Existence of Riemann Stieltjes Integral

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Theorem 1. Let $[a,b]$ be a closed and bounded interval with $a < b$. Suppose $g:[a,b]\to\mathbb{R}$ is an increasing non-constant function and $f:[a,b]\to\mathbb{R}$ is a bounded Borel function. If the function *f* is Riemann Stieltjes integrable with respect to *g*, then $\mu_g(D_f)=0$, where D_f is the set of discontinuities of *f* in [*a*, *b*].

Proof. Since *f* is Riemann Stieltjes integrable with respect to *g*, it is also Darboux Stieltjes integrable with respect to *g* and the Riemann Stieltjes integral and the Darboux Stieltjes integral are the same. (See Theorem 3 below.) Therefore, take a sequence of partition (P_n) of [*a*, *b*] such that $P_n \subseteq P_{n+1}$ and the norm of the partition $\|P_n\|$ tends to zero as *n* tends to infinity. If we denote the partition P_n by $P_n : a = x_0{}^n < x_1{}^n < \cdots < x_{N^n}{}^n = b$, then the lower Darboux Stieltjes sum, $L(f,P_n)=\sum_{i=1}^{N^n} m_i(f,P_n)\left(g(x_i{}^n)-g(x_{i-1}{}^n)\right)$ and the upper Darboux Stieltjes sum $U(f,P_n)=\sum_{i=1}^{N_n} M_i(f,P_n)\left(g(x_i{}^n)-g(x_{i-1}{}^n)\right)$, where $m_i(f,P_n)=\inf\{f(x): x\in[x_{i-1}{}^n, x_i{}^n]\}$ and $M_i(f,P_n)=\sup\{f(x): x\in[x_{i-1}{}^n, x_i{}^n]\}$. Plainly, $L(f,P_n)\le U(f,P_n)$. Moreover, $L(f,P_n)\to\int_a^b fdg$ and $U(f,P_n)\to\int_a^b fdg$.

Without loss of generality, we may assume that *g* is continuous at *a* and *b*. This is because if *g* is not continuous at *a*, then *f* is continuous at *a* and so the contribution of the point *a* to the integral is zero and we may redefine the value of *g* at *a* by its right limit at *a*. Similar argument for the situation at the point *b*.

By Lemma 16 of "Limit of the Lebesgue Stieltjes Integral and Change of Variable", for each *n* we can find an admissible partition Q_n of [*a*, *b*] such that $L(f,P_n)\le L(f,Q_n)+\frac{1}{2^n}$ and $U(f,g,P_n)\ge U(f,g,Q_n)-\frac{1}{2^n}$. We may assume that $Q_n\subseteq Q_{n+1}$. Thus, $L(f,Q_n)\to\int_a^b fdg$ and $U(f,Q_n)\to\int_a^b fdg$. We now define admissible step functions, $H_n(x)$ and $L_n(x)$ such that $\int_a^b L_n dg = L(f,Q_n)$ and $\int_a^b H_n dg = U(f,Q_n)$. If $Q: a = y_0 < y_1{}^n < \cdots < y_N = b$ is an admissible partition, define $L(x)=m_i(f,Q)$ for $x\in(y_{i-1},y_i)$ and $H(x)=M_i(f,Q)$ for $x\in(y_{i-1},y_i)$ for $1\le i\le N$ and $L(y_i)=H(y_i)=f(y_i)$ for $0\le i\le N$. Then $\int_a^b L_n dg = L(f,Q_n)$ and $\int_a^b H_n dg = U(f,Q_n)$.

Let $k_n = H_n - L_n$. Then $\lim_{n\to\infty}\int_a^b k_n dg = 0$. Moreover, $k_{n+1} \le k_n$ for $n \ge 1$. Now, each k_n is a non-negative admissible step function and so the sequence (k_n) converges to a function *G*. That is, $G = \lim_{n\to\infty} k_n$. Therefore, by the Lebesgue Monotone Convergence Theorem, $\int_a^g G d\mu_g = \lim_{n\to\infty}\int_a^b k_n d\mu_g = 0$. Therefore, $G = 0$ almost everywhere with respect to μ_g. Since $L_n(x) \le f(x) \le H_n(x)$ for all *x* in [*a*, *b*], $\lim_{n\to\infty} L_n(x) = f(x) = \lim_{n\to\infty} H_n(x)$ almost everywhere with respect to μ_g. Thus, there is a set *E* of μ_g measure zero such that $\lim_{n\to\infty} L_n(x) = f(x) = \lim_{n\to\infty} H_n(x)$ for $x \in [a,b] - E$. Let *F* be the collection of all the partition points of Q_n for all *n*, then *F* is at most denumerable and as *g* is continuous at each point of *F*, $\mu_g(F) = 0$. Let $E_1 = E \cup F$ We shall show that *f* is continuous on $x \in [a,b] - E_1$.

If $x \in [a,b] - E_1$, then $\lim_{n\to\infty} H_n(x) - f(x) = f(x) - \lim_{n\to\infty} L_n(x) = 0$. Therefore, given $\varepsilon > 0$, there exists an integer *N* such that $n > N \Rightarrow 0 \le H_n(x) - f(x) < \varepsilon$ and $n > N \Rightarrow 0 \le f(x) - L_n(x) < \varepsilon$ Take a fixed $n_0 > N$. Let the admissible partition corresponding to H_{n_0} be Q_{n_0}. Let the partition Q_{n_0} be denoted by. $Q_{n_0} : a = y_0 < y_1 < \cdots < y_{N_{n_0}} = b$. Now $x \in (y_j, y_{j-1})$ for some $1 \le j \le N_{n_0}$. Hence, there exists $\delta_1 > 0$ such that $(x - \delta_1, x - \delta_1) \subseteq (y_j, y_{j-1})$.

Take $y \in (x - \delta_1, x - \delta_1)$. If $f(y) > f(x)$, then $f(y) - f(x) \le H_n(x) - f(x) < \varepsilon$.

If $f(y) < f(x)$, then $f(x) - f(y) \le f(x) - L_n(x) < \varepsilon$. This shows that *f* is continuous at each point of $[a,b] - E_1$. Hence, the set D_f of discontinuities of *f* is contained in E_1, Hence, $\mu_g(D_f) = 0$.

Theorem 2. Let $[a,b]$ be a closed and bounded interval with *a* < *b*. Suppose $g : [a,b] \to \mathbb{R}$ is an increasing non-constant function and $f : [a,b] \to \mathbb{R}$ is a bounded Borel function. If $\mu_g(D_f) = 0$, where D_f is the set of discontinuities of *f* in [*a*, *b*], then the function *f* is Riemann Stieltjes integrable with respect to *g*.

Proof. If $\mu_g(D_f) = 0$, then *f* and *g* have no common set of discontinuities.

We shall use the fact that if *f* and *g* have no common set of discontinuities and if *f* is Darboux Stieltjes integrable, then *f* is Riemann Stieltjes integrable. (See Theorem 5 below.)

We shall now show that *f* is Darboux Stieltjes integrable. We shall assume without loss of generality that *g* is continuous at *a* and *b*.

We shall show that the upper and lower Darboux Stieltjes integrals are the same.

By the standard argument, there exists a sequence of partition (P_n) of $[a, b]$ such that $P_n \subseteq P_{n+1}$ and the norm of the partition $\|P_n\|$ tends to zero as *n* tends to infinity such that $L(f,P_n) \to LD\int_a^b fdg$ and $U(f,P_n) \to UD\int_a^b fdg$, where $LD\int_a^b fdg$ and $UD\int_a^b fdg$ are the lower and upper Darboux Stieltjes integrals. By Lemma 16 of "*Limit of the Lebesgue Stieltjes Integral and Change of Variable*", we can replaced the partitions (P_n) by a sequence of admissible partitions (Q_n) such that $L(f,Q_n) \to LD\int_a^b fdg$ and $U(f,Q_n) \to UD\int_a^b fdg$. Let $L_n(x)$ and $H_n(x)$ be the corresponding admissible step functions corresponding to the lower and upper Darboux Stieljes sums. Let $k_n = H_n - L_n$. Let *F* be the collection of all the partition points of Q_n for all *n*. Since *g* is continuous at these points, $\mu_g(F) = 0$. Let $E = F \cup D_f$. Note that *f* is continuous on $[a,b]-E$. Take $x \in [a,b]-E$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in (x-\delta, x+\delta) \cap [a,b] \Rightarrow |f(y) - f(x)| < \frac{\varepsilon}{2}$. Since $\|Q_n\| \to 0$, there exists an integer N_0 such that for all $n \geq N_0 \Rightarrow \|Q_n\| < \delta$. Let the partition Q_n be given by $Q_n : a = y_0 < y_1 < \cdots < y_{N_n} = b$. Then $x \in (y_j, y_{j-1})$ for some $1 \leq j \leq N_n$. It follows that

$k_n(x) = H_n(x) - L_n(x) = M_j(f,Q_n) - m_j(f,Q_n) \leq \varepsilon$. Hence, $\lim_{n\to\infty} k_n(x) = 0$ almost everywhere with respect to μ_g . Hence,

$LD\int_a^b fdg = \lim_{n\to\infty}\int_a^b L_n(x)dg = \lim_{n\to\infty}\int_a^b H_n(x)dg = UD\int_a^b fdg$. Therefore, *f* is Darboux Stieltjes integrable with respect to *g*, consequently by Theorem 5 below, *f* is Riemann Stieltjes integrable with respect to *g*.

Theorem 3. Let $[a,b]$ be a closed and bounded interval with $a < b$. Suppose $g:[a,b] \to \mathbb{R}$ is an increasing non-constant function and $f:[a,b] \to \mathbb{R}$ is a bounded Borel function. If *f* is Riemann Stieltjes integrable with respect to *g*, then *f* is Darboux Stieltjes integrable with respect to *g*.

Proof. Since *f* is Riemann Stieltjes integrable with respect to *g*, for any $\varepsilon > 0$, there exists a partition $P: a = x_0 < x_1 < \cdots < x_N = b$ such that for any $z_i \in [x_{i-1}, x_i]$,

$$\left|\sum_{i=1}^{n} f(z_i)(g(x_i) - g(x_{i-1})) - \int_a^b fdg\right| < \frac{\varepsilon}{2},$$

that is, $\int_a^b fdg-\frac{\varepsilon}{2}<\sum_{i=1}^{n} f(z_i)(g(x_i)-g(x_{i-1}))<\int_a^b fdg+\frac{\varepsilon}{2}$.

Let $m_i=\inf\{f(x):x\in[x_{i-1},x_i]\}$ and $M_i=\sup\{f(x):x\in[x_{i-1},x_i]\}$.

By the definition of infimum and supremum, for each *i* with $1\le i\le N$, there exist $z_i, z_i'\in[x_{i-1},x_i]$ such that

$$f(z_i)<m_i+\frac{\varepsilon}{2(g(b)-g(a))} \text{ and } f(z_i')>M_i-\frac{\varepsilon}{2(g(b)-g(a))}.$$

Therefore, $\sum_{i=1}^{n} f(z_i)(g(x_i)-g(x_{i-1}))<\sum_{i=1}^{n} m_i(g(x_i)-g(x_{i-1}))+\frac{\varepsilon}{2}=L(f,P)+\frac{\varepsilon}{2}$

and $\sum_{i=1}^{n} f(z_i')(g(x_i)-g(x_{i-1}))>\sum_{i=1}^{n} M_i(g(x_i)-g(x_{i-1}))-\frac{\varepsilon}{2}=U(f,P)-\frac{\varepsilon}{2}$.

Hence, $\int_a^b fdg<\sum_{i=1}^{n} f(z_i)(g(x_i)-g(x_{i-1}))+\frac{\varepsilon}{2}<L(f,P)+\varepsilon$. It follows that

$$\int_a^b fdg-\varepsilon<L(f,P)\le LD\int_a^b fdg.$$

Similarly, $\int_a^b fdg>\sum_{i=1}^{n} f(z_i')(g(x_i)-g(x_{i-1}))-\frac{\varepsilon}{2}>U(f,P)-\varepsilon$ and so

$$\int_a^b fdg+\varepsilon>U(f,P)\ge UD\int_a^b fdg.$$

Hence, $\int_a^b fdg-\varepsilon<LD\int_a^b fdg\le UD\int_a^b fdg<\int_a^b fdg+\varepsilon$, for any $\varepsilon>0$. It follows that $LD\int_a^b fdg=UD\int_a^b fdg$ and so *f* is Darboux Stieltjes integrable with respect to *g*.

Theorem 4. Let $[a,b]$ be a closed and bounded interval with *a* < *b*. Suppose $g:[a,b]\to\mathbb{R}$ is an increasing non-constant function and $f:[a,b]\to\mathbb{R}$ is a bounded Borel function. If *f* is Riemann Stieltjes integrable with respect to *g*, then *f* is Lebesgue Stieltjes integrable with respect to the Lebesgue Stieltjes measure, μ_g.

Proof.

In the proof of Theorem 1, we show that there exists admissible step functions, $L_n(x)$ and $H_n(x)$ such that $L_n(x)\le f(x)\le H_n(x)$ for all *x* in [*a*, *b*] and $\lim_{n\to\infty} L_n(x)=f(x)=\lim_{n\to\infty} H_n(x)$ almost everywhere with respect to μ_g. Moreover,

$(L_n(x))$ is an increasing sequence of μ_g measurable functions. Now $\lim_{n\to\infty}\int_a^b L_n d\mu_g = \lim_{n\to\infty}\int_a^b L_n dg = \int_a^b fdg$ and so by the Lebesgue Dominated Convergence Theorem, *f* is μ_g measurable and $\lim_{n\to\infty}\int_a^b L_n d\mu_g = \int_a^b fd\mu_g$, the Lebesgue Stieltjes integral of *f* with respect to μ_g. Hence, $\int_a^b fdg = \int_a^b fd\mu_g$.

Theorem 5. Let $[a,b]$ be a closed and bounded interval with *a* < *b*. Suppose $g:[a,b]\to\mathbb{R}$ is an increasing non-constant function and $f:[a,b]\to\mathbb{R}$ is a bounded Borel function. If *f* and g have no common set of discontinuities and *f* is Darboux Stieltjes integrable with respect to *g*, then *f* is Riemann Stieltjes integrable with respect to *g*.

Proof. Since *f* is Darboux Stieltjes integrable, given $\varepsilon > 0$, there exists a partition $P1: a = x_0 < x_1 < \cdots < x_N = b$ such that

$$0 \le U(f,P1) - L(f,P1) < \frac{\varepsilon}{2},$$

where $L(f,P1) = \sum_{i=1}^{N} m_i(f,P1)\big(g(x_i) - g(x_{i-1})\big)$, $U(f,P1) = \sum_{i=1}^{N} M_i(f,P1)\big(g(x_i) - g(x_{i-1})\big)$,

$m_i = \inf\{f(x): x\in[x_{i-1},x_i]\}$ and $M_i = \inf\{f(x): x\in[x_{i-1},x_i]\}$.

Suppose $Q: a = y_0 < y_1 < \cdots < y_L = b$ is a refinement of $P1$, then

$$L(f,P1) \le L(f,Q) \le U(f,Q) \le U(f,P1).$$

Let $S(f,g,Q) = \sum_{i=1}^{L} f(z_i)(g(y_i) - g(y_{i-1}))$ be the Riemann Stieltjes sum with respect to the partition *Q*. Then

$$L(f,P) \le L(f,Q) \le S(f,g,Q), DS\int_a^b fdg \le U(f,Q) \le U(f,P).$$

It follows that $\left|S(f,g,Q) - DS\int_a^b fdg\right| < \frac{\varepsilon}{2}$. ------------------------- (1)

Now we are going to find a special refinement of $P1$. Since the functions *f* and *g* are both bounded, take a common bound *M* of *f* and *g* on [*a*, *b*] such that $|f(x)|, |g(x)| \le M$ for all *x* in [*a*, *b*]. At the partition point x_j, $0 \le j \le N$, either *f* or *g* is continuous. Therefore, since there is only a finite number of partition points of $P1$, there exists a $\delta > 0$, such that for all x_j, $0 \le j \le N$,

$$|f(x) - f(x_j)| < \frac{\varepsilon}{8NM} \text{ or } |g(x) - g(x_j)| < \frac{\varepsilon}{8NM}. \text{ --------------- (2)}$$

Take a partition $P: a = y_0 < y_1 < \cdots < y_s = b$ with $\|P\| < \delta$. Take a Riemann Stieltjes sum with respect to the partition *P*,

$$S(f,g,P) = \sum_{k=1}^{s} f(z_k)(g(y_k) - g(y_{k-1})) .$$

Form the common refinement $Q1$ of $P1$ and *P*. The partition $P1$ is given by $\left\{ I_j = (x_{j-1}, x_j) : 1 \le j \le N \right\}$ and the partition *P* is given by $\left\{ J_k = (y_{k-1}, y_k) : 1 \le k \le s \right\}$.

Then the common refinement $Q1$ is given by $\left\{ I_j \cap J_k : 1 \le j \le N, 1 \le k \le s \right\}$. If $I_j \cap J_k \ne \varnothing$, let $I_j \cap J_k = (a_{jk}, b_{jk})$. Take a Riemann Stieltjes sum with respect to the refinement $Q1$,

$$S(f,g,Q1) = \sum_{j,k} f(z_{jk})(g(b_{jk}) - g(a_{jk})) ,$$

where the summation is over *j*, *k* where $I_j \cap J_k \ne \varnothing$, $z_{jk} = z_k$ if $I_j \cap J_k = J_k$,

$z_{jk} = x_{j-1}$ or x_j if $I_j \cap J_k \ne J_k$.

Then $$S(f,g,Q1) - S(f,g,P) = \sum_{j,k} f(z_{jk})(g(b_{jk}) - g(a_{jk})) - \sum_{k=1}^{s} f(z_k)(g(y_k) - g(y_{k-1}))$$

$$= \sum_{j,k} \left(f(z_{jk}) - f(z_k) \right)(g(b_{jk}) - g(a_{jk})) .$$

Note that $f(z_{jk}) - f(z_k) = 0$ if $I_j \cap J_k = J_k$ and the contribution of the terms in the above summation is zero when $I_j \cap J_k = J_k$. Thus, the above sum can have at most $2N$ terms. For each non-zero term, we have $z_{jk} = x_{j-1}$ or x_j and $a_{jk} = x_{j-1}$ or $b_{jk} = x_j$. Then by (2),

$$\left| S(f,g,Q1) - S(f,g,P) \right| \le \sum_{j,k} \left| \left(f(z_{jk}) - f(z_k) \right) \right| \left| g(b_{jk}) - g(a_{jk}) \right|$$

$$\le 2N \frac{\varepsilon}{8NM} 2M = \frac{\varepsilon}{2} .$$

Since $Q1$ is a refinement of $P1$, by (1) and (2),

$$\left| S(f,g,Q1) - DS\int_a^b f dg \right| \le \left| S(f,g,Q1) - S(f,g,P) \right| + \left| S(f,g,Q) - DS\int_a^b f dg \right|$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

It follows that f is Riemann Stieltjes integrable with respect to g and the Riemann Stieltjes integral $\int_a^b fdg = \text{Darboux Stieltjes } DS\int_a^b fdg$.

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