(1+t^2\right)} \text{. -- (4)}$$

Observe that for all $t \geq 0$,

$$\left|H_n'(t)\right| \leq \frac{e^{-tn}\left(1+t\right)+1}{\left(1+t^2\right)} \leq \frac{e^{-t}\left(1+t\right)+1}{\left(1+t^2\right)} \leq \frac{2}{\left(1+t^2\right)} .$$

Therefore, $H_n'(t)$ is dominated by $\dfrac{2}{\left(1+t^2\right)}$ on $[0, \infty)$ which is Lebesgue integrable on $[0, \infty)$.

Now we are going to employ Lebesgue Dominated Convergence Theorem.
For each integer $n \geq 1$, let

$$g_n = H_n' \chi_{[0,n]} ,$$

where $\chi_{[0,n]}$ is the characteristic function on the interval $[0, n]$.

Then plainly, for $t > 0$,

$$\lim_{n\to\infty} g_n(t) = \lim_{n\to\infty} H_n'(t) = -\frac{1}{1+t^2} .$$

Each g_n is a Lebesgue integrable function and g_n converges pointwise to $-\dfrac{1}{1+t^2}$ for all $t >$ 0.

Moreover, $\left|g_n(t)\right| \leq \left|H_n'(t)\right| \leq \dfrac{2}{\left(1+t^2\right)}$. Note that $\displaystyle\int_0^\infty \frac{2}{\left(1+t^2\right)} dt$ is absolutely convergent.

Hence, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^\infty g_n(t)dt = \lim_{n\to\infty} \int_0^n H_n'(t)dt = -\int_0^\infty \frac{1}{1+t^2} dt = -\frac{\pi}{2} .$$

But $\displaystyle\int_0^n H_n'(t)dt = H_n(n) - H_n(0)$. So, taking limit we have, by (3),

$$-\frac{\pi}{2} = \lim_{n\to\infty} \int_0^n H_n'(t)dt = \lim_{n\to\infty} H_n(n) - \lim_{n\to\infty} H_n(0) = 0 - \lim_{n\to\infty} H_n(0) = -\lim_{n\to\infty} H_n(0) .$$

This means $\displaystyle\lim_{n\to\infty} H_n(0) = \frac{\pi}{2}$. That is to say, $\displaystyle\lim_{n\to\infty} \int_0^n \frac{\sin(x)}{x} dx = \frac{\pi}{2}$. Hence, $\displaystyle\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$.

Thus, for all $t \geq 0$, $\displaystyle\int_0^\infty e^{-tx} \frac{\sin(x)}{x} dx = \frac{\pi}{2} - \tan^{-1}(t)$

Example 2. For any t in $\mathbb{R}$, $\int_0^\infty e^{-x^2}\cos(2tx)dx = \frac{\sqrt{\pi}}{2}e^{-t^2}$.

Let $f(x,t) = e^{-x^2}\cos(2xt)$. Then f is a continuous function on $[0,\infty)\times(-\infty,+\infty)$. Moreover, $|f(x,t)| \le e^{-x^2}$ for all $x \ge 0$ and all t in $\mathbb{R}$.

The partial derivative $\frac{\partial f}{\partial t}(x,t) = -2xe^{-x^2}\sin(2xt)$ exists for all $x \ge 0$ and all t in $\mathbb{R}$.

Since $\int_0^\infty e^{-x^2}dx$ is convergent, the improper Riemann integral $\int_0^\infty f(x,t)dx = \int_0^\infty e^{-x^2}\cos(2xt)dx$ is absolutely convergent and $f_t(x) = f(x,t)$ is Lebesgue integrable on [0, ∞).

Also, we have that $\left|\frac{\partial f}{\partial t}(x,t)\right| \le 2xe^{-x^2}$ for all x in [0, ∞) and all t in $\mathbb{R}$ and $g(x) = 2xe^{-x^2}$ is a Lebesgue integrable function on [0, ∞). It follows that the improper Riemann integral $\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ converges absolutely and uniformly for t in $(-\infty,\infty)$. In particular, it converges uniformly for t in $[-a,a]$ for any $a > 0$. Therefore, by Theorem 2, $F(t) = \int_0^\infty f(x,t)dx = \int_0^\infty e^{-x^2}\cos(2xt)dx$, $F'(t) = \int_0^\infty \frac{\partial f}{\partial t} f(x,t)dx = -\int_0^\infty 2xe^{-x^2}\sin(2xt)dx$ for any t in $[-a,a]$. Since a is arbitrary, for all $t \in \mathbb{R}$,

$$F'(t) = -\int_0^\infty 2xe^{-x^2}\sin(2xt)dx .$$

Now, for any $t > 0$,

$$\int_0^s -2xe^{-x^2}\sin(2xt)dx = \left[e^{-x^2}\sin(2xt)\right]_0^s - \int_0^s 2te^{-x^2}\cos(2xt)dx$$

$$= e^{-s^2}\sin(2st) - 2t\int_0^s e^{-x^2}\cos(2xt)dx .$$

Hence,

$$\int_0^\infty -2xe^{-x^2}\sin(2xt)dx = \lim_{s\to\infty}\int_0^s -2xe^{-x^2}\sin(2xt)dx = \lim_{s\to\infty} e^{-s^2}\sin(2st) - 2t\lim_{s\to\infty}\int_0^s e^{-x^2}\cos(2xt)dx$$

$$= 0 - 2t\int_0^\infty e^{-x^2}\cos(2xt)dx = -2t\int_0^\infty e^{-x^2}\cos(2xt)dx .$$

That is,

$$F'(t) = -2tF(t) .$$

Hence, $F(t)$ is a solution of the differential equation

$$\frac{dy}{dt} = -2ty .$$

Solving this equation by the method of variable separable gives

$$y = e^{-t^2}K \quad \text{for some constant } K$$

and $K = y(0)$.

Thus, we have $F(t) = F(0)e^{-t^2}$.

Now $F(0) = \int_0^\infty f(x,0)dx = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Therefore,

$$F(t) = \int_0^\infty e^{-x^2}\cos(2xt)dx = \frac{\sqrt{\pi}}{2}e^{-t^2}.$$

Example 3.

For $t > 0$, $\int_0^\infty e^{-tx}\frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2} - \tan^{-1}\left(\frac{t}{2}\right) - \frac{t}{4}\ln(4+t^2) + \frac{t\ln(t)}{2}$. In particular,
$\int_0^\infty e^{-x}\frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right) - \frac{\ln(5)}{4}$.

For $x \geq 0$, $t \geq 0$, let $f(x,t) = \begin{cases} e^{-tx}\dfrac{\sin^2(x)}{x^2}, & x > 0, \\ 1, & x = 0 \end{cases}$. Then f is a continuous function on $[0,\infty)\times[0,\infty)$.

For $x \geq 0$, $t > 0$, $|f(x,t)| \leq e^{-tx}$. Let $f_t(x) = f(x,t)$.

Then for $t > 0$, since $\int_0^\infty e^{-tx}dx = \lim_{s\to\infty}\left[-\frac{1}{t}e^{-tx}\right]_{x=0}^{x=s} = \frac{1}{t} - \lim_{s\to\infty}\frac{1}{t}e^{-ts} = \frac{1}{t} - 0 = \frac{1}{t}$, the improper Riemann integral $\int_0^\infty f_t(x)dx = \int_0^\infty f(x,t)dx$ is absolutely convergent for t > 0.

For $t = 0$, $\int_0^\infty f_0(x)dx = \int_0^\infty \frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2}$. Since $\frac{(\sin(x))^2}{x^2}$ is Lebesgue integrable on [0, ∞). this means $\int_0^\infty f_t(x)dx$ is improperly Riemann integrable and Lebesgue integrable for all $t \geq 0$.

Define $F(t) = \int_0^\infty f_t(x)dx = \int_0^\infty f(x,t)dx = \int_0^\infty e^{-tx}\frac{\sin^2(x)}{x^2}dx$.

Since $\int_0^\infty |f(x,t)|dx \leq \int_0^\infty e^{-tx}dx = \frac{1}{t}$ for $t > 0$ and $\lim_{t\to\infty}\frac{1}{t} = 0$, $\lim_{t\to\infty}F(t) = 0$.

It is easily seen that for $x > 0$, the partial derivative, $\frac{\partial f}{\partial t}(x,t) = -e^{-xt}\frac{\sin^2(x)}{x}$. We also have,
$\frac{\partial f}{\partial t}(0,t) = 0$, $\frac{\partial f}{\partial t}(0,0) = 0$, $\frac{\partial f}{\partial t}(x,0) = -\frac{\sin^2(x)}{x}$.

Hence, we can write for $x > 0$, $t > 0$, $\left|\frac{\partial f}{\partial t}(x,t)\right| = \left|-e^{-xt}\frac{\sin^2(x)}{x}\right| \leq e^{-xt}$. Take any $k > 0$.

Then for any $t \geq k$ and $x > 0$, $tx \geq kx$ and so $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq e^{-xk}$ and since $\frac{\partial f}{\partial t}(0,t) = 0$, $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq e^{-xk}$ for any $t \geq k$ and $x \geq 0$. Since $\int_0^\infty e^{-xk} dx$ is convergent, $\int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ converges absolutely and uniformly in t in $[k,\infty)$. Therefore, by Theorem 2, $F(t)$ is differentiable in $[k,\infty)$ and $F'(t) = \int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ t in $[k,\infty)$. Since k is arbitrarily chosen, we conclude that $F'(t) = \int_0^\infty \frac{\partial f}{\partial t}(x,t)dx$ for $t \in (0,\infty)$.

Let $g(x,t) = \begin{cases} -e^{-xt}\dfrac{\sin^2(x)}{x}, & \text{if } x > 0, \\ 0, & \text{if } x = 0 \end{cases}$

Then $F'(t) = \int_0^\infty g(x,t)dx = -\int_0^\infty e^{-xt}\frac{\sin^2(x)}{x}dx$.

Let $G(t) = \int_0^\infty g(x,t)dx = -\int_0^\infty e^{-xt}\frac{\sin^2(x)}{x}dx$. Observe that $|G(t)| = \left|\int_0^\infty e^{-xt}\frac{\sin^2(x)}{x}dx\right| \leq \int_0^\infty e^{-xt}dx \leq \int_0^\infty e^{-xk}dx$ for any $k > 0$. As $g_t(x) = g(x,t)$ tends pointwise to the zero constant function, by the Lebesgue Dominated Convergence Theorem, $\lim_{t\to\infty} G(t) = 0$.

Now, $\frac{\partial g}{\partial t}(x,t) = e^{-xt}\sin^2(x)$ for $x > 0$, $\frac{\partial g}{\partial t}(0,t) = 0$ for $t \geq 0$.

As for the case of $f(x,t)$, $\int_0^\infty \frac{\partial g}{\partial t}(x,t)dx = \int_0^\infty e^{-xt}\sin^2(x)dx$ converges absolutely and uniformly in t in $[k,\infty)$. Thus, by Theorem 2, $G(t)$ is differentiable on $[k,\infty)$ and hence on $(0,\infty)$ with

$$G'(t) = \int_0^\infty \frac{\partial g}{\partial t}(x,t)dx = \int_0^\infty e^{-xt}\sin^2(x)dx$$

Now, for $t > 0$,

$$\int_0^\infty e^{-xt}\sin^2(x)dx = \left[-\frac{1}{t}e^{-xt}\sin^2(x)\right]_0^\infty + \int_0^\infty \frac{1}{t}e^{-xt}2\sin(x)\cos(x)dx$$

$$= \int_0^\infty \frac{1}{t}e^{-xt}\sin(2x)dx$$

$$= \left[-\frac{1}{t^2}e^{-xt}\sin(2x)\right]_0^\infty + \int_0^\infty \frac{1}{t^2}e^{-xt}2\cos(2x)dx = \frac{2}{t^2}\int_0^\infty e^{-xt}\cos(2x)dx$$

$$= \frac{2}{t^2}\left[-\frac{1}{t}e^{-xt}\cos(2x)\right]_0^\infty + \frac{2}{t^3}\int_0^\infty e^{-xt}(-2\sin(2x))dx$$

$$= \frac{2}{t^3} - \frac{4}{t^3}\int_0^\infty e^{-xt}\sin(2x)dx .$$

Therefore, $\left(\frac{1}{t}+\frac{4}{t^3}\right)\int_0^\infty e^{-xt}\sin(2x)dx = \frac{2}{t^3}$.

Thus, $\int_0^\infty e^{-xt}\sin^2(x)dx = \int_0^\infty \frac{1}{t}e^{-xt}\sin(2x)dx = \frac{2}{t^3}\frac{t^2}{4+t^2} = \frac{2}{t(4+t^2)}$.

Hence, $G'(t) = \frac{2}{t^3}\frac{t^2}{4+t^2} = \frac{2}{t(4+t^2)}$ for $t > 0$. Now, $\frac{2}{t(4+t^2)} = \frac{1}{2t} - \frac{t}{2(4+t^2)}$, taking antiderivative, we get $G(t) = \frac{\ln(t)}{2} - \frac{\ln(4+t^2)}{4} + C$. Then

$$\lim_{t\to\infty} G(t) = \lim_{t\to\infty}\left(\frac{\ln(t)}{2} - \frac{\ln(4+t^2)}{4}\right) + C = \frac{1}{4}\lim_{t\to\infty}\ln\left(\frac{t^2}{4+t^2}\right) + C = \frac{1}{4}\ln(1) + C = C .$$

Since $\lim_{t\to\infty} G(t) = 0$, $C = 0$. Hence,

$$G(t) = \frac{\ln(t)}{2} - \frac{\ln(4+t^2)}{4} .$$

It follows that $F'(t) = \int_0^\infty \frac{\partial f}{\partial t}(x,t)dx = G(t) = \frac{\ln(t)}{2} - \frac{\ln(4+t^2)}{4}$ for $t > 0$. Taking antiderivative, we have,

$$F(t) = \frac{1}{2}t\ln(t) - \frac{1}{4}t\ln(4+t^2) - \tan^{-1}\left(\frac{t}{2}\right) + C .$$

Taking limit as t tends to infinity,

$$0 = \lim_{t\to\infty} F(t) = \lim_{t\to\infty}\left(\frac{1}{2}t\ln(t) - \frac{1}{4}t\ln(4+t^2)\right) - \lim_{t\to\infty}\left(\tan^{-1}\left(\frac{t}{2}\right)\right) + C$$

$$= \frac{1}{4}\lim_{t\to\infty}\left(t\ln\left(\frac{t^2}{4+t^2}\right)\right) - \frac{\pi}{2} + C = \frac{1}{4}\lim_{t\to\infty}\left(\frac{\frac{2}{t} - \frac{2t}{4+t^2}}{-\frac{1}{t^2}}\right) - \frac{\pi}{2} + C = -\frac{1}{4}\lim_{t\to\infty}\frac{8t}{(4+t^2)} - \frac{\pi}{2} + C$$

$$= 0 - \frac{\pi}{2} + C = C - \frac{\pi}{2} .$$

Thus, $C = \frac{\pi}{2}$ and so, for $t > 0$,

$$\int_0^\infty e^{-tx}\frac{\sin^2(x)}{x^2}dx = F(t) = \frac{\pi}{2} - \tan^{-1}\left(\frac{t}{2}\right) - \frac{t}{4}\ln(4+t^2) + \frac{t\ln(t)}{2} .$$

We may also use the limit $F(t)$ as t tends to 0 on the right. Note that $f_t(x)$ on the domain $[0,\infty)$ is monotone increasing and non-negative and each $f_t(x)$ is Lebesgue integrable on $[0,\infty)$ and so by the Lebesgue Monotone Convergence Theorem as $f_t(x)$ converges pointwise to $f_0(x)$, $F(t)=\int_0^\infty f_t(x)dx$ converges to $F(0)=\int_0^\infty f_0(x)dx=\int_0^\infty \frac{\sin^2(x)}{x^2}dx=\frac{\pi}{2}$.

This would give the integration constant C to be $\frac{\pi}{2}$. Taking the limit of $F(t)$ as t tends to infinity is better in the sense that we avoid evaluating $\int_0^\infty \frac{\sin^2(x)}{x^2}dx$ and the limit of $\tan^{-1}(t)$ as t tends to infinity is more familiar.

In particular,

$$\int_0^\infty e^{-x}\frac{\sin^2(x)}{x^2}dx=F(1)=\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{2}\right)-\frac{\ln(5)}{4} \text{ and } \int_0^\infty e^{-x}\frac{\sin^2(x)}{x}dx=G(1)=-\frac{\ln(5)}{4}.$$

References.

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[2] Mathematical Analysis, An Introduction, *Chapter 9 Uniform Convergence, Integration and Power Series*

[3] Introduction to Measure Theory