Integrating Using Power Series Expansion

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Introduction. Not all integral has a closed form evaluation. We give some typical examples where power series, or Fourier series is involved.

(1) The integral $\int_0^{\frac{\pi}{4}} \ln(\tan(x))dx$

$$\int_0^{\frac{\pi}{4}} \ln(\tan(x))dx = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

$\int_0^{\frac{\pi}{4}} \ln(\tan(x))dx = \int_0^1 \frac{\ln(t)}{1+t^2}dt$ by using the substitution $t = \tan(x)$.

By Newton's Binomial Theorem, $\frac{1}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n}$ with radius of convergence 1.

Hence, $\frac{\ln(t)}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n t^{2n}\ln(t)$ is convergence for $0 < t < 1$. Obviously when $t = 1$, $\sum_{n=0}^{\infty}(-1)^n t^{2n}\ln(t)$ is 0. Therefore, the series $\sum_{n=0}^{\infty}(-1)^n t^{2n}\ln(t)$ converges pointwise to $\frac{\ln(t)}{1+t^2}$ for $0 < t \leq 1$.

Now $\left|\sum_{n=0}^{k}(-1)^n t^{2n}\ln(t)\right| \leq \sum_{n=0}^{k} t^{2n}\left|\ln(t)\right| \leq \sum_{n=0}^{\infty} t^{2n}\left|\ln(t)\right| = \frac{\left|\ln(t)\right|}{1-t^2}$ for $0 < t < 1$.

Note that for $0 < t < 1$, $\sum_{n=0}^{\infty}(-1)^n t^{2n}\ln(t) \leq \sum_{n=0}^{\infty} t^{2n}\left|\ln(t)\right|$. Observe that the partial sums of $\sum_{n=0}^{\infty} t^{2n}\left|\ln(t)\right|$ is a monotone increasing sequence converging pointwise to $\frac{\left|\ln(t)\right|}{1-t^2}$ on (0, 1). Therefore, by the Lebesgue Monotone Convergence Theorem, $\sum_{n=0}^{\infty}\int_0^1 t^{2n}\left|\ln(t)\right|dt$ converges to $\int_0^1 \frac{\left|\ln(t)\right|}{1-t^2}dt$. Note that $\lim_{t\to 1^-}\frac{\left|\ln(t)\right|}{1-t^2} = \lim_{t\to 1^-}\frac{-\ln(t)}{1-t^2} = -\lim_{t\to 1^-}\frac{\frac{1}{t}}{-2t} = \frac{1}{2}$. Let

$$g(t) = \begin{cases} \dfrac{-\ln(t)}{1-t^2}, & \text{for } 0 < t < 1, \\ \dfrac{1}{2}, & \text{for } t = 1 \end{cases}.$$

Then g is continuous on (0, 1]. Moreover

$\left|\sum_{n=0}^{k}(-1)^n t^{2n}\ln(t)\right| \leq \sum_{n=0}^{k} t^{2n}\left|\ln(t)\right| \leq \sum_{n=0}^{\infty} t^{2n}\left|\ln(t)\right| = \frac{\left|\ln(t)\right|}{1-t^2} \leq g(t)$ for t in (0, 1]. Note that the improper Riemann integral of each $\int_0^1 t^{2n}\left|\ln(t)\right|dt$ is the Lebesgue integral over (0, 1) and the improper Riemann integral $\int_0^1 \frac{|\ln(t)|}{1-t^2}dt$ is the Lebesgue integral of $\frac{|\ln(t)|}{1-t^2}$ over (0, 1). Now

$$\int_s^1 t^{2n}\ln(t)dt = \left[\frac{1}{2n+1}t^{2n+1}\ln(t)\right]_s^1 - \int_s^1 \frac{1}{2n+1}t^{2n}dt = -\frac{1}{2n+1}s^{2n+1}\ln(s) - \left[\frac{t^{2n+1}}{(2n+1)^2}\right]_s^1$$

$$= -\frac{1}{2n+1}s^{2n+1}\ln(s) - \frac{1}{(2n+1)^2} + \frac{s^{2n+1}}{(2n+1)^2} \quad .$$

Therefore,

$$\int_0^1 t^{2n}\ln(t)dt = \lim_{s\to 0^+}\int_s^1 t^{2n}\ln(t)dt = -\frac{1}{(2n+1)^2} - \lim_{s\to 0^+}\frac{1}{2n+1}s^{2n+1}\ln(s) + \lim_{s\to 0^+}\frac{s^{2n+1}}{(2n+1)^2}$$

$$= -\frac{1}{(2n+1)^2} + 0 + 0 = -\frac{1}{(2n+1)^2} .$$

Hence, $\int_0^1 \frac{|\ln(t)|}{1-t^2}dt = -\sum_{n=0}^{\infty}\int_0^1 t^{2n}\ln(t)dt = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. By comparing with $\sum_{n=0}^{\infty}\frac{1}{(2n)^2}$, we note that $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$ is convergent and so $g(t)$ is a Lebesgue integrable function on [0,1]. Hence, $\sum_{n=0}^{\infty}(-1)^n t^{2n}\ln(t)$ is dominated by a Lebesgue integrable function and so by the Lebesgue Dominated Convergence Theorem, $\frac{\ln(t)}{1+t^2}$ is Lebesgue integrable over (0, 1) and the integral

$$\int_0^1 \frac{\ln(t)}{1+t^2}dt = \sum_{n=0}^{\infty}\int_0^1 (-1)^n t^{2n}\ln(t)dt = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} . \qquad \text{----------------} \ (1)$$

It follows that $\int_0^{\frac{\pi}{4}} \ln(\tan(x))dx = \int_0^1 \frac{\ln(t)}{1+t^2}dx = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$.

(2) The integrals $\int_0^{\frac{\pi}{4}} \ln(\sin(x))dx$ **and** $\int_0^{\frac{\pi}{4}} \ln(\cos(x))dx$**.**

(a) $\int_0^{\frac{\pi}{4}} \ln(\sin(x))dx = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \frac{\pi}{4}\ln(2)$,

(b) $\int_0^{\frac{\pi}{4}} \ln(\cos(x))dx = -\frac{\pi}{4}\ln(2) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$.

$$\int_0^{\frac{\pi}{4}} \ln(\sin(x))dx + \int_0^{\frac{\pi}{4}} \ln(\cos(x))dx = \int_0^{\frac{\pi}{4}} \ln(\sin(x)\cos(x))dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{\sin(2x)}{2}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\sin(2x)\right)dx - \int_0^{\frac{\pi}{4}} \ln\left(2\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\sin(x)\right)dx - \frac{\pi}{4}\ln(2)$$

$$= \frac{1}{2}\left(-\frac{\pi}{2}\ln(2)\right) - \frac{\pi}{4}\ln(2) = -\frac{\pi}{2}\ln(2), \quad \text{--------------------------- (1)}$$

since $\int_0^{\frac{\pi}{2}} \ln\left(\sin(x)\right)dx = -\frac{\pi}{2}\ln(2)$.

Now, $\int_0^{\frac{\pi}{4}} \ln(\sin(x))dx - \int_0^{\frac{\pi}{4}} \ln(\cos(x))dx = \int_0^{\frac{\pi}{4}} \ln(\tan(x))dx = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$. ------- (2)

Thus, adding (1) and (2) gives, $2\int_0^{\frac{\pi}{4}} \ln(\sin(x))dx = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \frac{\pi}{2}\ln(2)$ and so

(a) $\int_0^{\frac{\pi}{4}} \ln(\sin(x))dx = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \frac{\pi}{4}\ln(2)$ and

(b) $\int_0^{\frac{\pi}{4}} \ln(\cos(x))dx = -\frac{\pi}{2}\ln(2) - \int_0^{\frac{\pi}{4}} \ln(\sin(x))dx = -\frac{\pi}{2}\ln(2) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} + \frac{\pi}{4}\ln(2)$

$$= -\frac{\pi}{4}\ln(2) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \quad .$$

(3) The integral $\int_0^{\infty}\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}dx$

$$\int_0^{\infty}\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

$$\int_0^{\infty}\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}dx = \int_0^{\frac{\pi}{2}} \ln\left(\tan(y)+\sqrt{1+\tan^2(y)}\right)dy,$$

letting $x = \tan(y)$, so that $dx = (1+x^2)dy$,

$$= \int_0^{\frac{\pi}{2}} \ln\left(\tan(y)+\sec(y)\right)dy = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\sin(y)}{\cos(y)}\right)dy = \int_0^{\frac{\pi}{2}} \ln\left(1+\sin(y)\right)dy - \int_0^{\frac{\pi}{2}} \ln\left(\cos(y)\right)dy$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(1+\sin(y)\right)dy + \frac{\pi}{2}\ln(2) = \int_0^{1} \ln\left(1+\frac{2t}{1+t^2}\right)\frac{2}{1+t^2}dt + \frac{\pi}{2}\ln(2),$$

with $t = \tan\left(\frac{y}{2}\right)$ so that $\sin(y) = \frac{2t}{1+t^2}$ and $dy = \frac{2}{1+t^2}dt$,

$$= 2\int_0^1 \frac{\ln\left((1+t)^2\right)}{1+t^2}\,dt - 2\int_0^1 \frac{\ln\left(1+t^2\right)}{1+t^2}\,dt + \frac{\pi}{2}\ln(2)$$

$$= 4\int_0^1 \frac{\ln\left(1+t\right)}{1+t^2}\,dt - 2\int_0^1 \frac{\ln\left(1+t^2\right)}{1+t^2}\,dt + \frac{\pi}{2}\ln(2)$$

$$= 4\int_0^{\frac{\pi}{4}} \ln\left(1+\tan(u)\right)du - 2\int_0^1 \frac{\ln\left(1+t^2\right)}{1+t^2}\,dt + \frac{\pi}{2}\ln(2)$$

$$= 4\frac{\pi}{8}\ln(2) - 2\int_0^{\frac{\pi}{4}} \ln\left(1+\tan^2(y)\right)dy + \frac{\pi}{2}\ln(2),$$

since $\int_0^{\frac{\pi}{4}} \ln\left(1+\tan(u)\right)du = \frac{\pi}{8}\ln(2)$ (see **(5)** below) and letting $t = \tan(y)$

$$= \pi\ln(2) - 2\int_0^{\frac{\pi}{4}} \ln\left(\sec^2(y)\right)dy = \pi\ln(2) + 4\int_0^{\frac{\pi}{4}} \ln\left(\cos(y)\right)dy$$

$$= \pi\ln(2) + 4\left(-\frac{\pi}{4}\ln(2) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\ .$$

(4) The integral $\int_0^{\infty} \frac{\ln\left(1+\sqrt{1+x^2}\right)}{1+x^2}\,dx$

$$\int_0^{\infty} \frac{\ln\left(1+\sqrt{1+x^2}\right)}{1+x^2}\,dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\,.$$

$$\int_0^{\infty} \frac{\ln\left(1+\sqrt{1+x^2}\right)}{1+x^2}\,dx = \int_0^{\frac{\pi}{2}} \ln\left(1+\sqrt{1+\tan^2(y)}\right)dy\,,$$

letting $x = \tan(y)$, so that $dx = (1+x^2)dy$,

$$= \int_0^{\frac{\pi}{2}} \ln\left(1+\sec(y)\right)dy = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\cos(y)}{\cos(y)}\right)dy = \int_0^{\frac{\pi}{2}} \ln\left(1+\cos(y)\right)dy - \int_0^{\frac{\pi}{2}} \ln\left(\cos(y)\right)dy$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(1+\cos(y)\right)dy + \frac{\pi}{2}\ln(2) = \int_0^1 \ln\left(1+\frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}\,dt + \frac{\pi}{2}\ln(2)\,,$$

with $t = \tan\left(\frac{y}{2}\right)$ so that $\cos(y) = \frac{1-t^2}{1+t^2}$ and $dy = \frac{2}{1+t^2}\,dt$,

$$= 2\int_0^1 \frac{\ln\left(2\right)}{1+t^2}\,dt - 2\int_0^1 \frac{\ln\left(1+t^2\right)}{1+t^2}\,dt + \frac{\pi}{2}\ln(2)$$

$$= 2\ln(2)\tan^{-1}(1) - 2\int_0^1 \frac{\ln\left(1+t^2\right)}{1+t^2}\,dt + \frac{\pi}{2}\ln(2)$$

$$= \frac{\pi}{2}\ln(2) - 2\int_0^{\frac{\pi}{4}} \ln\left(1+\tan^2(y)\right)dy + \frac{\pi}{2}\ln(2)$$, letting $t = \tan(y)$,

$$= \pi\ln(2) - 2\int_0^{\frac{\pi}{4}} \ln\left(\sec^2(y)\right)dy = \pi\ln(2) + 4\int_0^{\frac{\pi}{4}} \ln\left(\cos(y)\right)dy$$

$$= \pi\ln(2) + 4\left(-\frac{\pi}{4}\ln(2) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \; .$$

(5) The integrals $\int_0^{\frac{\pi}{4}} \ln(1+\tan(x))dx$**,** $\int_0^{\frac{\pi}{4}} \ln(1-\tan(x))dx$**.**

(a) $\int_0^{\frac{\pi}{4}} \ln\left(1+\tan(x)\right)dx = \frac{\pi}{8}\ln(2)$**, (b)** $\int_0^{\frac{\pi}{4}} \ln(1-\tan(x))dx = \frac{\pi}{8}\ln(2) - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$**.**

(a) $\int_0^{\frac{\pi}{4}} \ln(1+\tan(x))dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\tan(\frac{\pi}{4})+\tan(-x)}{1-\tan(\frac{\pi}{4})(\tan(-x)}\right)dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan(x)}{1+\tan(x)}\right)dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan(x)}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(2\right)dx - \int_0^{\frac{\pi}{4}} \ln\left(1+\tan(x)\right)dx \, .$$

Therefore, $\int_0^{\frac{\pi}{4}} \ln\left(1+\tan(x)\right)dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\left(2\right)dx = \frac{\pi}{8}\ln(2)$**.**

(b) $\int_0^{\frac{\pi}{4}} \ln(1-\tan(x))dx = \int_0^{\frac{\pi}{4}} \ln\left(1-\tan\left(\frac{\pi}{4}-x\right)\right)dx = \int_0^{\frac{\pi}{4}} \ln\left(1-\frac{1-\tan(x)}{1+\tan(x)}\right)dx$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2\tan(x)}{1+\tan(x)}\right)dx = \int_0^{\frac{\pi}{4}} \ln(2)dx + \int_0^{\frac{\pi}{4}} \ln\left(\tan(x)\right)dx - \int_0^{\frac{\pi}{4}} \ln\left(1+\tan(x)\right)dx$$

$$= \frac{\pi}{4}\ln(2) - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - \frac{\pi}{8}\ln(2) = \frac{\pi}{8}\ln(2) - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \, .$$

(6) The integrals $\int_0^1 \frac{(\ln(x))^2}{1+x^2}dx$**,** $\int_0^{\infty} \frac{(\ln(x))^2}{1+x^2}dx$**,** and $\int_0^{\frac{\pi}{4}} \left(\ln(\tan(x))\right)^2 dx$.

(a) $\int_0^1 \frac{(\ln(x))^2}{1+x^2}dt = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$, **(b)** $\int_0^{\infty} \frac{(\ln(x))^2}{1+x^2}dx = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$,

(c) $\int_0^{\frac{\pi}{4}} \left(\ln(\tan(x))\right)^2 dx = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$

(a) and (b)

By a change of variable, $x = \tan(t)$, $\int_0^{\infty} \frac{\left(\ln(x)\right)^2}{1+x^2} dx = \int_0^{\frac{\pi}{2}} \left(\ln(\tan(x))\right)^2 dx$.

Now, $\int_0^{\infty} \frac{\left(\ln(x)\right)^2}{1+x^2} dx = \int_0^{1} \frac{\left(\ln(x)\right)^2}{1+x^2} dx + \int_1^{\infty} \frac{\left(\ln(x)\right)^2}{1+x^2} dx$. By a change of variable $y = \frac{1}{x}$,

$\int_1^{\infty} \frac{\left(\ln(x)\right)^2}{1+x^2} dx = -\int_1^{0} \frac{\left(\ln(\frac{1}{y})\right)^2}{1+\left(\frac{1}{y}\right)^2} \frac{1}{y^2} dy = \int_0^{1} \frac{\left(\ln(y)\right)^2}{1+y^2} dy$.

Hence, $\int_0^{\infty} \frac{\left(\ln(x)\right)^2}{1+x^2} dx = \int_0^{1} \frac{\left(\ln(x)\right)^2}{1+x^2} dx + \int_1^{\infty} \frac{\left(\ln(y)\right)^2}{1+y^2} dy = 2\int_0^{1} \frac{\left(\ln(x)\right)^2}{1+x^2} dx$. --------- (1)

As with the case of $\int_0^{\infty} \frac{\ln(x)}{1+x^2} dx$, we can show that

$$\int_0^1 \frac{(\ln(t))^2}{1+t^2} dt = \sum_{n=0}^{\infty} \int_0^1 (-1)^n t^{2n} (\ln(t))^2 dt .$$

Now,

$$\int_s^1 t^{2n} (\ln(t))^2 dt = \left[\frac{1}{2n+1} t^{2n+1} (\ln(t))^2 \right]_s^1 - \int_s^1 \frac{1}{2n+1} t^{2n} 2\ln(t) dt$$

$$= -\frac{1}{2n+1} s^{2n+1} (\ln(s))^2 - 2\int_s^1 \frac{1}{2n+1} t^{2n} \ln(t) dt$$

$$= -\frac{1}{2n+1} s^{2n+1} (\ln(s))^2 - 2\left[\frac{1}{(2n+1)^2} t^{2n+1} \ln(t) \right]_s^1 + 2\int_s^1 \frac{1}{(2n+1)^2} t^{2n} dt$$

$$= -\frac{1}{2n+1} s^{2n+1} (\ln(s))^2 + 2\frac{1}{(2n+1)^2} s^{2n+1} \ln(s) + 2\left[\frac{1}{(2n+1)^3} t^{2n+1} \right]_s^1$$

$$= -\frac{1}{2n+1} s^{2n+1} (\ln(s))^2 + 2\frac{1}{(2n+1)^2} s^{2n+1} \ln(s) - 2\frac{1}{(2n+1)^3} s^{2n+1} + \frac{2}{(2n+1)^3} .$$

Therefore,

$$\int_0^1 t^{2n} (\ln(t))^2 dt = \lim_{s \to 0^+} \int_s^1 t^{2n} (\ln(t))^2 dt$$

$$= -\frac{1}{2n+1}\lim_{s\to 0^+} s^{2n+1}(\ln(s))^2 + 2\lim_{s\to 0^+}\frac{1}{(2n+1)^2}s^{2n+1}\ln(s) - 2\lim_{s\to 0^+}\frac{1}{(2n+1)^3}s^{2n+1} + \frac{2}{(2n+1)^3}$$

$$= -\frac{1}{2n+1}\cdot 0 + 2\cdot 0 - 2\cdot 0 + \frac{2}{(2n+1)^3} = \frac{2}{(2n+1)^3}.$$

Hence,

$$\int_0^1 \frac{(\ln(t))^2}{1+t^2}\,dt = \sum_{n=0}^{\infty}\int_0^1 (-1)^n t^{2n}(\ln(t))^2\,dt = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}. \quad \text{------------}(2)$$

It follows from (1) that

$$\int_0^\infty \frac{(\ln(t))^2}{1+t^2}\,dt = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}.$$

(c) By a change of variable, $\int_0^{\frac{\pi}{2}}\left(\ln(\tan(x))\right)^2 dx = \int_0^\infty \frac{\left(\ln(x)\right)^2}{1+x^2}\,dx = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}$. Also,

$$\int_0^{\frac{\pi}{4}}\left(\ln(\tan(x))\right)^2 dx = \int_0^1 \frac{\left(\ln(x)\right)^2}{1+x^2}\,dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}.$$

Note that $\int_0^\infty \frac{\left(\ln(x)\right)^2}{1+x^2}\,dx = \int_0^1 \frac{\left(\ln(x)\right)^2}{1+x^2}\,dx + \int_1^\infty \frac{\left(\ln(x)\right)^2}{1+x^2}\,dx = \int_0^1 \frac{\left(\ln(x)\right)^2}{1+x^2}\,dx - \int_0^1 \frac{\left(\ln(x)\right)^2}{1+x^2}\,dx = 0$.

(7) Use of Known Series and Power Series

(a) $\int_0^{\frac{\pi}{2}}(\ln(\tan(t))^2\,dt = \frac{\pi^3}{8}$, **(b)** $\int_0^1 \frac{(\ln(t))^k}{1+t^2}\,dt = \int_0^{\frac{\pi}{4}}\left(\ln(\tan(x))\right)^k dx = k!(-1)^k\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{k+1}}$,

(c) $\int_0^{\frac{\pi}{4}}\left(\ln(\tan(x))\right)^2 dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{16}$,

(d) $\int_0^{\frac{\pi}{2}}\left(\ln(\sin(x))\right)^2 dx = \int_0^{\frac{\pi}{2}}\left(\ln(\cos(x))\right)^2 dx = \frac{\pi^3}{24} + \frac{\pi}{2}\left(\ln(2)\right)^2$,

(e) $\int_0^{\frac{\pi}{2}}\ln(\sin(x))\ln(\cos(x))dx = \frac{\pi}{2}\left(\ln(2)\right)^2 - \frac{\pi^3}{48}$.

(f) $\int_0^\infty \frac{x}{\sinh(x)}\,dx = \frac{\pi^2}{4}$.

(g) $\int_0^\infty \frac{x}{\cosh(x)}\,dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^2}$.

(h) $\int_0^\infty \frac{x^2}{\cosh(x)}dx = \frac{\pi^3}{8}$

(i) $\int_0^\infty \frac{x^2}{\sinh(x)}dx = \frac{7}{4}\sum_{n=0}^{\infty}\frac{1}{n^3} = \frac{7}{4}\zeta(3)$

(j) $\int_0^\infty \frac{x^k}{\sinh(x)}dx = 2k!\left(1-\frac{1}{2^{k+1}}\right)\zeta(k+1)$**,** where $\zeta(n)$ is the Riemann Zeta function

(k) $\int_0^\infty \frac{(\ln(x))^2}{\sqrt{x}(1-x)^2}dx = 2\pi^2$

(a) The Dirichlet beta function is given by $\beta(k) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^k}$.

Recall the well-known series, $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Now $\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$. Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8} .$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{(2n)^2} + \sum_{n=1}^{\infty}\frac{(-1)^{2n-2}}{(2n-1)^2} = -\sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$= -\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = -\frac{1}{4}\frac{\pi^2}{6} + \frac{\pi^2}{8} = \frac{\pi^2}{12} .$$

We shall make us of the Fourier series of the function, (see [4]),

$$J(x) = \begin{cases} \frac{1}{2}(\pi - x), \ 0 < x < 2\pi, \\ 0, \ x = 0 \end{cases} ,$$

$S(x) = \sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$. $S(x)$ converges uniformly in any closed interval free from multiples of 2π and converges boundedly to $J(x)$. It is well known that for any $0 < s < 2\pi$, the formally integrated series of $S(x)$ converges to the integral of $J(x)$ on $[0,s]$. That is, $\sum_{n=1}^{\infty}\int_0^s \frac{\sin(nx)}{n}dx = \int_0^s J(x)dx = \frac{1}{2}\left[\pi x - \frac{x^2}{2}\right]_0^s = \frac{1}{2}\left(\pi s - \frac{s^2}{2}\right)$ and plainly we can also perform the formal integration over the interval $[s,\pi]$ to give

$$\sum_{n=1}^{\infty}\int_s^{\pi}\frac{\sin(nx)}{n}dx=\int_s^{\pi}J(x)dx=\frac{1}{2}\left[\pi x-\frac{x^2}{2}\right]_s^{\pi}=\frac{\pi^2}{4}-\frac{1}{2}\left(\pi s-\frac{s^2}{2}\right)=\frac{\pi^2+s^2-2s}{4}$$. Hence,

$$\sum_{n=1}^{\infty}\left[-\frac{\cos(nx)}{n^2}\right]_s^{\pi}=\sum_{n=1}^{\infty}\left(\frac{\cos(ns)}{n^2}-\frac{\cos(n\pi)}{n^2}\right)=\frac{(\pi-s)^2}{4}$$. That is,

$$\sum_{n=1}^{\infty}\frac{\cos(ns)}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2+s^2-2s}{4}.$$

Thus, $\sum_{n=1}^{\infty}\frac{\cos(ns)}{n^2}+\frac{\pi^2}{12}=\frac{(\pi-s)^2}{4}$ and so $\sum_{n=1}^{\infty}\frac{\cos(ns)}{n^2}=\frac{(\pi-s)^2}{4}-\frac{\pi^2}{12}$. Note that the series converges to $\frac{(\pi-s)^2}{4}-\frac{\pi^2}{12}$ for any s in $[0, 2\pi]$. Note that the function $\frac{(\pi-s)^2}{4}-\frac{\pi^2}{12}$ is continuous on $[0, 2\pi]$. Therefore, $\sum_{n=1}^{\infty}\frac{\cos(ns)}{n^2}$ is the Fourier series of $\frac{(\pi-s)^2}{4}-\frac{\pi^2}{12}$.

We may now integrate the series $\sum_{n=1}^{\infty}\frac{\cos(ns)}{n^2}$ formally and the integrated series converges to the integral of $\frac{(\pi-s)^2}{4}-\frac{\pi^2}{12}$. That is, for any $0<s<2\pi$,

$$\sum_{n=1}^{\infty}\int_s^{\pi}\frac{\cos(nx)}{n^2}dx=\int_s^{\pi}\left(\frac{(\pi-x)^2}{4}-\frac{\pi^2}{12}\right)dx,$$

Hence,

$$\sum_{n=1}^{\infty}\left[\frac{\sin(nx)}{n^3}\right]_s^{\pi}=-\left[\frac{(\pi-x)^3}{12}\right]_s^{\pi}-\frac{\pi^2}{12}(\pi-s)=\frac{(\pi-s)^3}{12}-\frac{\pi^2}{12}(\pi-s)=\frac{(\pi-s)}{12}\left((\pi-s)^2-\pi^2\right).$$

Thus, $-\sum_{n=1}^{\infty}\frac{\sin(ns)}{n^3}=\frac{(\pi-s)}{12}\left((\pi-s)^2-\pi^2\right)$. Taking $s=\frac{\pi}{2}$, we get

$$-\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^3}=-\frac{\pi}{24}\left(\frac{3\pi^2}{4}\right)=-\frac{\pi^3}{32}.$$

That is,

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^3}=\frac{\pi^3}{32}. \quad \text{-------------------} \quad (1)$$

Hence,

(a) $\int_0^{\infty} \frac{(\ln(t))^2}{1+t^2}\,dt = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = 4\frac{\pi^3}{32} = \frac{\pi^3}{8}$.

Thus, $\int_0^{\frac{\pi}{2}} (\ln(\tan(t))^2\,dt = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{8}$.

(b) Following the method in **(6)**,

$$\int_0^1 \frac{(\ln(t))^k}{1+t^2}\,dt = \sum_{n=0}^{\infty}\int_0^1 (-1)^n t^{2n}(\ln(t))^k\,dt = k!(-1)^k\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{k+1}}.$$

Hence,

$\int_0^{\frac{\pi}{4}} \left(\ln(\tan(x))\right)^k dx = k!(-1)^k\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{k+1}}$. Thus,

(c) $$\int_0^{\frac{\pi}{4}} \left(\ln(\tan(x))\right)^2 dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2+1}} = \frac{\pi^3}{16}$$

(d) $\int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx = \int_0^{\frac{\pi}{2}} \left(\ln(2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)^2 dx$

$$= 2\int_0^{\frac{\pi}{4}} \left(\ln(2\sin(u)\cos(u)\right)^2 du = 2\int_0^{\frac{\pi}{4}} \left(\ln(\sin(2u)\right)^2 du.$$

Now, $\int_0^{\frac{\pi}{2}} \left(\ln(\sin(2u)\right)^2 du = \int_0^{\frac{\pi}{4}} \left(\ln(\sin(2u)\right)^2 du + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\ln(\sin(2u)\right)^2 du$

$$= \int_0^{\frac{\pi}{4}} \left(\ln(\sin(2u)\right)^2 du + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\ln(\sin\left(2\left(\frac{\pi}{2}-u\right)\right)\right)^2 du$$

$$= \int_0^{\frac{\pi}{4}} \left(\ln(\sin(2u)\right)^2 du - \int_{\frac{\pi}{4}}^{0} \left(\ln(\sin(2(y))\right)^2 dy = 2\int_0^{\frac{\pi}{4}} \left(\ln(\sin(2u)\right)^2 du.$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx = \int_0^{\frac{\pi}{2}} \left(\ln(\sin(2x)\right)^2 dx = \int_0^{\frac{\pi}{2}} \left(\ln(2)+\ln(\sin(x))+\ln(\cos(x))\right)^2 dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\ln(2)\right)^2 dx + \int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx + \int_0^{\frac{\pi}{2}} \left(\ln(\cos(x))\right)^2 dx$$

$$+2\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ln(\cos(x))dx + 2\ln(2)\int_0^{\frac{\pi}{2}} \ln(\sin(x))dx + 2\ln(2)\int_0^{\frac{\pi}{2}} \ln(\cos(x))dx$$

$$= \left(\ln(2)\right)^2 \frac{\pi}{2} + 2\int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx + 2\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ln(\cos(x))dx$$

$$-\pi\left(\ln(2)\right)^2 - \pi\left(\ln(2)\right)^2,$$

since $\int_0^{\frac{\pi}{2}} \ln(\sin(x))dx = \int_0^{\frac{\pi}{2}} \ln(\cos(x))dx = -\frac{\pi}{2}\ln(2)$.   .

Thus,

$$\int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx + 2\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ln(\cos(x))dx = \frac{3\pi}{2}\left(\ln(2)\right)^2 \quad \text{------------ (2)}$$

We also have,

$$\int_0^{\frac{\pi}{2}} \left(\ln(\tan(x))\right)^2 dx = \int_0^{\frac{\pi}{2}} \left(\ln(\sin(x)) - \ln(\cos(x))\right)^2 dx$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx - 2\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ln(\cos(x))dx .$$

Thus, by part **(a)**,

$$2\int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx - 2\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ln(\cos(x))dx = \frac{\pi^3}{8} . \quad \text{--------------- (3)}$$

From (2) and (3) we have, $3\int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx = \frac{\pi^3}{8} + \frac{3\pi}{2}\left(\ln(2)\right)^2$ giving

(d) $\int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx = \int_0^{\frac{\pi}{2}} \left(\ln(\cos(x))\right)^2 dx = \frac{\pi^3}{24} + \frac{\pi}{2}\left(\ln(2)\right)^2$ and

(e) $\int_0^{\frac{\pi}{2}} \ln(\sin(x))\ln(\cos(x))dx = \int_0^{\frac{\pi}{2}} \left(\ln(\sin(x))\right)^2 dx - \frac{\pi^3}{16} = \frac{\pi^3}{24} + \frac{\pi}{2}\left(\ln(2)\right)^2 - \frac{\pi^3}{16}$

$$= \frac{\pi}{2}\left(\ln(2)\right)^2 - \frac{\pi^3}{48} .$$

(f) $\int_0^{\infty} \frac{x}{\sinh(x)} dx = \int_0^{\infty} \frac{2x}{e^x - e^{-x}} dx = 2\int_0^{\infty} \frac{xe^{-x}}{1-e^{-2x}} dx$. We note that $\int_1^{\infty} \frac{xe^{-x}}{1-e^{-2x}}dx$ is convergent since $\frac{xe^{-x}}{1-e^{-2x}}$ is non-negative, $\frac{xe^{-x}}{1-e^{-2x}} \le \frac{xe^{-x}}{1-e^{-2}}$ for $x \ge 1, \int_1^{\infty} xe^{-x} = \left[-xe^{-x}\right]_1^{\infty} + \int_1^{\infty} e^{-x}dx = 2e^{-1}$ and $\frac{xe^{-x}}{1-e^{-2x}}$ is continuous on [0, 1] with its value 0 taken to be $\frac{1}{2}$. Furthermore, for $x > 0$, $\frac{xe^{-x}}{1-e^{-2x}} = xe^{-x}\sum_{n=0}^{\infty} e^{-2nx}$ and so the partial sums of the series $\sum_{n=0}^{\infty} xe^{-x}e^{-2nx}$ converges pointwise to $\frac{xe^{-x}}{1-e^{-2x}}$. Since each term $xe^{-x}e^{-2nx}$ is non-negative and improperly Riemann integrable on [0, ∞) and so is Lebesgue integrable on [0, ∞), by the Lebesgue Monotone Convergence Theorem,

$$2\int_0^{\infty} \frac{xe^{-x}}{1-e^{-2x}} dx = 2\sum_{n=0}^{\infty} \int_0^{\infty} xe^{-x}e^{-2nx}dx .$$

Note that

$$\int_0^\infty xe^{-x}e^{-2nx}dx = \int_0^\infty xe^{-(2n+1)x}dx = \left[\frac{-x}{(1+2n)}e^{-(2n+1)x}\right]_0^\infty + \int_0^\infty \frac{1}{(1+2n)}e^{-(2n+1)x}dx$$

$$= 0 + \int_0^\infty \frac{1}{(1+2n)}e^{-(2n+1)x}dx = \left[\frac{-1}{(1+2n)^2}e^{-(2n+1)x}\right]_0^\infty = \frac{1}{(1+2n)^2}.$$

Therefore, $\int_0^\infty \frac{x}{\sinh(x)}dx = 2\int_0^\infty \frac{xe^{-x}}{1-e^{-2x}}dx = 2\sum_{n=0}^\infty \frac{1}{(1+2n)^2} = 2\frac{\pi^2}{8} = \frac{\pi^2}{4}$.

(g) Similarly, $\int_0^\infty \frac{x}{\cosh(x)}dx = 2\int_0^\infty \frac{xe^{-x}}{1+e^{-2x}}dx = 2\sum_{n=0}^\infty \frac{(-1)^n}{(1+2n)^2}$.

(h) $\int_0^\infty \frac{x^2}{\cosh(x)}dx = 2\int_0^\infty \frac{x^2e^{-x}}{1+e^{-2x}}dx = 4\sum_{n=0}^\infty \frac{(-1)^n}{(1+2n)^3} = 4\frac{\pi^3}{32} = \frac{\pi^3}{8}$

(i) $\int_0^\infty \frac{x^2}{\sinh(x)}dx = 2\int_0^\infty \frac{x^2e^{-x}}{1-e^{-x}}dx = 4\sum_{n=0}^\infty \frac{1}{(1+2n)^3} = \frac{7}{4}\sum_{n=0}^\infty \frac{1}{n^3}$

(j) $\int_0^\infty \frac{x^k}{\sinh(x)}dx = 2\int_0^\infty \frac{x^ke^{-x}}{1-e^{-x}}dx = 2k!\left(1-\frac{1}{2^{k+1}}\right)\sum_{n=0}^\infty \frac{1}{n^{k+1}} = 2k!\left(1-\frac{1}{2^{k+1}}\right)\zeta(k+1)$, where $\zeta(n)$ is the Riemann Zeta function.

(k) $\int_0^\infty \frac{(\ln(x))^2}{\sqrt{x}(1-x)^2}dx = \int_{-\infty}^\infty \frac{y^2e^{\frac{y}{2}}}{(1-e^y)^2}dy$, by using the substitution $y=\ln(x)$,

$$= \int_{-\infty}^\infty \frac{y^2e^{-\frac{y}{2}}}{(e^{\frac{y}{2}}-e^{-\frac{y}{2}})^2}dy = \frac{1}{4}\int_{-\infty}^\infty \frac{y^2e^{-\frac{y}{2}}}{(\sinh(\frac{y}{2}))^2}dy = \frac{1}{4}\int_{-\infty}^\infty y^2\left(\frac{\cosh(\frac{y}{2})-\sinh(\frac{y}{2})}{(\sinh(\frac{y}{2}))^2}\right)dy$$

$$= \frac{1}{4}\int_{-\infty}^\infty y^2\left(\coth(\tfrac{y}{2})\operatorname{csch}(\tfrac{y}{2}) - \frac{1}{\sinh(\frac{y}{2})}\right)dy$$

$$= \frac{1}{4}\int_{-\infty}^\infty y^2\coth(\tfrac{y}{2})\operatorname{csch}(\tfrac{y}{2})dy - \frac{1}{4}\int_{-\infty}^\infty \frac{y^2}{\sinh(\frac{y}{2})}dy$$

$$= \frac{1}{4}\int_{-\infty}^\infty y^2\coth(\tfrac{y}{2})\operatorname{csch}(\tfrac{y}{2})dy - 0,$$

since $\frac{y^2}{\sinh(\frac{y}{2})}$ is an odd function and is integrable over [0, ∞),

$$= \frac{1}{4}\left[-2y^2\operatorname{csch}\left(\tfrac{y}{2}\right)\right]_{-\infty}^\infty + \int_{-\infty}^\infty y\operatorname{csch}(\tfrac{y}{2})dy = 0 + \int_{-\infty}^\infty \frac{y}{\sinh(\frac{y}{2})}dy$$

$$=\int_{-\infty}^{\infty}\frac{4x}{\sinh(x)}dx=8\int_0^{\infty}\frac{x}{\sinh(x)}dx=8\frac{\pi^2}{4}=2\pi^2\,.$$

(8) The integrals,

(a) $\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)-\sin(x)\right)dx=-\frac{\pi}{8}\ln(2)-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$,

(b) $\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)+\sin(x)\right)dx=-\frac{\pi}{8}\ln(2)+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$,

(c) $\int_0^{\frac{\pi}{4}}\ln\left(\sin(x)\right)\ln(\cos(x))dx=\frac{\pi}{4}\left(\ln(2)\right)^2-\frac{\pi^3}{96}$,

(d) $\int_0^{\frac{\pi}{4}}(\ln(\sin(x))^2\,dx$

$$=\frac{1}{\sqrt{2}}+\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\left(\frac{1}{2n+1}\frac{\left(\ln\left(2\right)\right)^2}{2^{n+2}\sqrt{2}}+\frac{1}{(2n+1)^2}\frac{\ln(2)}{2^n\sqrt{2}}+\frac{1}{(2n+1)^3}\frac{1}{2^{n-1}\sqrt{2}}\right),$$

(e) $\int_0^{\frac{\pi}{4}}(\ln(\cos(x))^2\,dx$

$$=1-\frac{1}{\sqrt{2}}+\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\left(-\frac{1}{2n+1}\frac{\left(\ln\left(\sqrt{2}\right)\right)^2}{2^n\sqrt{2}}-\frac{1}{(2n+1)^2}\frac{\ln(2)}{2^n\sqrt{2}}-\frac{1}{2^n\sqrt{2}}\frac{2}{(2n+1)^3}+\frac{2}{(2n+1)^3}\right).$$

(a) We start with the following relation,

$$\int_0^{\frac{\pi}{4}}\ln(1-\tan(x))dx+\int_0^{\frac{\pi}{4}}\ln(1+\tan(x))dx=\frac{\pi}{8}\ln(2)-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+\frac{\pi}{8}\ln(2)$$

$$=\frac{\pi}{4}\ln(2)-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\ .$$

Now,

$$\int_0^{\frac{\pi}{4}}\ln(1-\tan(x))dx+\int_0^{\frac{\pi}{4}}\ln(1+\tan(x))dx$$

$$=\int_0^{\frac{\pi}{4}}\ln\left(\frac{\cos(x)-\sin(x)}{\cos(x)}\right)dx+\int_0^{\frac{\pi}{4}}\ln\left(\frac{\cos(x)+\sin(x)}{\cos(x)}\right)dx$$

$$=\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)-\sin(x)\right)dx+\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)+\sin(x)\right)dx-2\int_0^{\frac{\pi}{4}}\ln(\cos(x))dx$$

$$=\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)-\sin(x)\right)dx+\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)+\sin(x)\right)dx-2\left(-\frac{\pi}{4}\ln(2)+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\right)$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\cos(x) - \sin(x)\right) dx + \int_0^{\frac{\pi}{4}} \ln\left(\cos(x) + \sin(x)\right) dx + \frac{\pi}{2}\ln(2) - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

Let $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$. We have,

$$\int_0^{\frac{\pi}{4}} \ln\left(\cos(x) - \sin(x)\right) dx + \int_0^{\frac{\pi}{4}} \ln\left(\cos(x) + \sin(x)\right) dx = \frac{\pi}{4}\ln(2) - G - \frac{\pi}{2}\ln(2) + G$$

$$= -\frac{\pi}{4}\ln(2). \qquad \text{-- (1)}$$

Note we can derive (1) as follows too.

$$\int_0^{\frac{\pi}{4}} \ln\left(\cos(x) - \sin(x)\right) dx + \int_0^{\frac{\pi}{4}} \ln\left(\cos(x) + \sin(x)\right) dx = \int_0^{\frac{\pi}{4}} \ln\left(\cos^2(x) - \sin^2(x)\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\cos(2x)\right) dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\cos(y)\right) dy = \frac{1}{2}\left(-\frac{\pi}{2}\ln(2)\right) = -\frac{\pi}{4}\ln(2).$$

Also,

$$\int_0^{\frac{\pi}{4}} \ln(1-\tan(x))dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan(x))dx = \frac{\pi}{8}\ln(2) - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} - \frac{\pi}{8}\ln(2) = -G.$$

Hence,

$$\int_0^{\frac{\pi}{4}} \ln\left(\cos(x) - \sin(x)\right) dx - \int_0^{\frac{\pi}{4}} \ln\left(\cos(x) + \sin(x)\right) dx = -G. \qquad \text{-------------- (2)}$$

Adding (1) and (2) gives,

$$2\int_0^{\frac{\pi}{4}} \ln\left(\cos(x) - \sin(x)\right) dx = -G - \frac{\pi}{4}\ln(2).$$

and so,

(a) $$\int_0^{\frac{\pi}{4}} \ln\left(\cos(x) - \sin(x)\right) dx = -\frac{G}{2} - \frac{\pi}{8}\ln(2) = -\frac{\pi}{8}\ln(2) - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

(1) – (2) gives,

$$2\int_0^{\frac{\pi}{4}} \ln\left(\cos(x) + \sin(x)\right) dx = -\frac{\pi}{4}\ln(2) + G.$$

It follows that,

(b) $$\int_0^{\frac{\pi}{4}} \ln\left(\cos(x) + \sin(x)\right) dx = -\frac{\pi}{8}\ln(2) + \frac{G}{2} = -\frac{\pi}{8}\ln(2) + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

$$\int_0^{\frac{\pi}{4}}\left(\ln\left(\tan(x)\right)\right)^2 dx=\int_0^{\frac{\pi}{4}}\left(\ln\left(\sin(x)\right)-\ln(\cos(x)\right)^2 dx$$

$$=\int_0^{\frac{\pi}{4}}\left(\ln\left(\sin(x)\right)\right)^2 dx+\int_0^{\frac{\pi}{4}}\left(\ln\left(\cos(x)\right)\right)^2 dx-2\int_0^{\frac{\pi}{4}}\ln\left(\sin(x)\right)\ln(\cos(x))dx \quad \text{-------------} \ (3)$$

Now, using the substitution $x=\frac{\pi}{2}-y$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\ln\left(\sin(x)\right)\right)^2 dx=-\int_{\frac{\pi}{4}}^{0}\left(\ln\left(\sin(\tfrac{\pi}{2}-y)\right)\right)^2 dy=\int_0^{\frac{\pi}{4}}\left(\ln\left(\cos(y)\right)\right)^2 dy$$ so that

$$\int_0^{\frac{\pi}{4}}\left(\ln\left(\sin(x)\right)\right)^2 dx+\int_0^{\frac{\pi}{4}}\left(\ln\left(\cos(x)\right)\right)^2 dx=\int_0^{\frac{\pi}{4}}\left(\ln\left(\sin(x)\right)\right)^2 dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\ln\left(\sin(x)\right)\right)^2 dx$$

$$=\int_0^{\frac{\pi}{2}}\left(\ln\left(\sin(x)\right)\right)^2 dx=\frac{\pi^3}{24}+\frac{\pi}{2}\left(\ln(2)\right)^2 .$$

Hence, it follows from (3) that

$$\frac{\pi^3}{16}=\int_0^{\frac{\pi}{4}}\left(\ln(\tan(x))\right)^2 dx=\frac{\pi^3}{24}+\frac{\pi}{2}\left(\ln(2)\right)^2-2\int_0^{\frac{\pi}{4}}\ln\left(\sin(x)\right)\ln(\cos(x))dx .$$

Therefore,

$$2\int_0^{\frac{\pi}{4}}\ln\left(\sin(x)\right)\ln(\cos(x))dx=\frac{\pi^3}{24}+\frac{\pi}{2}\left(\ln(2)\right)^2-\frac{\pi^3}{16}=-\frac{\pi^3}{48}+\frac{\pi}{2}\left(\ln(2)\right)^2$$, so that

(c) $\int_0^{\frac{\pi}{4}}\ln\left(\sin(x)\right)\ln(\cos(x))dx=\frac{\pi}{4}\left(\ln(2)\right)^2-\frac{\pi^3}{96}$.

Alternatively, since

$$\int_0^{\frac{\pi}{4}}\ln\left(\sin(x)\right)\ln(\cos(x))dx=-\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\ln\left(\sin\left(\frac{\pi}{2}-y\right)\right)\ln\left(\cos\left(\frac{\pi}{2}-y\right)\right)dx ,$$

using the substitution, $y=\frac{\pi}{2}-x$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\left(\sin(y)\right)\ln(\cos(y))dy=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\left(\sin(x)\right)\ln(\cos(x))dx .$$

$$\int_0^{\frac{\pi}{4}}\ln\left(\sin(x)\right)\ln(\cos(x))dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\left(\sin(x)\right)\ln(\cos(x))dx=\frac{\pi}{4}\left(\ln(2)\right)^2-\frac{\pi^3}{96} .$$

(d) For the integral $\int_0^{\frac{\pi}{4}}(\ln(\sin(x))^2 dx$, we shall use a power series approach.

$$\int_0^{\frac{\pi}{4}}(\ln(\sin(x))^2 dx=\int_0^{\sin(\frac{\pi}{4})}\frac{(\ln(t)^2}{\sqrt{1-t^2}}dt \quad \text{with } t=\sin(x) .$$

We shall see if we can use a power series to determine this integral.

Newton's Binomial Series expansion for $(1-x^2)^{-\frac{1}{2}}$([2] Theorem 19, Chapter 9) gives:

$$(1-x^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}(-x^2)^k \quad \text{for} \quad |x^2|<1\ ,$$

$$=1+\sum_{k=1}^{\infty}\frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-(k-1))}{1\cdot 2\cdot 3\cdots k}x^{2k}(-1)^k$$

$$=1+\sum_{k=1}^{\infty}\frac{\frac{1}{2}(\frac{3}{2})(\frac{5}{2})\cdots(\frac{2k-1}{2})}{1\cdot 2\cdot 3\cdots k}x^{2k} = 1+\sum_{k=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}x^{2k}\ .$$

Thus, this is a power series of radius of convergence 1 as

$$\lim_{k\to\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k+1}{2k+2}\Big/\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k} = \lim_{k\to\infty}\frac{2k+1}{2k}=1\ .$$

Therefore, by Theorem 11 Chapter 8, for any K such that $0 < K < 1$,

$$1+\sum_{k=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}x^{2k}$$

converges uniformly to $(1-x^2)^{-\frac{1}{2}}$ on $[-K, K]$.

By the Lebesgue Monotone Convergence Theorem and since each term of the series $\left(\ln(t)\right)^2+\sum_{k=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2k-1}{2k}t^{2k}\left(\ln(t)\right)^2$ is non-negative and the series converges pointwise on (0, 1) to a Lebesgue integrable function $\frac{(\ln(t)^2}{\sqrt{1-t^2}}$ on (0 1), we may thus integrate $\frac{(\ln(t)^2}{\sqrt{1-t^2}}$ term by term from 0 to $\frac{1}{\sqrt{2}}$. Note that the Lebesgue integral of each term is just the improper integral.

Now we evaluate the relevant integrals of $t^{2n}(\ln(t)^2$.

$$\int_s^{\frac{1}{\sqrt{2}}} t^{2n}(\ln(t))^2\,dt = \left[\frac{1}{2n+1}t^{2n+1}(\ln(t))^2\right]_s^{\frac{1}{\sqrt{2}}} - \int_s^{\frac{1}{\sqrt{2}}}\frac{1}{2n+1}t^{2n}\,2\ln(t)\,dt$$

$$=\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2-\frac{1}{2n+1}s^{2n+1}(\ln(s))^2-2\int_s^{\frac{1}{\sqrt{2}}}\frac{1}{2n+1}t^{2n}\ln(t)\,dt$$

$$=\frac{1}{2n+1}\frac{1}{2^{n+2}\sqrt{2}}\left(\ln(2)\right)^2-\frac{1}{2n+1}s^{2n+1}(\ln(s))^2-2\int_s^{\frac{1}{\sqrt{2}}}\frac{1}{2n+1}t^{2n}\ln(t)\,dt$$

$$=\frac{1}{2n+1}\frac{1}{2^{n+2}\sqrt{2}}\left(\ln(2)\right)^2-\frac{1}{2n+1}s^{2n+1}(\ln(s))^2-2\left[\frac{1}{(2n+1)^2}t^{2n+1}\ln(t)\right]_s^{\frac{1}{\sqrt{2}}}+2\int_s^{\frac{1}{\sqrt{2}}}\frac{1}{(2n+1)^2}t^{2n}\,dt$$

$$=\frac{1}{2n+1}\frac{1}{2^{n+2}\sqrt{2}}\left(\ln(2)\right)^2-\frac{1}{2n+1}s^{2n+1}(\ln(s))^2$$

$$+\frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2)+\frac{2}{(2n+1)^2}s^{2n+1}\ln(s)+2\int_s^{\frac{1}{\sqrt{2}}}\frac{1}{(2n+1)^2}t^{2n}dt$$

$$=\frac{1}{2n+1}\frac{1}{2^{n+2}\sqrt{2}}\left(\ln\left(2\right)\right)^2-\frac{1}{2n+1}s^{2n+1}(\ln(s))^2+\frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2)+\frac{2}{(2n+1)^2}s^{2n+1}\ln(s)$$

$$+\left[\frac{2}{(2n+1)^3}t^{2n+1}\right]_s^{\frac{1}{\sqrt{2}}}$$

$$=\frac{1}{2n+1}\frac{1}{2^{n+2}\sqrt{2}}\left(\ln\left(2\right)\right)^2+\frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2)+\frac{2}{(2n+1)^3}\frac{1}{2^n\sqrt{2}}$$

$$-\frac{1}{2n+1}s^{2n+1}(\ln(s))^2+\frac{2}{(2n+1)^2}s^{2n+1}\ln(s)-\frac{2}{(2n+1)^3}s^{2n+1}$$

Therefore,

$$\int_0^{\frac{1}{\sqrt{2}}}t^{2n}(\ln(t))^2dt=\lim_{s\to0^+}\int_s^{\frac{1}{\sqrt{2}}}t^{2n}(\ln(t))^2dt=\frac{1}{2n+1}\frac{1}{2^{n+2}\sqrt{2}}\left(\ln\left(2\right)\right)^2+\frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2)+\frac{1}{(2n+1)^3}\frac{1}{2^{n-1}\sqrt{2}}$$

Hence,

(d) $$\int_0^{\frac{\pi}{4}}(\ln(\sin(x))^2dx=\int_0^{\sin(\frac{\pi}{4})}\frac{(\ln(t))^2}{\sqrt{1-t^2}}dt=\frac{1}{\sqrt{2}}+\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\int_0^{\sin(\frac{\pi}{4})}(\ln(t)^2t^{2n}dt$$

$$=\frac{1}{\sqrt{2}}+\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\left(\frac{1}{2n+1}\frac{1}{2^{n+2}\sqrt{2}}\left(\ln\left(2\right)\right)^2+\frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2)+\frac{1}{(2n+1)^3}\frac{1}{2^{n-1}\sqrt{2}}\right).$$

Similarly,

$$\int_0^{\frac{\pi}{4}}(\ln(\cos(x))^2dx=-\int_1^{\cos(\frac{\pi}{4})}\frac{(\ln(t))^2}{\sqrt{1-t^2}}dt=\int_{\cos(\frac{\pi}{4})1}^{1}\frac{(\ln(t))^2}{\sqrt{1-t^2}}dt=1-\frac{1}{\sqrt{2}}+\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\int_{\cos(\frac{\pi}{4})}^{1}(\ln(t)^2t^{2n}dt$$

$$\int_{\frac{1}{\sqrt{2}}}^{1}t^{2n}(\ln(t))^2dt=\left[\frac{1}{2n+1}t^{2n+1}(\ln(t))^2\right]_{\frac{1}{\sqrt{2}}}^{1}-\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{2n+1}t^{2n}2\ln(t)dt$$

$$=-\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2-2\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{2n+1}t^{2n}\ln(t)dt$$

$$=-\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2-2\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{2n+1}t^{2n}\ln(t)dt$$

$$=-\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2-2\left[\frac{1}{(2n+1)^2}t^{2n+1}\ln(t)\right]_{\frac{1}{\sqrt{2}}}^{1}+2\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{(2n+1)^2}t^{2n}dt$$

$$=-\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2-\frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2)+2\int_{\frac{1}{\sqrt{2}}}^{1}\frac{1}{(2n+1)^2}t^{2n}dt$$

$$= -\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2 - \frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2) + \left[\frac{2}{(2n+1)^3}t^{2n+1}\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= -\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2 - \frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2) + \frac{2}{(2n+1)^3}\left(1-\frac{1}{2^n\sqrt{2}}\right)$$

$$= -\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2 - \frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2) - \frac{1}{2^n\sqrt{2}}\frac{2}{(2n+1)^3} + \frac{2}{(2n+1)^3}$$

Therefore,

$$\int_{\frac{1}{\sqrt{2}}}^{1} t^{2n}(\ln(t))^2\,dt = -\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2 - \frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2) - \frac{1}{2^n\sqrt{2}}\frac{2}{(2n+1)^3} + \frac{2}{(2n+1)^3}.$$

Thus,

(e) $\int_0^{\frac{\pi}{4}}(\ln(\cos(x))^2\,dx = 1-\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\int_{\cos(\frac{\pi}{4})}^{1}(\ln(t)^2 t^{2n}\,dt$

$$=1-\frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\left(-\frac{1}{2n+1}\frac{1}{2^n\sqrt{2}}\left(\ln\left(\sqrt{2}\right)\right)^2 - \frac{1}{(2n+1)^2}\frac{1}{2^n\sqrt{2}}\ln(2) - \frac{1}{2^n\sqrt{2}}\frac{2}{(2n+1)^3} + \frac{2}{(2n+1)^3}\right)$$

Therefore,

$$\int_0^{\frac{\pi}{4}}(\ln(\cos(x))^2\,dx + \int_0^{\frac{\pi}{4}}(\ln(\sin(x))^2\,dx$$

$$=1+\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\frac{2}{(2n+1)^3} = \int_0^{\frac{\pi}{2}}(\ln(\sin(x))^2\,dx = \frac{\pi^3}{24}+\frac{\pi}{2}\left(\ln(2)\right)^2 \;.$$

Thus, we have the following identity,

$$1+\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\frac{2}{(2n+1)^3} = \frac{\pi^3}{24}+\frac{\pi}{2}\left(\ln(2)\right)^2 \;.$$

Use of the Following Fourier series

Recall from Theorem 11 of "*Riemann Summable Everywhere Series*, *Two Special Cosine Series And Abel Summable series*" that the series

$$\cos(\theta)+\frac{\cos(2\theta)}{2}+\frac{\cos(3\theta)}{3}\cdots=\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n}$$

is a Fourier series which converges pointwise and dominatedly to the continuous function $\ln\left(\frac{1}{2\sin\left(\frac{\theta}{2}\right)}\right)$ for θ in $(0, 2\pi)$ in the L^1 norm.

That is, $\ln\left(\sin\left(\frac{\theta}{2}\right)\right) = -\ln(2) - \sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n}$ for θ in $(0, 2\pi)$. ----------- (C)

(9) (a) $\int_0^{\frac{\pi}{2}}\frac{x^2}{\sin(x)}dx$ **,** $\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx$**,** $\int_0^{\frac{\pi}{4}}x\ln(\sin(x))dx$ **(b)** $\int_0^{\frac{\pi}{2}}\sin(x)\ln(\sin(x))dx$ **(c)** $\int_0^{\frac{\pi}{2}}\sin^2(x)\ln(\sin(x))dx$

(a) $\int_0^{\frac{\pi}{2}}\frac{x^2}{\sin(x)}dx = \left[\left(\ln(1-\cos(x)) - \ln(\sin(x))\right)x^2\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}2x\left(\ln(1-\cos(x)) - \ln(\sin(x))\right)dx$

$= 0 + 2\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx - 2\int_0^{\frac{\pi}{2}}x\left(\ln(1-\cos(x))\right)dx$

$= 2\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx - 2\int_0^{\frac{\pi}{2}}x\ln\left(2\sin^2\left(\frac{x}{2}\right)\right)dx$

$= 2\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx - 2\ln(2)\int_0^{\frac{\pi}{2}}xdx - 4\int_0^{\frac{\pi}{2}}x\ln\left(\sin\left(\frac{x}{2}\right)\right)dx$

$= 2\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx - \ln(2)\frac{\pi^2}{4} - 4\int_0^{\frac{\pi}{2}}x\ln\left(\sin\left(\frac{x}{2}\right)\right)dx$. ------------------- (1)

From (C), we get

$$\ln\left(\sin\left(x\right)\right) = -\ln(2) - \sum_{n=1}^{\infty}\frac{\cos(2nx)}{n} \quad \text{---------------- (2)}$$

so that by the Lebesgue Dominated Convergence Theorem,

$\int_0^{\frac{\pi}{2}}x\ln(\sin(x))dx = -\ln(2)\int_0^{\frac{\pi}{2}}xdx - \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}}x\cos(2nx)dx$

$= -\ln(2)\frac{\pi^2}{8} - \sum_{n=1}^{\infty}\frac{1}{n}\left\{\left[x\frac{\sin(2nx)}{2n}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\frac{\sin(2nx)}{2n}dx\right\}$

$= -\ln(2)\frac{\pi^2}{8} - \sum_{n=1}^{\infty}\frac{1}{n}\left\{0 - \frac{1}{2n}\left[-\frac{\cos(2nx)}{2n}\right]_0^{\frac{\pi}{2}}\right\} = -\frac{1}{8}\ln(2)\pi^2 - \sum_{n=1}^{\infty}\frac{1}{4n^3}\left\{\cos(n\pi) - 1\right\}$

$= -\frac{1}{8}\ln(2)\pi^2 - \sum_{n=1}^{\infty}\frac{1}{4n^3}(-1)^n + \sum_{n=1}^{\infty}\frac{1}{4n^3}$

$$=-\frac{1}{8}\ln(2)\pi^2+\frac{1}{4}\eta(3)+\frac{1}{4}\zeta(3) \text{ ------------------------------------ } (2)$$

$$=\frac{1}{4}\zeta(3)-\frac{1}{8}\ln(2)\pi^2+\frac{1}{4}(1-2^{-2})\zeta(3)=\frac{7}{16}\zeta(3)-\frac{1}{8}\ln(2)\pi^2 .$$

$$\int_0^{\frac{\pi}{2}} x\ln\left(\sin\left(\frac{x}{2}\right)\right)dx=-\ln(2)\int_0^{\frac{\pi}{2}} xdx-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} x\cos(nx)dx$$

$$=-\ln(2)\frac{\pi^2}{8}-\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left[x\frac{\sin(nx)}{n}\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{\sin(nx)}{n}dx\right\}$$

$$=-\ln(2)\frac{\pi^2}{8}-\sum_{n=1}^{\infty}\frac{1}{n}\left\{\frac{\pi}{2}\frac{\sin\left(n\frac{\pi}{2}\right)}{n}-\frac{1}{n}\left[-\frac{\cos(nx)}{n}\right]_0^{\frac{\pi}{2}}\right\}$$

$$=-\ln(2)\frac{\pi^2}{8}-\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin\left((2n+1)\frac{\pi}{2}\right)-\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\cos\left(n\frac{\pi}{2}\right)-1\right)$$

$$=-\ln(2)\frac{\pi^2}{8}-\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}(-1)^n-\sum_{n=1}^{\infty}\frac{1}{8n^3}\cos(n\pi)+\sum_{n=1}^{\infty}\frac{1}{n^3}$$

$$=-\ln(2)\frac{\pi^2}{8}-\frac{\pi}{2}G-\sum_{n=1}^{\infty}\frac{1}{8n^3}(-1)^n+\sum_{n=1}^{\infty}\frac{1}{n^3}\text{, where } G=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}(-1)^n ,$$

$$=-\ln(2)\frac{\pi^2}{8}-\frac{\pi}{2}G+\frac{1}{8}\eta(3)+\zeta(3). \text{ -------------------- } (3).$$

It follows from (1) that

$$\int_0^{\frac{\pi}{2}}\frac{x^2}{\sin(x)}dx=-\ln(2)\frac{\pi^2}{4}+2\left(-\frac{1}{8}\ln(2)\pi^2+\frac{1}{4}\eta(3)+\frac{1}{4}\zeta(3)\right)-4\left(-\ln(2)\frac{\pi^2}{8}-\frac{\pi}{2}G+\frac{1}{8}\eta(3)+\zeta(3)\right)$$

$$=2\pi G-\frac{7}{2}\zeta(3).$$

We may use (C) to compute $\int_0^{\frac{\pi}{4}}\ln(\sin(x))dx=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\frac{\pi}{4}\ln(2)$

$$\int_0^{\frac{\pi}{4}}\ln(\sin(x))dx=-\ln(2)\int_0^{\frac{\pi}{4}}dx-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{4}}\cos(2nx)dx$$

$$=-\frac{\pi}{4}\ln(2)-\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\sin(2nx)}{2n}\right]_0^{\frac{\pi}{4}}=-\frac{\pi}{4}\ln(2)-\sum_{n=1}^{\infty}\frac{1}{2n^2}\sin\left(n\frac{\pi}{2}\right)$$

$$=-\frac{\pi}{4}\ln(2)-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin\left((2n+1)\frac{\pi}{2}\right)=-\frac{\pi}{4}\ln(2)-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}(-1)^n .$$

$\int_0^{\frac{\pi}{2}} x\ln\left(\sin\left(\frac{x}{2}\right)\right)dx=4\int_0^{\frac{\pi}{4}} u\ln\left(\sin(u)\right)du$, using substitution $u=\frac{x}{2}$.

Therefore,

$$\int_0^{\frac{\pi}{4}} x\ln(\sin(x))dx=\frac{1}{4}\int_0^{\frac{\pi}{2}} x\ln\left(\sin\left(\frac{x}{2}\right)\right)dx=\frac{1}{4}\left(-\ln(2)\frac{\pi^2}{8}-\frac{\pi}{2}G+\frac{1}{8}\eta(3)+\zeta(3)\right)$$

$$=\frac{1}{32}\eta(3)+\frac{1}{4}\zeta(3)-\ln(2)\frac{\pi^2}{32}-\frac{\pi}{8}G .$$

(b) $\int_0^{\frac{\pi}{2}}\sin(x)\ln(\sin(x))dx=-\int_1^0\ln(\sqrt{1-u^2}du$ **,** using substitution $u=\cos(x)$,

$$=\frac{1}{2}\int_0^1\ln(1-u^2)du=\frac{1}{2}\int_0^1\left(\ln(1-u)+\ln(1+u)\right)du$$

$$=\frac{1}{2}\left(\left[(1+u)\ln(1+u)-1\right]_0^1+\left[-(1-u)\ln(1-u)-1\right]_0^1\right)$$

$$=\frac{1}{2}\left(2\ln(2)-1+0-1\right)=\ln(2)-1\textbf{.}$$

(c) $\int_0^{\frac{\pi}{2}}\sin^2(x)\ln(\sin(x))dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1-\cos(2x)\right)\ln(\sin(x))dx$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sin(x))dx-\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2x)\ln(\sin(x))dx=\frac{1}{2}\left(-\frac{\pi}{2}\ln(2)\right)-\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2x)\ln(\sin(x))dx$$

$$=\frac{1}{2}\left(-\frac{\pi}{2}\ln(2)\right)-\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2x)\left(-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right)dx$$

$$=-\frac{\pi}{4}\ln(2)-\frac{\ln(2)}{2}\int_0^{\frac{\pi}{2}}\cos(2x)dx+\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos(2x)\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}dx$$

$$=-\frac{\pi}{4}\ln(2)-\frac{\ln(2)}{2}\left[\frac{\sin(2x)}{2}\right]_0^{\frac{\pi}{2}}+\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^2(2x)dx+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}}\cos(2x)\cos(2nx)dx$$

$$=-\frac{\pi}{4}\ln(2)-0+\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{2}\left(\cos(4x)+1\right)dx+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}}\cos(2x)\cos(2nx)dx$$

$=-\frac{\pi}{4}\ln(2)+\frac{\pi}{8}$**,** since $\int_0^{\frac{\pi}{2}}\cos(2x)\cos(2nx)dx=0$ for $n>1$.

(10) A recursive formula for the integral $\int_0^{\frac{\pi}{2}}\sin^n(x)\ln(\sin(x))dx$

We note that $\cos(x)$ is absolutely continuous on the interval $\left[0,\frac{\pi}{2}\right]$ and that for $n\geq 2$, the function $\sin^{n-1}(x)\ln(\sin(x))$ is absolutely continuous on $\left(0,\frac{\pi}{2}\right]$. Since for $n\geq 2$ $\lim_{x\to 0^+}\sin^{n-1}(x)\ln(\sin(x))=0$, we may define the function $\sin^{n-1}(x)\ln(\sin(x))$ to take the value 0 at 0. We may thus regard $\sin^{n-1}(x)\ln(\sin(x))$ to be absolutely continuous on the interval $\left[0,\frac{\pi}{2}\right]$ for $n\geq 2$.

If we let $I_n=\int_0^{\frac{\pi}{2}}\sin^n(x)\ln(\sin(x))dx$, then for $n\geq 2$, by integration by parts,

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n(x)\ln(\sin(x))dx=\int_0^{\frac{\pi}{2}}\sin(x)\left(\sin^{n-1}(x)\ln(\sin(x))\right)dx$$

$$=\left[-\cos(x)\sin^{n-1}(x)\ln(\sin(x))\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos(x)\left((n-1)\sin^{n-2}(x)\cos(x)\ln(\sin(x))+\sin^{n-1}(x)\frac{\cos(x)}{\sin(x)}\right)dx$$

$$=0+\int_0^{\frac{\pi}{2}}\left((n-1)\sin^{n-2}(x)\cos^2(x)\ln(\sin(x))+\sin^{n-2}(x)\cos^2(x)\right)dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}(x)\ln(\sin(x))dx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n(x)\cos^2(x)\ln(\sin(x))dx+\int_0^{\frac{\pi}{2}}\sin^{n-2}(x)\cos^2(x)dx$$

$$=(n-1)I_{n-2}-(n-1)I_n+\int_0^{\frac{\pi}{2}}\sin^{n-2}(x)\cos^2(x)dx$$

$$=(n-1)I_{n-2}-(n-1)I_n+\frac{1}{n-1}\int_0^{\frac{\pi}{2}}\sin^n(x)dx. \text{ ------------------ (1)}$$

Hence, $I_n=\frac{n-1}{n}I_{n-2}+\frac{1}{n(n-1)}\int_0^{\frac{\pi}{2}}\sin^n(x)dx$ for $n\geq 2$.

Using (1), we have another way to compute $\int_0^{\frac{\pi}{2}}\sin^2(x)\ln(\sin(x))dx$ without using Fourier series.

$$\int_0^{\frac{\pi}{2}}\sin^2(x)\ln(\sin(x))dx=I_2=\frac{1}{2}I_0+\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2(x)dx$$

$$=\frac{1}{2}\left(-\frac{\pi}{2}\ln(2)\right)+\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{2}\left(1-\cos(2x)\right)dx=-\frac{\pi}{4}\ln(2)+\frac{\pi}{8}-\frac{1}{4}\int_0^{\frac{\pi}{2}}\left(\cos(2x)\right)dx$$

$$=-\frac{\pi}{4}\ln(2)+\frac{\pi}{8}-0=\frac{\pi}{8}-\frac{\pi}{4}\ln(2).$$

Using the formula for $\int_0^{\frac{\pi}{2}}\sin^n(x)dx$,

$$\int_0^{\frac{\pi}{2}} \sin^n(x)dx = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{2}{3}, & n \text{ odd,} \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{1}{2}, & n \text{ even} \end{cases},$$

we may compute, $I_n = \int_0^{\frac{\pi}{2}} \sin^n(x)\ln(\sin(x))dx$ recursively for any $n \geq 2$.

For example,

$$\int_0^{\frac{\pi}{2}} \sin^3(x)\ln(\sin(x))dx = I_3 = \frac{2}{3}I_1 + \frac{1}{6}\int_0^{\frac{\pi}{2}} \sin^3(x)dx$$

$$= \frac{2}{3}\left(\ln(2)-1\right) + \frac{1}{6}\bullet\frac{2}{3} = \frac{2}{3}\ln(2) - \frac{2}{3} + \frac{1}{9} = \frac{2}{3}\ln(2) - \frac{5}{9}.$$

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