Discourse on Monotone Functions

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This is the fourth and final part of the series of articles towards Denjoy Saks Young Theorem. We give some application of the theorems in the previous articles and some interesting results concerning monotone functions.

The first result is a technical one, of interest in itself.

Theorem 1. Suppose $f : A \to \mathbb{R}$ is a strictly increasing bounded function, A is a subset of $\mathbb{R}$ and p is a non-negative number. If at every point x of a subset E in A, there exists at least one derived number, $Df(x)$ such that $Df(x) \le p$, then $m^*(f(E)) \le pm^*(E)$, where m^* is the Lebesgue outer measure.

Proof.

Suppose $m^*(E) < \infty$. Therefore, given any $\varepsilon > 0$, there exists an open set U containing E such that

$$m^*(U) < m^*(E) + \varepsilon .$$

Let q be any number such that $q > p$. Let $x \in E$. Then there exists a sequence (h_n) such that $x + h_n \in A$, $h_n \neq 0$, $h_n \to 0$ and

$$\lim_{n\to\infty} \frac{f(x+h_n) - f(x)}{h_n} = Df(x) \le p < q.$$

Since U is open and $m^*(U) < \infty$, it is at most a countable disjoint union of bounded open intervals. Therefore, x must belong to one of these bounded open intervals. It follows that for sufficiently large n, $[x, x+h_n] \subseteq U$ if $h_n > 0$ or $[x+h_n, x] \subseteq U$ if $h_n < 0$ and

$$\frac{f(x+h_n) - f(x)}{h_n} < q . \quad \text{-----------------------} \ (1)$$

Let $I_n(x) = [x, x+h_n]$ or $[x+h_n, x]$ depending on the sign of h_n. Let $\Delta_n(x) = [f(x), f(x+h_n)]$ if $h_n > 0$ and $\Delta_n(x) = [f(x+h_n), f(x)]$ if $h_n < 0$.

As $h_n \to 0$, there exists arbitrary such small closed interval, $I_n(x)$, containing *x* and $I_n(x) \subseteq U$. Since *f* is strictly increasing, $\Delta_n(x)$ is a non-degenerate closed interval and $f(I_n(x) \cap A) \subseteq \Delta_n(x)$. Note that $f(x) \in \Delta_n(x)$ and $m(\Delta_n(x)) = |f(x+h_n) - f(x)| < q|h_n|$ by inequality (1). As $qh_n \to 0$, we conclude that there are arbitrary such small closed intervals $\Delta_n(x)$ containing $f(x)$ with the requirement that the corresponding closed interval $I_n(x) \subseteq U$. Hence, this collection of intervals for each *x* in *E*, $\{\Delta_n(x) : n \in \mathbb{N}, x \in E\}$, forms a Vitali covering for $f(E)$. It follows by the Vitali Covering Theorem that there exists a countable number of pairwise disjoint closed intervals, $\{\Delta_{n_i}(x_i)\}$ such that $m^*\left(f(E) - \bigcup_{i=1}^{\infty} \Delta_{n_i}(x_i)\right) = 0$. Therefore,

$$m^*(f(E)) \le m^*\left(\bigcup_{i=1}^{\infty} \Delta_{n_i}(x_i)\right) + m^*\left(f(E) - \bigcup_{i=1}^{\infty} \Delta_{n_i}(x_i)\right) = m^*\left(\bigcup_{i=1}^{\infty} \Delta_{n_i}(x_i)\right)$$

$$\le \sum_{i=1}^{\infty} m^*\left(\Delta_{n_i}(x_i)\right) = \sum_{i=1}^{\infty} \left| f(x_i + h_{n_i}) - f(x_i) \right|$$

$$\le \sum_{i=1}^{\infty} q\left|h_{n_i}\right| = q\sum_{i=1}^{\infty} m^*\left(I_{n_i}(x_i)\right), \text{ by inequality (1).} \quad \text{---------------------} \ (2)$$

Since *f* is strictly increasing, the corresponding countable collection, $\{I_{n_i}(x_i)\}$, is also a pairwise disjoint collection of closed intervals. It follows that

$$\sum_{i=1}^{\infty} m^*\left(I_{n_i}(x_i)\right) = m^*\left(\bigcup_{i=q}^{\infty} I_{n_i}(x_i)\right) \le m^*(U) < m^*(E) + \varepsilon .$$

Therefore, $m^*(f(E)) \le q(m^*(E) + \varepsilon)$. Letting $q \searrow p$ and $\varepsilon \searrow 0$, we get $m^*(f(E)) \le pm^*(E)$.

If $m^*(E) = \infty$ and $p > 0$, we have nothing to prove.

If $m^*(E) < \infty$ and $p = 0$, then for each positive integer *n*, as $Df(x) \le 0 < \frac{1}{n}$ $m^*(f(E)) \le \frac{1}{n} m^*(E)$, by what we have just proved. Therefore, letting $n \to \infty$, we get $m^*(f(E)) \le 0$ and so $m^*(f(E)) = 0 \le pm^*(E)$.

If $p = 0$ and $m^*(E) = \infty$. Partition E into countable pieces by setting $E_n = D \cap [n, n+1]$. Then E is a countable union of $\{E_n\}$. Since each E_n has finite outer measure, it follows that $m^*(f(E_n)) = 0$ and so $m^*(f(E)) = 0$ and the inequality is trivially true if we set the multiplication rule $0 * \infty = 0$.

Remark. Theorem 1 appears to be stronger than Theorem 10 of *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*, in that it uses only derived number instead of the Dini derivates. Strict monotonicity plays a very important and crucial role in the proof of Theorem 1 and this makes its case somewhat weaker as Theorem 10 cited above is for a general function.

Theorem 2. Suppose $f : A \to \mathbb{R}$ is an increasing bounded function, A is a subset of $\mathbb{R}$ and q is a non-negative number. If at every point x of a subset E of finite outer measure in A, there exists at least one derived number, $Df(x)$ finite or infinite such that $Df(x) \geq q$, then $m^*\left(f(E)\right) \geq q m^*(E)$.

Proof.

The proof is almost similar to that of Theorem 1.

If $q = 0$, we have nothing to prove. So, we now assume that $q > 0$.

Let p be a number such that $0 < p < q$.

As $f(E)$ is bounded, given any $\varepsilon > 0$, there exists a bounded open set U containing $f(E)$ such that

$$m^*(U) < m^*(f(E)) + \varepsilon .$$

Let $S \subseteq E$ be the set of points at which f is continuous. Then by Theorem 4 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, $E - S$ is at most denumerable. We may remove these points in $E - S$ from E without affecting the conclusion of the theorem. We now assume that f is continuous at every point in E.
Let $x \in E$. Then there exists a sequence (h_n) such that $x + h_n \in A$, $h_n \neq 0$, $h_n \to 0$ and

$$\lim_{n\to\infty}\frac{f(x+h_n)-f(x)}{h_n} = Df(x) \geq q > p.$$

Therefore, there exists an integer N_x such that

$$n > N_x \Rightarrow \frac{f(x+h_n)-f(x)}{h_n} > p.$$

Now, the open set U is a disjoint union of open intervals, $U = \bigcup_i I_i$, where each I_i is a bounded open interval. Note that $f(x) \in I_i$ for some i. Let $d_n(x) = [x, x+h_n]$ or $[x+h_n, x]$ depending on the sign of h_n. Let $\Delta_n(x) = \left[f(x), f(x+h_n)\right]$ if $h_n > 0$ and $\Delta_n(x) = \left[f(x+h_n), f(x)\right]$ if $h_n < 0$. Since f is continuous at x, for sufficiently large $n > N_x$, $\Delta_n(x) = \left[f(x+h_n), f(x)\right] \subseteq I_i \subseteq U$ and

$$\frac{f(x+h_n)-f(x)}{h_n} > p. \text{ -----------------------(1)}$$

Thus, for sufficiently large n,

$$m^*\left(\Delta_n(x)\right) = \left|f(x+h_n) - f(x)\right| > ph_n = pm^*(d_n(x)) \text{ ------------ (2)}$$

and $f(d_n(x)) \subseteq \Delta_n(x) \subseteq U$.

Hence, this collection of such small intervals for each x in E, $\{d_n(x) : n \in \mathbb{N}, x \in E\}$, forms a Vitali covering for E. Note that in this collection $f(d_n(x)) \subseteq \Delta_n(x) \subseteq U$ for every member, $d_n(x)$, of this collection. It follows by the Vitali Covering Theorem, that there exists a countable number of pairwise disjoint closed intervals, $\{d_{n_i}(x_i)\}$ such that $m^*\left(E - \bigcup_{i=1}^{\infty} d_{n_i}(x_i)\right) = 0$. It follows that

$$m^*(E) \leq m^*\left(\bigcup_{i=1}^{\infty} d_{n_i}(x_i)\right) + m^*\left(E - \bigcup_{i=1}^{\infty} d_{n_i}(x_i)\right) = m^*\left(\bigcup_{i=1}^{\infty} d_{n_i}(x_i)\right)$$

$$= \sum_{i=1}^{\infty} m^*\left(d_{n_i}(x_i)\right) < \frac{1}{p}\sum_{i=1}^{\infty} m^*\left(\Delta_{n_i}(x_i)\right), \text{ by inequality (2).}$$

Since f is increasing, the corresponding countable collection $\{\Delta_{n_i}(x_i)\}$ is a pairwise non-overlapping collection of closed intervals. Therefore,

$$\sum_{i=1}^{\infty} m^*\left(\Delta_{n_i}(x_i)\right) = m^*\left(\bigcup_{i=1}^{\infty}\Delta_{n_i}(x_i)\right) \le m^*(U) < m^*\left(f(E)\right)+\varepsilon .$$

It follows that $m^*(E) < \dfrac{1}{p}\left(m^*\left(f(E)\right)+\varepsilon\right)$. Taking limit as $p \nearrow q$ and $\varepsilon \searrow 0$, we get $m^*\left(f(E)\right) \ge q m^*\left(E\right)$.

Corollary 3. Suppose $f : A \to \mathbb{R}$ is an increasing bounded function, *A* is a subset of $\mathbb{R}$ and *q* is a positive number. There does not exist a subset *E* in *A* with infinite outer measure such that at every *x* in *E*, there exists at least one derived number, *Df*(*x*), finite or infinite such that $Df(x) \ge q$.

Proof.

Suppose $m^*(E) = \infty$. Let $E_n = [-n, n] \cap E$. Then by the continuity from below property of outer measure, $\lim_{n\to\infty} m^*(E_n) = m^*(E) = \infty$. Since $m^*(E_n) < \infty$, applying Theorem 2 to E_n , we get $m^*\left(f(E_n)\right) \ge q m^*(E_n)$. Note that $f(E_n) \subseteq f(E_{n+1})$ and $f(E) = \bigcup_{n=1}^{\infty} f(E_n)$. It follows that $m^*\left(f(E)\right) = \lim_{n\to\infty} m^*\left(f(E_n)\right) = \infty$ because $\lim_{n\to\infty} m^*(E_n) = \infty$. Hence, $f(E)$ cannot be bounded, contradicting that *f* is a bounded function.

Theorem 4. Suppose $f : A \to \mathbb{R}$ is an increasing bounded function and *A* is a subset in $\mathbb{R}$. Then there exists a subset *N* such that *f* is not differentiable, finitely or infinitely at every point in *N* and at every point *x* in *A*–*N*, *f* is differentiable with finite derivative or infinite derivative, + ∞, and $m(N) = m\left(f(N)\right) = 0$. Moreover, the set of points at which *f* has infinite derivative constitutes a null set. The function, *f*, has finite derivative almost everywhere on *A*.

Proof.

Note that *f* is a function of bounded variation on *A*. By Theorem 15, *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, there exists a subset *N* such that *f* is not differentiable, finite or infinitely at every point in *N*, whereas at every point in *A*–*N*, *f* is differentiable with finite derivative or infinite derivative,+ ∞, and that $m(N) = m\left(f(N)\right) = 0$. By Theorem 8 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, the set on which *f* has infinite derivative is of measure zero. Hence, *f* has finite derivative almost everywhere on *A*.

Theorem 5. Suppose $f: A \to \mathbb{R}$ is an increasing bounded function and *A* is a subset of $\mathbb{R}$. Let $E = \left\{x \in A: f \text{ has at least one infinite derived number at } x\right\}$. Then $m^*(E) = 0$.

Proof.

By Corollary 3, *E* cannot have infinite outer measure.

Thus, we may assume $m^*(E) < \infty$.

Suppose $m^*(E) > 0$

Then for any *K* > 0, by Theorem 2, $m^*\left(f(E)\right) \geq Km^*(E)$. Consequently, $m^*\left(f(E)\right) = \infty$, contradicting *f* is bounded on *E*. Therefore, $m^*(E) = 0$.

Theorem 6. Suppose $f: A \to \mathbb{R}$ is an increasing function and *A* is a subset of $\mathbb{R}$. Then the set of points, where *f* has infinite derivative is of measure zero.

Proof. Since *f* is increasing it can have only +∞, as infinite derived number. Let *E* be the set, where *f* has infinite derivative.

If *f* has infinite derivative at *x*, it means

$${}_A D^+ f(x) = {}_A D_- f(x) = {}_A D^- f(x) = {}_A D_+ f(x) = \infty \ .$$

Therefore, by Corollary 10 of *Denjoy Saks Young Theorem for Arbitrary Function*, *E* is of measure zero.

We state the analogue of Theorem 4 for decreasing function.

Theorem 7. Suppose $f: A \to \mathbb{R}$ is a decreasing bounded function and *A* is a subset in $\mathbb{R}$. Then there exists a subset *N* of *A*, on which *f* is not differentiable, finitely or infinitely, at every point in *A*–*N*, *f* is differentiable with finite

derivative or negative infinite derivative, $-\infty$, and $m(N) = m\big(f(N)\big) = 0$.
Moreover, the set of points at which *f* has negative infinite derivative, $-\infty$, constitutes a null set. The function *f* has finite derivative almost everywhere on *A*.

Combining Theorem 4 and Theorem 7 we have:

Theorem 8. Suppose $f : A \to \mathbb{R}$ is a monotone bounded function and *A* is a subset in $\mathbb{R}$. Then there exists a subset *N* at every point of which, *f* is not differentiable, finitely or infinitely, at every point in *A*–*N*, *f* is differentiable with finite derivative or infinite derivative, $+\infty$ or $-\infty$ and $m(N) = m\big(f(N)\big) = 0$.
Moreover, the set of points, at which *f* has negative or positive infinite derivative, constitutes a null set. The function *f* has finite derivative almost everywhere on *A*.

The next theorem is an application of Theorem 4.

Theorem 9. Suppose $f : A \to \mathbb{R}$ is an increasing bounded function and *A* is a subset in $\mathbb{R}$. Suppose $E \subseteq f(A)$ is a set of measure zero. Let $H = \{x \in A : {}_A Df(x) \text{ exists and } {}_A Df(x) \neq 0\}$. Then $m\Big(f^{-1}(E) \cap H\Big) = 0$. Hence, ${}_A Df(x) = 0$ almost everywhere on $f^{-1}(E)$.

Proof.

By Theorem 4, there exists a subset *N* such that *f* is not differentiable, finitely or infinitely at every point in *N* and at every point *x* in *A*–*N*, *f* is differentiable with finite derivative or infinite derivative, $+\infty$, and $m(N) = m\big(f(N)\big) = 0$. We may assume that $H = \{x \in A : {}_A Df(x) \text{ exists and } {}_A Df(x) \neq 0\} \subseteq A - N$. Note that for every *x* in $f^{-1}(E) \cap H$, ${}_A Df(x) \neq 0$. Since *f* is increasing all derived number of *f* is non-negative. Hence, any *x* in $f^{-1}(E) \cap H$ has a unique positive derived number. Let $H_n = \left\{x \in A - N : {}_A Df(x) > \dfrac{1}{n}\right\}$ for each positive integer *n*. Then $H = \bigcup_{n=1}^{\infty} H_n$ and $f^{-1}(E) \cap H = \bigcup_{n=1}^{\infty} f^{-1}(E) \cap H_n$. We claim that $m\Big(f^{-1}(E) \cap H_n\Big) = 0$. By Corollary 3,

$f^{-1}(E)\cap H_n$ cannot have infinite outer measure. Suppose $m\left(f^{-1}(E)\cap H_n\right)>0$. Then by Theorem 2,

$$m^*\left(f\left(f^{-1}(E)\cap H_n\right)\right)\geq \frac{1}{n}m^*(f^{-1}(E)\cap H_n)>0.$$

But $f\left(f^{-1}(E)\cap H_n\right)\subseteq E$ and so $m^*\left(f\left(f^{-1}(E)\cap H_n\right)\right)=0$ and we have a contradiction. It follows that $m\left(f^{-1}(E)\cap H_n\right)=0$. Hence,

$$m^*\left(f^{-1}(E)\cap H\right)=m^*\left(\bigcup_{n=1}^{\infty} f^{-1}(E)\cap H_n\right)\leq \sum_{n=1}^{\infty} m^*\left(f^{-1}(E)\cap H_n\right)=0$$

implying that $m\left(f^{-1}(E)\cap H\right)=0$.

Remark.

Theorem 9 is not the most general result. Indeed, we can do away with the monotone condition on *f* in Theorem 9. The next theorem is a much more general result.

Theorem 10. Suppose $f: A\to\mathbb{R}$ is a finite valued function and *A* is a subset in $\mathbb{R}$. Suppose *f* has derivative (finite or infinite) on a subset *E* with $m(f(E))=0$. Then ${}_A Df(x)=0$ almost everywhere on *E*.

Proof.
We may assume that every point in *E* is a two-sided limit point of *A*.
Let $B=\{t\in E: |{}_A Df(t)|>0\}$. Let $C_n=\left\{t\in B: |{}_A Df(t)|\geq \frac{1}{n}\right\}$ and
$B_n=\left\{t\in B: |f(s)-f(t)|\geq \frac{|s-t|}{n},\ \text{for } s\in A \text{ and } |s-t|<\frac{1}{n}\right\}$, for each positive integer *n*.

Note that $B=\bigcup_{n=1}^{\infty} C_n$. Let $x\in C_n$. We claim that there exists an integer *k* such that $x\in B_k$. Note that either $|{}_A Df(x)|$ is finite and $|{}_A Df(x)|\geq \frac{1}{n}$ or $|{}_A Df(t)|=\infty$. If $|{}_A Df(x)|$ is finite, then there exists $\delta>0$ such that

$$\left|\left|\frac{f(s)-f(x)}{s-x}\right|-|{}_A Df(x)|\right|<\frac{1}{2n},$$

for all *s* in *A* with $0<|s-x|<\delta$. Take any integer *k* such that $k > 2n$ and $\frac{1}{k}<\delta$.
Then we have

$$0<|s-x|<\frac{1}{k} \text{ and } s\in A \Rightarrow \left|\frac{f(s)-f(x)}{s-x}\right| > |{}_A Df(x)| - \frac{1}{2n} \geq \frac{1}{2n} \geq \frac{1}{k}.$$

This means that $x\in B_k$.

If $|{}_A Df(x)|$ is infinite, then there exists $\delta > 0$ such that

$$\left|\frac{f(s)-f(x)}{s-x}\right| > 1,$$

for all *s* in *A* with $0<|s-x|<\delta$. In this case, just take any integer *k* with $\frac{1}{k}<\delta$.
Then we have

$$0<|s-x|<\frac{1}{k} \text{ and } s\in A \Rightarrow \left|\frac{f(s)-f(x)}{s-x}\right| > 1 \geq \frac{1}{k}.$$

It follows that $x\in B_k$. This implies that $B=\bigcup_{n=1}^{\infty} C_n \subseteq \bigcup_{n=1}^{\infty} B_n \subseteq B$ and so $B=\bigcup_{n=1}^{\infty} B_n$.
We shall show that the measure of *B* is 0 by showing that the measure of B_n is zero. Fix an integer *n* and consider any interval *I* of length $1/n$ and its intersection with B_n, $C = I\cap B_n$. We claim that the measure of *C* is zero. Since $m(f(E)) = 0$, $m(f(B)) = 0$ and so $m(f(C)) = 0$. As $m(f(C)) = 0$, given any $\varepsilon > 0$, we can cover $f(C)$ by a countable union of disjoint intervals I_k such that

$$f(C)\subseteq \bigcup_k I_k \text{ and } m\left(\bigcup_k I_k\right) = \sum_k m(I_k) < \varepsilon. \quad \text{-------- (1)}$$

Let $A_k = f^{-1}(I_k)\cap C$. Then $C=\bigcup_k A_k$.

$$m^*(C) \leq \sum_k m^*(A_k) \leq \sum_k \sup\{|s-t| : s,t\in A_k\}. \text{ -------------------- (2)}$$

Note that $\sup\{|s-t| : s,t\in A_k\}$ exists because A_k is bounded. Observe that $A_k \subseteq C = I\cap B_n \subseteq I$ and *I* is an interval of length less than $\frac{1}{n}$ and so for any *s*, *t* in A_k, $|s - t| < 1/n$. Thus, by the definition of B_n, for *s*, *t* in A_k,

$$|s-t| \leq n|f(s)-f(t)|.$$

Hence,

$$\sup\{|s-t| : s,t\in A_k\} \leq n\sup\{|f(s)-f(t)| : s,t\in A_k\} \leq nm^*(I_k) \text{----- (3)}$$

Because $f(A_k)\subseteq I_k$. It follows from inequalities (1), (2) and (3) that

$$m^*(C) \leq n\sum_k m^*(I_k) < n\varepsilon \ .$$

Since ε is arbitrary and *n* is fixed, we conclude that $m^*(C)=0$. Now we can cover B_n by a countable number of non-overlapping intervals *I*, each of length less than $\frac{1}{n}$. Thus, by the above argument, the measure of each of the intersection of B_n with the intervals has measure zero. It follows that the measure of B_n is zero. Hence, the measure of B is zero. Therefore, ${}_A Df(x)$ is zero almost everywhere on E.

Theorem 11. Suppose $f : A \to \mathbb{R}$ is a finite-valued function and A is a subset in $\mathbb{R}$. Suppose $E \subseteq f(A)$ is a set of measure zero. Let

$$H = \{x \in A : {}_A Df(x) \text{ exists finitely or infinitely and } {}_A Df(x) \neq 0\}.$$

Then $m\left(f^{-1}(E) \cap H\right) = 0$. Hence, ${}_A Df(x) = 0$ almost everywhere on $f^{-1}(E)$.

Proof.
If $H = \varnothing$, then we have nothing to prove.
So, we now assume that $H \neq \varnothing$.
If $f^{-1}(E) \cap H = \varnothing$, then we have nothing to prove.
Suppose now that $f^{-1}(E) \cap H \neq \varnothing$.
By hypothesis, f is differentiable (finite or infinitely) on $f^{-1}(E) \cap H$. Moreover, since $f\left(f^{-1}(E) \cap H\right) \subseteq E$ and E is of measure 0, $m\left(f\left(f^{-1}(E) \cap H\right)\right) = 0$. Then by Theorem 10, ${}_A Df(x) = 0$ almost everywhere on $f^{-1}(E) \cap H$. But as there does not exists an x in $f^{-1}(E) \cap H$ such that ${}_A Df(x) = 0$, $m\left(f^{-1}(E) \cap H\right) = 0$. Therefore, ${}_A Df(x) = 0$ almost everywhere on $f^{-1}(E)$.

Remark.

Theorem 11 was stated in *Change of Variables Theorem* when the domain is an interval. It is used in the proof of the chain rule used in the proof of the change of variable theorem.
Theorem 10 is a converse to Theorem 11 in *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem.*

We now discuss some condition for monotonicity for function on a bounded interval. For simplicity, we fix the domain of the function to be the unit interval [0, 1].

Theorem 12. If $f:[0,1]\to\mathbb{R}$ is continuous and $D^+f(x)>0$ for all *x* in $[0,1)$, then *f* is increasing on [0,1] and consequently, *f* is differentiable almost everywhere with finite derivative on [0, 1].

Proof.

Note that *f* is a bounded function since it is continuous on [0, 1] and [0,1] is a compact set so that its image is also compact and hence closed and bounded by the Heine-Borel Theorem.

Suppose *f* is not increasing on [0, 1). Then there exists *a*, *b* in [0, 1) with $0\le a<b<1$ such that $f(a)>f(b)$. Let

$$m=\frac{f(b)-f(a)}{b-a}<0\ .$$

Define $G:[a,b]\to\mathbb{R}$ by $G(x)=f(x)-\frac{m}{2}(x-a)$. Then *G* is continuous on [*a*, *b*].

We have $G(a)=f(a)$ and $G(b)=f(b)-\frac{m}{2}(b-a)=\frac{1}{2}\left(f(a)+f(b)\right)<f(a)=G(a)$.

Since *G* is continuous on [*a*, *b*], by the Extreme Value Theorem, *G* attains a maximum at some point *k* in [*a*, *b*) since $G(b)<G(a)$. Moreover, $D^+G(x)=D^+f(x)-\frac{m}{2}\le 0$, since $\frac{G(k+h)-G(k)}{h}\le 0$ for all $h>0$ with $k+h<b$ so that $\limsup_{h\to 0^+}\frac{G(k+h)-G(k)}{h}\le 0$. Therefore, $D^+f(k)\le\frac{m}{2}<0$, contradicting that $D^+f(k)>0$. It follows that *f* is increasing on [0, 1). Since *f* is continuous on [0, 1], *f* is increasing on [0,1]. By Theorem 4, *f* is differentiable almost everywhere with finite derivative on [0, 1].

Theorem 13. Suppose $f:[0,1]\to\mathbb{R}$ is continuous and $D^+f(x)$ is bounded below for all *x* in $[0,1)$. Then *f* is differentiable almost everywhere with finite derivative on [0, 1].

Proof.

Suppose $D^+f(x)\ge C$ for all *x* in $[0,1)$. Let $H(x)=f(x)+(1-C)x$. Then H is continuous on [0, 1] and $D^+H(x)=D^+f(x)+(1-C)\ge 1>0$ for all *x* in $[0,1)$. Therefore, by Theorem 12, *H* is increasing on [0, 1]. Therefore, by Theorem 4,

H has finite derivative almost everywhere on [0, 1]. It follows that $f(x) = H(x) - (1-C)x$ has finite derivative almost everywhere on [0, 1].

Theorem 14. If $f:[0,1] \to \mathbb{R}$ is continuous and $D_+ f(x) < 0$ for all x in $[0,1)$, then f is decreasing on [0,1] and consequently f is differentiable almost everywhere with finite derivative on [0, 1].

Proof.

The proof is similar to that for Theorem 12.

Since f is continuous on [0, 1], f is decreasing on [0, 1] if, and only if, f is decreasing on [0, 1).

Suppose f is not decreasing on [0, 1). Then there exists a, b in [0, 1) with $0 \le a < b < 1$ such that $f(a) < f(b)$. Let

$$m = \frac{f(b) - f(a)}{b - a} > 0 \ .$$

Define $G:[a,b] \to \mathbb{R}$ by $G(x) = f(x) - \frac{m}{2}(x-a)$. Then G is continuous on [a, b].

We have $G(a) = f(a)$ and $G(b) = f(b) - \frac{m}{2}(b-a) = \frac{1}{2}\big(f(a) + f(b)\big) > f(a) = G(a)$.

Since G is continuous on [a, b], by the Extreme Value Theorem, G attains its minimum at some point k in [a, b) since $G(b) > G(a)$. Moreover, $D_+G(x) = D_+ f(x) - \frac{m}{2} > 0$, since $\frac{G(k+h) - G(k)}{h} \ge 0$ for all $h > 0$ with $k + h < b$ so that $\liminf_{h \to 0^+} \frac{G(k+h) - G(k)}{h} \ge 0$. Therefore, $D_+ f(k) \ge \frac{m}{2} > 0$, contradicting that $D_+ f(k) < 0$. It follows that f is decreasing on [0, 1). Since f is continuous on [0, 1], f is decreasing on [0,1]. By Theorem 4, f is differentiable almost everywhere with finite derivative on [0, 1].

Similarly, we can prove the next theorem as for theorem 12.

Theorem 15. If $f:[0,1] \to \mathbb{R}$ is continuous and $D^- f(x) > 0$ for all x in $(0,1]$, then f is increasing on [0,1] and consequently, f is differentiable almost everywhere with finite derivative on [0, 1].

Theorem 16. If $f:[0,1]\to\mathbb{R}$ is continuous and $D_{-}f(x)<0$ for all *x* in $(0,1]$, then *f* is decreasing on [0,1] and consequently *f* is differentiable almost everywhere with finite derivative on [0, 1].

The proof of Theorem 16 is similar to that for Theorem 14.

We have the analogues of Theorem 12, deduced from Theorem 14,15 and 16.

We summarize the resulting conclusion as follows.

Theorem 17. Suppose $f:[0,1]\to\mathbb{R}$ is continuous and satisfies any one of the following conditions:

(1) $D^{+}f(x)$ is bounded below for all *x* in $[0,1)$,

(2) $D^{-}f(x)$ is bounded below for all *x* in $(0,1]$,

(3) $D_{+}f(x)$ is bounded above for all *x* in $[0,1)$,

(3) $D_{-}f(x)$ is bounded above for all *x* in $(0,1]$.

Then, *f* is differentiable almost everywhere with finite derivative on [0, 1].

We end this article with the following interesting result.

Theorem 18. Suppose $f:A\to\mathbb{R}$ is a function of bounded variation and *A* is a subset in $\mathbb{R}$. Suppose *E* is a subset of *A* such that at each point *x* of *E*, ${}_{A}Df(x)$ is finite. If $m\big(f(E)\big)=0$, then $m\big(v_f(E)\big)=0$.

Proof.

By Theorem 10, ${}_{A}Df(x)=0$ almost everywhere on *E*. By Theorem 18, *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, there is a subset *N* of *A* such that $m\big(v_f(N)\big)=m\big(f(N)\big)=m(N)=0$ and for each $x\in A-N$, ${}_{A}Df(x)$ and ${}_{A}Dv_f(x)$ exist (finitely or infinitely) and that ${}_{A}Dv_f(x)=\left|{}_{A}Df(x)\right|$. We may thus assume that $E \subseteq A - N$. Since ${}_A Df(x)$ is finite for x in E, we may further assume that $E \subseteq A - N - H$, where H is the subset of $A - N$, where ${}_A Dv_f(x) = |{}_A Df(x)| = +\infty$. Note that by Theorem 8 of *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, $m^*(H) = 0$. Hence, v_f is differentiable finitely on $E \subseteq A - N - H$. By Theorem 17, *Functions of Bounded Variation and de La Vallée Poussin's Theorem*, v_f is a Lusin function on $A - N - H$ and hence on E. As ${}_A Dv_f(x) = 0$ almost everywhere on E, there is a subset $B \subseteq E$, such that $m^*(B) = 0$ and ${}_A Dv_f(x) = 0$ for all x in $E - B$. Therefore, by Theorem 11 of *Arbitrary Function, Limit Superior, Dini Derivative and Lebesgue Density Theorem*, $m\left(v_f(E - B)\right) = 0$. As v_f is a Lusin function on E, $m^*\left(v_f(B)\right) = 0$. Hence, $m\left(v_f(E)\right) = m^*\left(v_f(E)\right) = 0$.

Remark.

Actually, we may not need to specify that E be a subset, where the derivative of f is finite. In *Functions of Bounded Variation and Johnson's Indicatrix*, I proved a stronger result (Theorem 10, in the above cited article) that when $f : A \to \mathbb{R}$ is of bounded variation, $m(f(E)) = 0 \Rightarrow m\left(v_f(E)\right) = 0$. We use the idea of the Johnson's Indicatrix in the proof of this result.