Introduction To Measure Theory

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We begin by asking several questions.

Question 1. Let $C[0,1]$ be the collection of all continuous real-valued functions on the unit interval [0,1]. We endow it with the uniform norm, $\|f\| = \sup_{t\in[0,1]} |f(t)|$ for $f \in C[0,1]$. Then $C[0,1]$ is a normed vector space. This gives rise to a metric in $C[0,1]$ associated with this norm. With this metric, $C[0,1]$ is a complete metric space and so is a Banach space. Note that $C[0,1]$ is a real linear space. Observe that the Riemann integral on $C[0,1]$ is a real linear functional on $C[0,1]$. Our question is: What is the conjugate space or space of all real linear functionals on $C[0,1]$?

Question 2. Now we give the linear space $C[0,1]$ a different norm, $\|f\|_1 = \int_0^1 |f(t)|\,dt$ for $f \in C[0,1]$. Then we have the associated metric space $C[0,1]$, where the metric d_1 is now given by

$$d_1(f,g) = \|f-g\|_1 = \int_0^1 |f(t)-g(t)|\,dt .$$

The linear space $C[0,1]$ with this metric is not complete. There is a general construction, which gives a Banach space $\widehat{X}$ from a normed linear space *X*. This is called the *completion* of the vector space *X*. It is the equivalent classes of Cauchy sequences of elements of *X*.

Can we represent the completion $\widehat{C[0,1]}$ of $C[0,1]$ with the norm given above as a function space?

Question 3. Then what is the conjugate space or dual of $\widehat{C[0,1]}$?

Practical Lebesgue integration does everything that Riemann integration does, better and easier and it does more.

For instance:

One aim of integration is to prove theorems like this,

given some hypothesis on the function *f*,

$$\frac{d}{dx}\int f(x,t)\,dt = (?)\int \frac{d}{dx} f(x,t)\,dt .$$

Theorems like this can be reduced to asking:

If $f_n \to f$, then $\int f_n \to \int f$?

So we want to prove theorem of the form:

If $f_n \to f$ in some appropriate sense, then f is integrable (in some appropriate sense) and $\int f_n \to \int f$ (in some appropriate sense).

The basic result for Riemann integration is:

Theorem. If each $f_n : [a,b] \to \mathbb{R}$ is continuous (and therefore Riemann integrable) and $f_n \to f$ uniformly on the closed and bounded interval $[a,b]$ (and consequently f is continuous and so Riemann integrable), then $\int_a^b f_n \to \int_a^b f$.

Most theorems on convergence of Riemann integrals are merely elaborations of this.

The basic theorem for Lebesgue integration is:

Theorem. If E is a measurable subset, $f_n : E \to \mathbb{R}^+$ is Lebesgue integrable and $f_n \nearrow f$ (monotonic increasing) pointwise or almost everywhere on E, and if f_n is Lebesgue integrable, then f is Lebesgue integrable and $\int_E f_n \to \int_E f$.

For example, $\left(1-\frac{t}{n}\right)^n \to e^{-t}$ as $n \to \infty$ and so for integer $m > 0$, $t^m\left(1-\frac{t}{n}\right)^n \to t^m e^{-t}$ as $n \to \infty$. For integer $n \geq 1$, let $f_n(t) = \begin{cases} t^m\left(1-\frac{t}{n}\right)^n, 0 \leq t \leq n \\ 0, \quad t > n \end{cases}$. Then $f_n(t) \to t^m e^{-t}$ pointwise on $[0, \infty)$ and the sequence of functions (f_n) is monotonic increasing. It follows by the above result that $\int_{[0,\infty)} f_n(t)dt = \int_0^n t^m\left(1-\frac{t}{n}\right)^n dt \to \int_{[0,\infty)} t^m e^{-t} dt$.

We verify that $t^m e^{-t}$ is Lebesgue integrable on $[0, \infty)$. Observe that for $t \geq 0$,

$$t^m e^{-t} = 2^m m! \frac{1}{m!}\left(\frac{t}{2}\right)^m e^{-t} \leq 2^m m! e^{\frac{1}{2}t} e^{-t} \leq 2^m m! e^{-\frac{1}{2}t} .$$

Therefore, since $2^m m! e^{-\frac{1}{2}t}$ is non-negative and improperly Riemann integrable on $[0, \infty)$ and so is Lebesgue integrable on $[0, \infty)$, it follows that $t^m e^{-t}$ is Lebesgue integrable on $[0, \infty)$.

Now, $\int_0^n t^m \left(1-\frac{t}{n}\right)^n dt = m!\frac{n}{n+1}\cdot\frac{n}{n+2}\cdots\frac{n}{n+m}\cdot\frac{n}{n+m+1} \to m!$. Therefore, $\int_{[0,\infty)} t^m e^{-t} dt = m!$.

In the sequel, $\overline{\mathbb{R}}$ denotes the extended real numbers $\mathbb{R}\cup\{-\infty,+\infty\}$. Here we note that many propositions are often simpler when the extended non-negative real numbers $\overline{\mathbb{R}^+} = [0,\infty]$ is involved.

Measurable Sets and Functions

Definition 1. Suppose *X* is a set. A *σ-algebra A* in *X* is a collection $\mathscr{M}$ of subsets of *X* such that

(i) $X \in \mathscr{M}$,

(ii) if $A \in \mathscr{M}$, then its complement $A^c \in \mathscr{M}$, and

(iii) if $A_n \in \mathscr{M}$, for *n* = 1, 2, …. , then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{M}$.

If $\mathscr{M}$ is a σ-algebra in *X*, then *X* or $(X, \mathscr{M})$ is called a *measure space* and the elements of $\mathscr{M}$ are called *measurable sets*.

If $(X, \mathscr{M})$ is a measure space and *Y* is a topological space, a function $f: X \to Y$ is said to be *measurable* if $f^{-1}(U) \in \mathscr{M}$ for any open set *U* in *Y*.

Some Immediate Results and Remarks

(1) $\varnothing \in \mathscr{M}$ as $X^c \in \mathscr{M}$.

(2) If $A_1, A_2 \in \mathscr{M}$, then $A_1 \cup A_2 \in \mathscr{M}$.

(3) If $A_n \in \mathscr{M}$, for *n* = 1, 2, …. , then $\bigcap_{n=1}^{\infty} A_n^c = \left(\bigcup_{n=1}^{\infty} A_n\right)^c \in \mathscr{M}$. Therefore, $\mathscr{M}$ is also closed under countable intersection.

(4) If $A, B \in \mathscr{M}$, then $A - B \in \mathscr{M}$ since $A - B = A \cap B^c$.

(5) If $A, B \in \mathscr{M}$, then $A \Delta B = A \cup B - A \cap B \in \mathscr{M}$.

Some Related Definitions

(1) If we replaced (iii) by (iii)* if $A_1, A_2 \in \mathscr{M}$, then $A_1 \cup A_2 \in \mathscr{M}$, then we get an *algebra of sets*.

(2) If we drop (i) and replace (ii) by (ii)* if $A, B \in \mathscr{M}$, then $A - B \in \mathscr{M}$, then $\mathscr{M}$ is a *σ-ring* of subsets of *X*.

(3) If we drop (i) and replace (ii) by (ii)*, (iii) by (iii)*, then $\mathscr{M}$ is a *ring* of subsets of *X*.

Note the following implications.

$$\begin{array}{ccc} \mathscr{M} \text{ is a } \sigma\text{-algebra} & \Rightarrow & \mathscr{M} \text{ is an algebra} \\ \Downarrow & & \Downarrow \\ \mathscr{M} \text{ is a } \sigma\text{-ring} & \Rightarrow & \mathscr{M} \text{ is a ring} \end{array}$$

The prefix '*σ*' is always connected somehow with countable sum operations. For example, a topological space *Y* is *σ-compact* if $Y = \bigcup_{n=1}^{\infty} K_n$ and each K_n is compact.

The following is an easy consequence of the definition.

Proposition 2. If Y and Z are two topological spaces and $(X, \mathscr{M})$ is a measure space and if $f : X \to Y$ is measurable and $g : Y \to Z$ is continuous, then the composition $g \circ f : X \to Z$ is measurable.

Just like the case of continuous functions, forming sum and product of measurable functions is a means of investigating measurable functions from simpler easier defined measurable functions. That we can do so is because sum and product of measurable functions are measurable.

The next result is used to prove that sum and product of measurable functions are measurable.

Lemma 3. Suppose $(X, \mathscr{M})$ is a measure space. Suppose $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$ are measurable functions. Suppose $\Phi : \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \to Y$ is continuous. Define $h : X \to Y$ by $h(x) = \Phi(u(x), v(x))$ for *x* in *X*. Then *h* is measurable. If *X* is a topological space and *u* and *v* are continuous, then *h* is continuous.

Proof. For the last statement concerning continuity, the function $\Gamma : X \to \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ defined by $\Gamma(x) = \big(u(x), v(x)\big)$ for *x* in *X*, is continuous since the projections onto each factor are *u* and *v* and are continuous. Therefore, $h = \Phi \circ \Gamma$ is a composition of continuous functions and so is continuous.

We now show that $\Gamma : X \to \mathbb{R} \times \mathbb{R}$ is measurable.

Suppose I_1 and I_2 are open intervals in $\mathbb{R}$. We shall show that $\Gamma^{-1}(I_1 \times I_2) \in \mathscr{M}$.

$\Gamma^{-1}(I_1 \times I_2) = u^{-1}(I_1) \cap v^{-1}(I_2) \in \mathscr{M}$. This is because *u* and *v* are measurable, $u^{-1}(I_1)$ and $v^{-1}(I_2) \in \mathscr{M}$. Hence, for any open rectangle, Γ^{-1}(open rectangle) is measurable.

Take any open set *U* in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Then *U* is a countable union of open rectangles, say $U = \bigcup_{i=1}^{\infty} U_i$, where each U_i is an open rectangle. Therefore, $\Gamma^{-1}(U) = \Gamma^{-1}\left(\bigcup_{i=1}^{\infty} U_i\right) = \bigcup_{i=1}^{\infty} \Gamma^{-1}(U_i)$ is a countable union of measurable sets in $\mathscr{M}$ and so is in $\mathscr{M}$ as $\mathscr{M}$ is a σ-algebra. Thus, Γ is measurable. Therefore, by Proposition 2, *h* is measurable.

Corollary 4. Suppose $u, v : X \to \mathbb{R}$ or $\mathbb{C}$ are measurable, then $u + v$ and $u \cdot v$ are measurable.

For real valued functions *u* and *v*, using the continuous function $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x, y) = x + y$ or $\Phi(x, y) = x \cdot y$, it follows from Lemma 3, that $u + v$ and $u \cdot v$ are measurable.

For complex functions *u* and *v*, we use the following to reduce the argument to the real case.

Corollary 5. The complex function $f = u + iv : X \to \mathbb{C}$, where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, is measurable if, and only if, both *u* and *v* are measurable.

Proof. Since the projection maps, $\operatorname{Re} : \mathbb{C} \to \mathbb{R}$ and $\operatorname{Im} : \mathbb{C} \to \mathbb{R}$ are continuous, it follows by Lemma 3, that if f is measurable, then *u* and *v* are measurable.

Suppose *u* and *v* are measurable. Define $\Phi : \mathbb{R}^2 \to \mathbb{C}$ by $\Phi(x, y) = x + iy$. Then plainly, Φ is continuous. Then by Lemma 3, $f(x) = \Phi(u(x), v(x))$ is measurable.

Completion of the proof of Corollary 4.

Suppose $u, v : X \to \mathbb{C}$ are measurable. Then by Corollary 5, $\operatorname{Re}(u + v) = \operatorname{Re} u + \operatorname{Re} v$ and $\operatorname{Im}(u + v) = \operatorname{Im} u + \operatorname{Im} v$ are measurable. Hence $u + v$ are measurable. Now $\operatorname{Re}(u \cdot v) = \operatorname{Re} u \cdot \operatorname{Re} v - \operatorname{Im} u \cdot \operatorname{Im} v$ and $\operatorname{Im}(u \cdot v) = \operatorname{Re} u \cdot \operatorname{Im} v + \operatorname{Im} u \cdot \operatorname{Re} v$. As $\operatorname{Re} u \cdot \operatorname{Re} v - \operatorname{Im} u \cdot \operatorname{Im} v$ and $\operatorname{Re} u \cdot \operatorname{Im} v + \operatorname{Im} u \cdot \operatorname{Re} v$ are measurable, it follows that $\operatorname{Re}(u \cdot v)$ and $\operatorname{Im}(u \cdot v)$ are measurable and so $u \cdot v$ is measurable.

An immediate consequence of the definition of measurable function is:

Corollary 6. Suppose $(X, \mathscr{M})$ is a measure space. If *E* is subset of *X*, then the characteristic function of *E*, χ_E , defined by $\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E \end{cases}$, is measurable, if and only if, *E* is measurable, i.e., $E \in \mathscr{M}$.

Corollary 7. Suppose $(X, \mathscr{M})$ is a measure space. A function $f : X \to \mathbb{R}$ or $\mathbb{C}$ is measurable implies that $|f|$ is measurable.

Proof. Since the modulus function $|\ |: \mathbb{R}$ or $\mathbb{C} \to \mathbb{R}$ is continuous, it follows from Proposition 2 that $|f|$ is measurable.

A question naturally arises is that, if we have a collection of subsets of *X*, then can there be a smallest σ-algebra on *X* that contains this collection? The collection of all subsets of *X* is the extreme case it is a σ-algebra that contains all σ-algebra in *X*. The next proposition gives the existence of such a smallest σ-algebra.

Proposition 8. Suppose Ω is a collection of subsets of *X*. Then there is a smallest σ-algebra $\mathscr{M}$ in *X* containing Ω.

Proof. Consider the collection of all σ-algebras containing Ω. Obviously, this collection is not empty as it contains the σ-algebra of all subsets of *X*. Then let

$$\mathscr{M} = \bigcap_{\Psi \text{ is a } \sigma-\text{algebra} \supseteq \Omega} \Psi .$$

This is obviously a σ-algebra contained in any σ-algebra containing Ω and so is the smallest σ-algebra containing Ω.

We call this algebra the *σ-algebra generated by* Ω.

A very important example is when $(X, \mathscr{T})$ is a topological space and $\mathscr{T}$ is its topology. Then the σ-algebra in *X* generated by $\mathscr{T}$ is called the *Borel measure* of *X*, more precisely it is the σ-algebra in *X* generated by the open sets of *X* and the elements in $\mathscr{M}$ are called the *Borel subsets* of *X*.

Suppose $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ are topological spaces. Suppose $f : (X, \mathscr{T}) \to (Y, \mathscr{S})$ is continuous. Let $\mathscr{M}$ be the *σ-algebra* of all *Borel subsets* of $(X, \mathscr{T})$. For any *U* open in $(Y, \mathscr{S})$, i.e., $U \in \mathscr{S}$, $f^{-1}(U)$ is open in $(X, \mathscr{T})$, i.e., $f^{-1}(U) \in \mathscr{T}$, a fortiori, Borel. Thus, f is $\mathscr{M}$-measurable or *Borel measurable*.

Definition 9. Suppose $(X, \mathscr{T})$ is a topological space. A Borel measurable real, or extended real, or complex function is called a *Borel function*.

We have the following criterion for the measurability of a function.

Proposition 10. Suppose $(X, \mathscr{M})$ is a measure space and $f : X \to Y$ is a function.

(1) $\Omega = \{ E \subseteq Y : f^{-1}(E) \in \mathscr{M} \}$ is a σ-algebra.

(2) If Y is a topological space, f is measurable and $E \subseteq Y$ is a Borel subset of *Y*, then $f^{-1}(E) \in \mathscr{M}$.

(3) If $Y = \overline{\mathbb{R}}$, then if $f^{-1}\left((a,\infty]\right) \in \mathscr{M}$ for any $a \in \mathbb{R}$, then f is measurable.

Proof.

(1) (i) As $f^{-1}(Y) = X \in \mathscr{M}$, $Y \in \Omega$.

(ii) If $A \in \Omega$, then $f^{-1}(Y - A) = f^{-1}(Y) - f^{-1}(A) = X - f^{-1}(A) \in \mathscr{M}$ since $f^{-1}(A) \in \mathscr{M}$. Therefore, $Y - A \in \Omega$.

(iii) If $A_n \in \Omega$ for n =1 ,2, … , then $f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n) \in \mathscr{M}$, since each $f^{-1}(A_n) \in \mathscr{M}$ and $\mathscr{M}$ is a σ-algebra. Therefore, $\bigcup_{n=1}^{\infty} A_n \in \Omega$.

Hence Ω is a σ-algebra.

(2) Define Ω as in part (1). Since f is measurable, Ω contains all open subsets of Y. Since by part (1) Ω is a σ-algebra and contains all open subsets of Y, it contains all Borel subsets of Y. Hence $f^{-1}(E) \in \mathscr{M}$.

(3) Let $\Omega = \{ E \subseteq \overline{\mathbb{R}} : f^{-1}(E) \in \mathscr{M} \}$. Then by part (1) is a σ-algebra and contains all open subsets of $\overline{\mathbb{R}}$. Thus for any open subset U of $\overline{\mathbb{R}}$, $f^{-1}(U) \in \mathscr{M}$. By hypothesis, $(a,\infty] \in \Omega$ for any a in $\mathbb{R}$. Now $[-\infty, a) = \bigcup_{n=1}^{\infty}\left(a - \frac{1}{n}, \infty\right]^c \in \Omega$ since each $\left(a - \frac{1}{n}, \infty\right]^c$ is in Ω and Ω is a σ-algebra. Therefore, for any $a < b$ in $\mathbb{R}$, the open interval $(a,b) = [-\infty, b) \cap (a, \infty] \in \Omega$. It follows that Ω contains all open intervals and so it contains all open sets in $\overline{\mathbb{R}}$. Hence, f is measurable.

Limit operations

Definition 11. Suppose (a_n) is a sequence in $\overline{\mathbb{R}}$, where $a_n \in \mathbb{R}$. Then

$\left(\sup\{a_n, \cdots\}\right)$ is a monotonic decreasing sequence and so tends to a limit in $\overline{\mathbb{R}}$. Call the limit of this sequence, which is the infimum of $\left(\sup\{a_n, \cdots\}\right)$, the *limit superior* of (a_n) and is denoted by $\limsup_{n\to\infty} a_n$ or simply $\limsup a_n$. That is,

$$\limsup a_n = \lim_{n\to\infty}\left\{\sup\{a_n, a_{n+1}, \cdots\}\right\} = \lim_{n\to\infty}\sup_p\left\{a_{n+p}, p = 0,1,2,\cdots\right\} .$$

$\left(\inf\{a_n,\cdots\}\right)$ is a monotonic increasing sequence and so tends to a limit in $\overline{\mathbb{R}}$. Call the limit of this sequence, which is the supremum of $\left(\inf\{a_n,\cdots\}\right)$, the *limit inferior* of (a_n) and is denoted by $\liminf\limits_{n\to\infty} a_n$ or simply $\liminf a_n$. That is,

$$\liminf a_n = \lim_{n\to\infty}\left\{\inf\{a_n, a_{n+1},\cdots\}\right\} = \lim_{n\to\infty}\inf_p\left\{a_{n+p}, p=0,1,2,\cdots\right\}.$$

Note that $\limsup a_n$ and $\liminf a_n$ may be $\pm\infty$.

Theorem 12. The sequence (a_n) converges in $\overline{\mathbb{R}}$ if and only if $\limsup a_n = \liminf a_n$.

Suppose there are infinite number of a_n such that $a_n = +\infty$. Then $\limsup a_n = +\infty$. If (a_n) converges, then plainly it cannot converge to a finite value or $-\infty$. This means that if it converges, it must tend to $+\infty$. This implies that $\left(\inf\{a_n,\cdots\}\right)$ is not bounded above and so $\liminf a_n = +\infty$.

Similarly, suppose there are infinite number of a_n such that $a_n = -\infty$. Then $\liminf a_n = -\infty$. If (a_n) converges, it must converge to $-\infty$. Consequently, $\left(\sup\{a_n,\cdots\}\right)$ is not bounded below and so $\limsup a_n = -\infty$. Therefore, $\limsup a_n = \liminf a_n = -\infty$.

Thus, we are left with the case that there are only finite number of a_n that assumes the value ∞ or $-\infty$. Thus, we may assume without loss of generality that $(a_n) \subseteq \mathbb{R}$.

The conclusion of the theorem now is Theorem 2 of *All About Lim Sup and Lim Inf.* The proof is given there.

Suppose $\left(f_n : X \to \overline{\mathbb{R}}\right)$ is a sequence of extended real valued functions. Define

$\sup\limits_n f_n : X \to \overline{\mathbb{R}}$ by $\left(\sup\limits_n f_n\right)(x) = \sup\limits_n\{f_n(x)\}$ for $x \in X$,

$\limsup f_n : X \to \overline{\mathbb{R}}$ by $\left(\limsup f_n\right)(x) = \limsup\limits_{n\to\infty} f_n(x)$ for x in X,

$\liminf f_n : X \to \overline{\mathbb{R}}$ by $\left(\liminf f_n\right)(x) = \liminf\limits_{n\to\infty} f_n(x)$ for x in X,

$\lim f_n : X \to \overline{\mathbb{R}}$ by $\left(\lim f_n\right)(x) = \lim\limits_{n\to\infty} f_n(x)$ for x in X.

If $\lim f_n : X \to \overline{\mathbb{R}}$ exists, i.e., $\lim\limits_{n\to\infty} f_n(x)$ exists for every x in X, then we say $\lim f_n$ is the pointwise limit of the sequence (f_n).

Note that if $\left(f_n : X \to \overline{\mathbb{R}}\right)$ is a sequence of extended real valued functions, then $\limsup f_n : X \to \overline{\mathbb{R}}$ and $\liminf f_n : X \to \overline{\mathbb{R}}$ always exist and may take the value $\pm\infty$. (Here, we include $\pm\infty$ as limit.)

Likewise, if $\left(f_n : X \to \overline{\mathbb{R}^+}\right)$ is a sequence of extended non-negative real-valued functions, then $\limsup f_n : X \to \overline{\mathbb{R}^+}$ and $\liminf f_n : X \to \overline{\mathbb{R}^+}$ always exist and may take the value $+\infty$. (Here, we include $+\infty$ as limit.)

Proposition 13. Suppose $(X, \mathscr{M})$ is a measure space and $\left(f_n : X \to \overline{\mathbb{R}}\right)$ is a sequence of measurable extended real valued functions. Then $g = \sup_n f_n$ and $h = \limsup f_n$ are measurable. Indeed, $\inf_n f_n$ and $\liminf f_n$ are also measurable.

Proof. By Proposition 10 part (3), we need only show that $g^{-1}((a,\infty])$ is measurable for any a in $\mathbb{R}$. Observe that $g^{-1}((a,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}\left((a,\infty]\right) \in$ since $f_n^{-1}\left((a,\infty]\right) \in \mathscr{M}$ for each integer $n \geq 1$. Hence g is measurable. Note that for any function $k : X \to \overline{\mathbb{R}}$,
$k^{-1}([-\infty,a]) = \left(k^{-1}\left((a,\infty]\right)\right)^c$, $k^{-1}\left((a,\infty]\right) = \left(k^{-1}\left((-\infty,a]\right)\right)^c$ and as $\mathscr{M}$ is a σ-algebra,
$k^{-1}\left((a,\infty]\right) \in \mathscr{M}$ if, and only if, $k^{-1}([-\infty,a]) \in \mathscr{M}$. If $g = \inf_n f_n$, then
$g^{-1}((-\infty,a]) = \bigcup_{n=1}^{\infty} f_n^{-1}\left((-\infty,a]\right) \in \mathscr{M}$ for any a in $\mathbb{R}$. Therefore, $\inf_n f_n$ is measurable.

Hence, $\limsup f_n = \inf_n\left(\sup_{p\geq 0} f_{n+p}\right)$ is measurable. Similarly, $\liminf f_n = \sup_n\left(\inf_{p\geq 0} f_{n+p}\right)$ is measurable.

Corollary 14. (1) The pointwise limit of a sequence of measurable (real or complex) functions is measurable.

(2) If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then max $\{f, g\}$ and min $\{f, g\}$ are measurable. In particular, $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\} = -\min\{f, 0\}$ are measurable.

Note that by proposition 13, The pointwise limit of a sequence of measurable real functions is measurable. Since a complex function is measurable if, and only, its real and imaginary parts are measurable. It follows that the pointwise limit of a sequence of measurable complex functions is measurable.

We next consider a class of easier to visualize measurable functions, namely, the simple functions. These are used to investigate measurable functions as well as to develop a theory of integration as we shall see in the next few sections.

Simple Functions

Definition 15. Suppose X is a non-empty set. Then a *simple function* on X is a non-negative real valued function, $s : X \to \mathbb{R}^+$, whose range consists of finite set of points in $\mathbb{R}^+$. Likewise, we say a complex function on X is simple if its range consists of finite set of points in $\mathbb{C}$. A *real valued simple function* on X is a function $s : X \to \mathbb{R}$, whose range consists of finite set of points in $\mathbb{R}$. For now, as we shall define the Lebesgue integral on non-negative functions, all simple functions are assumed to be non-negative unless otherwise stated. We may specify that the simple function is non-negative whenever we wish to emphasize that the result stated is only for non-negative simple function.

If $X = A_1 \cup A_2 \cup \cdots \cup A_n$ is a disjoint union such that $s\big|_{A_i} = \alpha_i$ for i =1, …, n, where the α_i's are distinct, then $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ is a simple function (real or complex) and all simple function is of this form.

A trivial example is $s : [0,1] \to \mathbb{R}^+$, given by $s(x) = \begin{cases} 0 \text{ if } x \in \mathbb{Q}, \\ 1, \text{ otherwise} \end{cases}$. The function s is a simple function.

If (X, $\mathscr{M}$) is a measure space, then the real or complex simple function s is measurable if, and only if, all A_i are measurable, i.e., $A_i \in \mathscr{M}$. The collection of real valued simple functions forms a real vector space or a linear space. The collection of complex simple functions forms a complex vector space.

The restriction of the range of s to $\mathbb{R}^+$ is purely technical as we shall first consider integrating non-negative functions f and then extend to complex function f by writing $f = \operatorname{Re} f + i \operatorname{Im} f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i(\operatorname{Im} f)^+ - i(\operatorname{Im} f)^-$, where $(\operatorname{Re} f)^+$ and $(\operatorname{Re} f)^-$ and $(\operatorname{Im} f)^+$ and $(\operatorname{Im} f)^-$ are respectively the positive and negative parts of Re f and Im f respectively.

Theorem 16. Let (X, $\mathscr{M}$) be a measure space and $f : X \to \overline{\mathbb{R}^+}$ is a non-negative measurable function. Then there exists a monotone increasing sequence of (non-negative) measurable simple functions (s_n) converging pointwise to f. If f is bounded, then (s_n) converges uniformly to f.

Proof.

We construct the sequence (s_n) as follows. For each integer $n \geq 1$, divide the interval [0, n] into $n \times 2^n$ sub-intervals of length $\frac{1}{2^n}$.

Let $E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)$, $i = 1, 2, \cdots, n2^n$, $F_n = f^{-1}([n, \infty))$ and $s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}$.

Since f is measurable, the sets $E_{n,i}$ and F_n are measurable.

Note that $E_{n,i} = E_{n+1,j} \cup E_{n+1,j+1}$, where $\frac{j-1}{2^{n+1}} = \frac{i-1}{2^n}$ or $j = 2i - 1$. On the set $E_{n,i}$, $s_{n+1}(x)$ takes on the value $\frac{j-1}{2^{n+1}} = \frac{i-1}{2^n}$ when x is in $E_{n+1,j}$ and the value $\frac{j}{2^{n+1}} > \frac{i-1}{2^n}$ when x is in $E_{n+1,j+1}$. Observe also that

$$F_n = f^{-1}([n, \infty)) = f^{-1}([n+1, \infty)) \cup f^{-1}([n, n+1)) = F_{n+1} \cup f^{-1}([n, n+1))$$

and $f^{-1}([n, n+1)) = \bigcup\left\{E_{n+1,i} : i = n2^{n+1} + 1 \text{ to } (n+1)2^{n+1}\right\}$.

Thus, on the set F_{n+1}, $s_{n+1}(x)$ takes on the value $n+1$ when x is in $E_{n+1,j}$ and on the set $f^{-1}([n, n+1))$, $s_{n+1}(x)$ takes on values $\geq n$, when $s_n(x)$ is defined and is equal to n. Therefore, $s_{n+1} \geq s_n$.

Since $f(x) < \infty$, take an integer N such that $N > f(x)$, then for all $n \geq N$, $s_{n+1}(x) \leq N$ and so the sequence is pointwise convergence. Moreover, for each integer $n > f(x)$, $f(x)$ lies in $\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ for some i such that $1 \leq i \leq n2^n$ and so $s_n(x) \leq f(x)$. Furthermore, $s_n(x) \geq f(x) - \frac{1}{2^n}$. Hence $\lim_{n\to\infty} s_n(x) = f(x)$.

Now, suppose f is bounded such that $0 \leq f < K$ and $K \geq 1$.

First of all, note that $F_n = \varnothing$ for all integer $n \geq K$. For any integer $n > K$, $E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) = \varnothing$ if $2^n K + 1 \leq i \leq n2^n$.

This means for $0 \le f < K$, we effectively partition the interval [0, *K*] into $2^n K$ sub-intervals each of length $\frac{1}{2^n}$.

Observe that since $f(x) < K$, for any integer $N \ge K$, $N > f(x)$ for all *x*, and so for all $n \ge N$, $s_{n+1}(x) \le N$ for all *x* and so the sequence is uniformly bounded. Moreover, for each integer $n \ge N$, $f(x)$ lies in $\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ for some i such that $1 \le i \le n2^n$ so that, $s_n(x) \ge f(x) - \frac{1}{2^n}$ for all *x*. Hence, for all $n \ge N$ and for all *x*, $f(x) \ge s_n(x) \ge f(x) - \frac{1}{2^n}$. This means that (s_n) converges uniformly to f.

Definition 17. Suppose $(X, \mathscr{M})$ is a measure space, i.e., *X* is a non-empty set and $\mathscr{M}$ is a σ-algebra on X. A function $\mu : \mathscr{M} \to \overline{\mathbb{R}}$ is *countably additive*, if for any countable collection of disjoint sets, $\{A_n\}$, of $\mathscr{M}$, we have that $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$. The function μ is *finitely additive*, if for any finite collection $\{A_i\}_{i=1}^{k}$ of disjoint sets of $\mathscr{M}$, then $\mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i)$.

A *positive measure* μ on $\mathscr{M}$ is a countably additive function $\mu: \mathscr{M} \to \overline{\mathbb{R}^+}$ mapping the σ-algebra $\mathscr{M}$ into the extended positive real numbers, a *real measure* on $\mathscr{M}$ is a countably additive function $\mu: \mathscr{M} \to \mathbb{R}$ mapping the σ-algebra $\mathscr{M}$ into the real numbers and a *complex measure* on $\mathscr{M}$ is a countably additive function $\mu: \mathscr{M} \to \mathbb{C}$ mapping the σ-algebra $\mathscr{M}$ into the complex numbers. Hence, a real measure is a complex measure but a positive measure is not necessarily a real measure nor a complex measure. For a positive measure μ, we shall assume that for at least one *A* in $\mathscr{M}$, $\mu(A) < \infty$, otherwise, μ is a trivial positive measure taking only ∞ as its value.

If $\mathscr{M}$ is an algebra of sets, then a finitely additive set function on $\mathscr{M}$ is also called a *content*, it is a *real content*, if it is real valued, a *positive content*, if its range is $\overline{\mathbb{R}^+}$, a *complex content* if it is complex valued.

Suppose $(X, \mathscr{T})$ is a topological space and $\mathscr{T}$ is its topology. Let $\mathscr{M}$ be the σ-algebra of *Borel subsets* of *X*, i.e., $\mathscr{M}$ is the collection of Borel measurable sets in *X*. Then a measure defined on $\mathscr{M}$ is called the *Borel measure* of *X*. It is a *real Borel measure*, if it is real-valued, a *positive Borel measure*, if its range is $\overline{\mathbb{R}^+}$, a *complex Borel measure*, if it is complex valued.

Suppose μ is a positive measure. Then since $\mu(A) = \mu(A \cup \varnothing) = \mu(A) + \mu(\varnothing)$ for some A with $\mu(A) < \infty$, $\mu(\varnothing) = 0$. For a real or complex measure, plainly $\mu(\varnothing) = 0$.

Positive Measures.

Proposition 18. Suppose $(X, \mathscr{M})$ is a measure space and $\mu\colon \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure on $(X, \mathscr{M})$. Then

(1) μ is monotonic, i.e., if $A \subseteq B$ and $A, B \in \mathscr{M}$, then $\mu(A) \le \mu(B)$;

(2) μ is continuous from below, i.e., if $A_1 \subseteq A_2 \subseteq \cdots$, $A_n \in \mathscr{M}$ for all integer $n \ge 1$, with $A_n \subseteq A_{n+1}$ and $A = \bigcup_{n=1}^{\infty} A_n$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ or $\mu(A_n) \to \mu(A)$ and

(3) μ is continuous from above, i.e., if $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, $A_n \in \mathscr{M}$ for all integer $n \ge 1$, with $A_n \supseteq A_{n+1}$ and for some integer i, $\mu(A_i) < \infty$ and $A = \bigcap_{n=1}^{\infty} A_n$, then $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ or $\mu(A_n) \to \mu(A)$.

Proof.

(1) Suppose $A \subseteq B$ and $A, B \in \mathscr{M}$. Now $B = A \cup (B - A)$ a disjoint union. Therefore, $\mu(B) = \mu(A) + \mu(B - A) \ge \mu(A)$ as $\mu(B - A) \ge 0$.

(2) Let $B_1 = A_1$, $B_2 = A_2 - A_1$, …, $B_n = A_n - A_{n-1}$, …. .

Then $A_n = B_1 \cup B_2 \cup \cdots \cup B_n$ a disjoint union. Therefore, by additivity,

$$\mu(A_n) = \sum_{i=1}^{n} \mu(B_i) .$$

Moreover, $A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ is a countable disjoint union of sets in $\mathscr{M}$. Hence, by countable additivity, $\mu(A) = \sum_{i=1}^{\infty} \mu(B_i)$. It follows that $\mu(A_n) = \sum_{i=1}^{n} \mu(B_i) \to \sum_{i=1}^{\infty} \mu(B_i) = \mu(A)$.

(3) We may suppose without loss of generality that $\mu(A_1) < \infty$. Otherwise, if $\mu(A_i) < \infty$, we may just re-index the A_i's to start with A_i as A_1 and discard the previous A_i's .

Let $C_n = A_1 - A_n$ for $n \geq 1$. Then obviously, $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots \subseteq C_n \subseteq \cdots$, i.e., $C_n \subseteq C_{n+1}$. Note that $\mu(C_n) = \mu(A_1) - \mu(A_n) \leq \mu(A_1) < \infty$ and $\bigcup_{n=1}^{\infty} C_n = A_1 - \bigcap_{n=1}^{\infty} A_n = A_1 - A$. Therefore, by part (2), $\mu(C_n) = \mu(A_1) - \mu(A_n) \to \mu(A_1 - A) = \mu(A_1) - \mu(A)$. It follows that $\mu(A_n) \to \mu(A)$.

Examples. 1. Unit mass concentrated at x_0, $x_0 \in X$, X is a non-empty set.

$\mu(A) = 1$ if $x_0 \in A$, otherwise $\mu(A) = 0$.

2. Counting measure.

X is a non-empty set. $\mu(A) = \begin{cases} \text{number of points in } A \text{ if } A \text{ is finite,} \\ \infty, \text{ otherwise} \end{cases}$.

3. Restriction of a measure.

Suppose $(X, \mathscr{M})$ is a measure space and $E \in \mathscr{M}$. Suppose μ is a *positive measure* on $\mathscr{M}$. Then the function $\mu_E : \mathscr{M} \to \overline{\mathbb{R}^+}$ defined by $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathscr{M}$ is also a positive measure. Indeed, if we let $\mathscr{M}_E = \{A \cap E : A \in \mathscr{M}\}$, then $\mathscr{M}_E$ is a σ-algebra on E with $\mathscr{M}_E \subseteq \mathscr{M}$ and the restriction of μ to $\mathscr{M}_E$ is a positive measure on $\mathscr{M}_E$.

Remark. The condition that for some integer i, $\mu(A_i) < \infty$ in part (3) of Proposition 18 is necessary. We have the following example. Let $X = \{1, 2, 3, \cdots\}$ be the set of positive integers. Let $\mathscr{M}$ be the collection of all subsets of X and $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ be the counting measure. For each integer $n \geq 1$, let $A_n = \{n, n+1, n+2, \cdots\}$. Then for each integer $n \geq 1$, $A_n \supseteq A_{n+1}$, i.e., $A_1 \supseteq A_2 \supseteq \cdots$. Note that $A = \bigcap_{n=1}^{\infty} A_n = \varnothing$ and so $\mu(A) = 0$. But $\mu(A_n) = \infty$ for each integer $n \geq 1$ and so $\mu(A_n)$ cannot converge to $\mu(A)$.

Integration of Non-negative Functions

Definition 19. Suppose $(X, \mathscr{M})$ is a measure space and $E \in \mathscr{M}$. Let $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ be a positive measure on X. Let $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ be a (non-negative) measurable simple function on X. Define

$$\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E).$$

If $f: X \to \overline{\mathbb{R}^+}$ is a measurable function, i.e., a $\mathcal{M}$ - measurable function, then

$$\int_E f\,d\mu = \sup\left\{\int_E s\,d\mu : s \text{ is a measurable simple function and } 0 \le s \le f\right\}$$

is called the *Lebesgue integral* of *f* over *E*.

We may omit the word "Lebesgue", write $\int_E f$ for $\int_E f\,d\mu$ when no confusion arises.

Plainly, the definition of the integral includes the definition of the integral for the measurable simple function.

We have the following obvious properties.

Properties 20. Suppose *E* is a measurable set in $\mathcal{M}$.

(1) If $0 \le f \le g$ are measurable functions, then $\int_E f \le \int_E g$.

(2) If $f \ge 0$ is measurable and $E_1 \subseteq E_2$ are measurable, then $\int_{E_1} f \le \int_{E_2} f$.

(3) If $f \ge 0$ is measurable and $c \in \mathbb{R}^+$, then $\int_E (cf) = c\int_E f$.

(4) If $f(x) = 0$ for all *x* in *E*, then $\int_E f = 0$, even by convention when $\mu(E) = \infty$.

(By convention for multiplication in $\overline{\mathbb{R}}$, $0 \times \infty = 0$.)

(5) If $\mu(E) = 0$, then $\int_E f = 0$, even by convention when $f(x) = \infty$ for all *x* in *E*.

(6) If $f \ge 0$ is measurable, then $\int_E f\,d\mu = \int_X \chi_E f\,d\mu$. (We could have started with the definition of the integral of f over *X* and use this to define the integral of *f* over *E*.)

In the sequel, by a measure we shall always mean a positive measure unless otherwise stated.

Proposition 21. Suppose $(X, \mathcal{M})$ is a measure space. Let $\mu : \mathcal{M} \to \overline{\mathbb{R}^+}$ be a positive measure on *X*. Suppose *s* is a (non-negative) measurable simple function on *X*. For $E \in \mathcal{M}$, define

$$\varphi(E) = \int_E s\,d\mu .$$

Then φ is a positive measure on *X*.

Proof.

Suppose $s=\sum_{i=1}^{n}\alpha_i\,\chi_{A_i}$ is a non-negative measurable simple function on *X*. We must show that φ is countably additive and non-trivial, i.e, not identically equal to ∞.

Suppose $\{C_i\}_{i=1}^{\infty}$ is a countable collection of pairwise disjoint sets in the σ-algebra $\mathscr{M}$. Let $C=\bigcup_{k=1}^{\infty}C_k$. Then

$$\varphi(C)=\int_C s\,d\mu=\sum_{i=1}^{n}\alpha_i\mu\left(A_i\cap C\right) \text{ by definition of } \int_C s\,d\mu,$$

$$=\sum_{i=1}^{n}\alpha_i\times\left(\sum_{k=1}^{\infty}\mu(A_i\cap C_k)\right) \text{ by countably additivity of } \mu,$$

$$=\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n}\alpha_i\mu(A_i\cap C_k)\right) \text{ by rearrangement}$$

$$=\sum_{k=1}^{\infty}\int_{C_k}s\,d\mu=\sum_{k=1}^{\infty}\varphi\left(C_k\right).$$

It follows that φ is countably additive. Moreover, by definition, $\varphi(\varnothing)=0$ and so φ is not identically equal to ∞.

The next result says that Lebesgue integration is linear on the collection of non-negative measurable simple functions.

Proposition 22. Suppose $(X,\mathscr{M})$ is a measure space and *s* and *t* are two (non-negative) measurable simple functions on *X*. Then *s* + *t* is a measurable function. Suppose $\mu:\mathscr{M}\to\overline{\mathbb{R}^+}$ is a positive measure on *X*. Then

$$\int_X(s+t)\,d\mu=\int_X s\,d\mu+\int_X t\,d\mu\ .$$

Proof. Suppose $s=\sum_{i=1}^{n}\alpha_i\,\chi_{A_i}$ and $t=\sum_{i=1}^{m}\beta_i\,\chi_{B_i}$. Let $D_{i,j}=A_i\cap B_j$, $1\le i\le n$, $1\le j\le m$. Then *s*, *t* and *s* + *t* are constant on $D_{i,j}$. Let $E_{i,j}=X\cap D_{i,j}$ for $1\le i\le n$, $1\le j\le m$.

Then $\varphi_{s+t}\left(E_{i,j}\right)=\int_{E_{i,j}}(s+t)\,d\mu=\left(\alpha_i+\beta_j\right)\mu\left(D_{i,j}\right)$ and

$\int_{E_{i,j}}s\,d\mu+\int_{E_{i,j}}t\,d\mu=\alpha_i\mu\left(D_{i,j}\right)+\beta_j\mu\left(D_{i,j}\right)$.

Thus $\varphi_{s+t}\left(E_{i,j}\right)=\varphi_s\left(E_{i,j}\right)+\varphi_t\left(E_{i,j}\right)$.

Next observe that X is a disjoint union of $E_{i,j}$, i.e., $X = \bigcup_{1\le i\le n, 1\le j\le m} E_{i,j}$. Therefore, by Proposition 21, $\varphi_{s+t}\left(X\right) = \sum_{1\le i\le n, 1\le j\le m} \varphi_{s+t}\left(E_{i,j}\right) = \sum_{1\le i\le n, 1\le j\le m} \varphi_{s}\left(E_{i,j}\right) + \sum_{1\le i\le n, 1\le j\le m} \varphi_{t}\left(E_{i,j}\right)$

$$= \varphi_s\left(X\right) + \varphi_t\left(X\right) .$$

Thus, $\int_X (s+t)\, d\mu = \int_X s\, d\mu + \int_X t\, d\mu$ follows.

The basic convergence theorem is the Monotone Convergence Theorem. From this we deduce other convergence theorems and results.

Theorem 23. Lebesgue Monotone Convergence Theorem.

Suppose $(X, \mathscr{M})$ is a measure space and $\left(f_n\right)$ is an increasing sequence of non-negative measurable functions on X tending pointwise to a function f. Suppose $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X. Then

$$\int_X f_n\, d\mu \nearrow \int_X f\, d\mu .$$

Remarks.

We elaborate the hypothesis of this theorem.

First of all, we have

(1) $0 \le f_1(x) \le f_2(x) \le \cdots \le \infty$, for all x in X.

(2) Since $\left(f_n(x)\right)$ is an increasing sequence in $\overline{\mathbb{R}^+}$, it tends to a limit, $f(x)$, which may be ∞ .

(3) Since f is the pointwise limit of a sequence of measurable functions, by Corollary 14, f is measurable. Note that plainly, f is non-negative and so $\int_X f\, d\mu$ is defined.

Proof of Theorem 23.

Since $f_n \le f_{n+1}$, for integer $n \ge 1$, $\int_X f_n \le \int_X f_{n+1}$, the sequence $\left(\int_X f_n\right)$ is an increasing sequence in $\overline{\mathbb{R}^+}$ and so it converges to some limit $\alpha \in \overline{\mathbb{R}^+}$. Moreover, $f_n \le f$ for each integer $n \ge 1$ and so $\int_X f_n \le \int_X f$. It follows that $\alpha \le \int_X f$.

We shall now proceed to show that $\alpha \geq \int_X f$.

Let s be a measurable simple function with $0 \leq s \leq f$. Take a real number c such that $0 < c < 1$. Define for each integer $n \geq 1$, $E_n = \{x : f_n(x) \geq cs(x)\}$. Then for $x \in X$, *either* $f(x) = 0$, in which case, $f_n(x) = 0$ for all $n \geq 1$ and $s(x) = 0$ so that $x \in E_n$ for all $n \geq 1$, *or* $f(x) > 0$, in which case, $cs(x) < f(x)$ and as $f_n(x) \nearrow f(x)$, there exists an integer N such that $n \geq N$ implies that $cs(x) \leq f_n(x) \leq f(x)$ and so $x \in E_n$ for $n \geq N$.

Note that as (f_n) is an increasing sequence, $E_n \subseteq E_{n+1}$ for $n \geq 1$ and so by the above argument, $X = \bigcup_{n=1}^{\infty} E_n$. It follows that

$$\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} cs \, d\mu = c\int_{E_n} s \, d\mu = c\varphi_s(E_n), \quad \text{-----------------} \quad (*)$$

where $\varphi_s(E) = \int_E s \, d\mu$. By Proposition 21, φ_s is a positive measure. By Proposition 18 part (2), φ_s is continuous from below and so

$$c\varphi_s(E_n) \to c\varphi_s\left(\bigcup_{n=1}^{\infty} E_n\right) = c\varphi_s(X) = c\int_X s \, d\mu .$$

Therefore, it follows from (*) that $\alpha \geq c\int_X s \, d\mu$ for any c with $0 < c < 1$. It follows that $\alpha \geq \int_X s \, d\mu$. This is true for any measurable simple function with $0 \leq s \leq f$. As

$$\int_X f \, d\mu = \sup\left\{\int_X s \, d\mu : s \text{ is a measurable simple function and } 0 \leq s \leq f\right\},$$

$\alpha \geq \int_X f \, d\mu$.

This completes the proof of Theorem 23.

Next, in the following proposition, we show that the Lebesgue integral is linear on non-negative measurable functions.

Proposition 24. Suppose $(X, \mathscr{M})$ is a measure space and $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X. If $f, g : X \to \overline{\mathbb{R}^+}$ are measurable, then $\int_X (f+g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$.

Proof. Since f and g are measurable, by Theorem 16, there are two monotone increasing sequences of (non-negative) measurable simple functions, (s_n) and (t_n) such that $s_n \nearrow f$ and $t_n \nearrow g$. Then $(s_n + t_n) \nearrow f + g$. Therefore, by the Lebesgue Monotone Convergence

Theorem (Theorem 23), $\int_X (s_n + t_n)\, d\mu \nearrow \int_X (f+g)\, d\mu$. But by Proposition 22, for each integer $n \geq 1$, $\int_X (s_n + t_n)\, d\mu = \int_X s_n d\mu + \int_X t_n d\mu$ and by the Lebesgue Monotone Convergence Theorem, $\int_X s_n d\mu + \int_X t_n d\mu \nearrow \int_X f\, d\mu + \int_X g\, d\mu$. Therefore, $\int_X (f+g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$.

An immediate corollary is:

Corollary 25. If $f_1, f_2, \cdots, f_n : X \to \overline{\mathbb{R}^+}$ are measurable, then

$$\int_X (f_1 + f_2 + \cdots + f_n)\, d\mu = \int_X f_1\, d\mu + \int_X f_2\, d\mu + \cdots + \int_X f_n\, d\mu .$$

Theorem 26. Suppose $f_1, f_2, \cdots, f_n, \cdots : X \to \overline{\mathbb{R}^+}$ are measurable functions. Then

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu .$$

Proof. By the Lebesgue Monotone Convergence Theorem (Theorem 23),

$$\int_X \left(\sum_{k=1}^{n} f_k \right) d\mu \nearrow \int_X \left(\sum_{k=1}^{\infty} f_k \right) d\mu .$$

By Corollary 25, $\int_X \left(\sum_{k=1}^{n} f_k \right) d\mu = \sum_{k=1}^{n} \int_X f_k\, d\mu$, and so $\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu$.

The next theorem is an important and very useful result about the integral of lim inf of a sequence of non-negative measurable functions. This theorem is usually known as Fatou's Lemma.

Theorem 27. Fatou's Lemma.

Suppose $(X, \mathscr{M})$ is a measure space and $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X.

Suppose $f_1, f_2, \cdots, f_n, \cdots : X \to \overline{\mathbb{R}^+}$ are measurable functions. Then

$$\int_X \liminf f_n\, d\mu \leq \liminf \int_X f_n\, d\mu .$$

Before we give the proof, we give an example to illustrate this theorem.

Example. We suppose that we have already constructed Lebesgue measure μ on the unit interval [0, 1]. It is, of course, a positive measure.

For odd integer $n \geq 1$, let $f_n = \chi_{[0,\frac{1}{2}]}$ and for even integer $n \geq 2$, let $f_n = \chi_{(\frac{1}{2},1]}$.

Then $\left(\liminf f_n\right)(x) = 0$. Therefore, $\int_{[0,1]} \liminf f_n \, d\mu = 0$. For odd integer $n \geq 1$,

$\int_{[0,1]} f_n \, d\mu = \int_{[0,1]} \chi_{[0,\frac{1}{2}]} \, d\mu = \mu\left([0,\tfrac{1}{2}]\right) = \dfrac{1}{2}$, and for even $n \geq 2$,

$\int_{[0,1]} f_n \, d\mu = \int_{[0,1]} \chi_{(\frac{1}{2},1]} \, d\mu = \mu\left((\tfrac{1}{2},1]\right) = \dfrac{1}{2}$. It follows that $\liminf \int_X f_n \, d\mu = \dfrac{1}{2}$. Indeed

$$\int_{[0,1]} \liminf f_n \, d\mu = 0 \leq \liminf \int_X f_n \, d\mu = \frac{1}{2} .$$

Since we have not constructed the Lebesgue measure, for $X = [0, 1]$, we may take the σ-algebra $\mathscr{M} = \{ X, \varnothing, [0,\tfrac{1}{2}] , (\tfrac{1}{2},1] \}$ and the positive measure on $\mathscr{M}$, to be given by $\mu(\varnothing) = 0$, $\mu\left([0,1]\right) = 1$, $\mu\left([0,\tfrac{1}{2}]\right) = \dfrac{1}{2}$ and $\mu\left((\tfrac{1}{2},1]\right) = \dfrac{1}{2}$.

Proof of Theorem 27.

For each integer $n \geq 1$, let $g_n(x) = \inf\left\{ f_n(x), f_{n+1}(x), \cdots \right\} = \inf_{k \geq 0}\left\{ f_{n+k}(x) \right\}$. Then $g_n(x) \leq f_k(x)$ for all x in X and for all $k \geq n$. By Proposition 13, g_n is measurable. Moreover, $g_n(x) \leq g_{n+1}(x)$ for all x in X so that $\left(g_n\right)$ is a monotone increasing sequence of measurable functions and $\liminf_{n\to\infty} f_n = \lim_{n\to\infty} g_n$. Therefore, by the Lebesgue Monotone Convergence Theorem (Theorem 23),

$$\int_X g_n \, d\mu \nearrow \int_X \left(\liminf_{n\to\infty} f_n \right) d\mu .$$

But $\int_X g_n \, d\mu \leq \int_X f_k \, d\mu$ for all $k \geq n$ and so $\int_X g_n \, d\mu \leq \inf\left\{ \int_X f_n \, d\mu, \int_X f_{n+1} \, d\mu, \cdots \right\}$. Hence $\lim_{n\to\infty} \int_X g_n \, d\mu \leq \lim_{n\to\infty}\left(\inf\left\{ \int_X f_n \, d\mu, \int_X f_{n+1} \, d\mu, \cdots \right\} \right) = \liminf_{n\to\infty} \int_X f_n \, d\mu$. It follows that

$$\int_X \left(\liminf_{n\to\infty} f_n \right) d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu .$$

Proposition 28. Suppose $(X, \mathcal{M})$ is a measure space and $\mu : \mathcal{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X. Suppose $f : X \to \overline{\mathbb{R}^+}$ is measurable. Define for each E in $\mathcal{M}$, $\varphi(E) = \int_E f \, d\mu$. Then φ is a positive measure on $\mathcal{M}$ and for any $\mathcal{M}$-measurable function $g : X \to \overline{\mathbb{R}^+}$,

$$\int_X g \, d\varphi = \int_X g \cdot f \, d\mu \ .$$

Proof. Plainly, $\varphi(\varnothing) = \int_\varnothing f \, d\mu = 0$. (See Properties 20 (5)). Clearly φ is mon-negative. Now we show that φ is countably additive. Suppose $\{E_i\}_{i=1}^{\infty}$ is a countable collection of pairwise disjoint sets in the σ-algebra $\mathcal{M}$. Let $E = \bigcup_{i=1}^{\infty} E_i$. Note that as the sets in the collection, $\{E_i\}_{i=1}^{\infty}$, are pairwise disjoint, $\chi_E f = \sum_{i=1}^{\infty} \chi_{E_i} f$. Therefore,

$$\varphi(E) = \int_E f \, d\mu = \int_X \chi_E f \, d\mu = \int_X \left(\sum_{i=1}^{\infty} \chi_{E_i} f \right) d\mu$$

$$= \sum_{i=1}^{\infty} \int_X \chi_{E_i} f d\mu \text{, by Theorem 26,}$$

since $\chi_{E_i} f$ is measurable and non-negative for each $i \geq 1$,

$$= \sum_{i=1}^{\infty} \varphi(E_i).$$

It follows that φ is countably additive.

Suppose that g is a measurable function. Then there is a monotone sequence of (non-negative) measurable simple functions (s_n) such that $s_n \nearrow g$. Suppose $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ is a measurable simple function. Then

$$\int_X s \, d\varphi = \sum_{i=1}^{n} \alpha_i \, \varphi(A_i) = \sum_{i=1}^{n} \alpha_i \int_{A_i} f \, d\mu = \sum_{i=1}^{n} \alpha_i \int_X \chi_{A_i} f \, d\mu$$

$$= \sum_{i=1}^{n} \int_X \alpha_i \, \chi_{A_i} f \, d\mu = \int_X \left(\sum_{i=1}^{n} \alpha_i \, \chi_{A_i} f \right) d\mu = \int_X s \, f \, d\mu .$$

Hence, $\int_X s_n \, d\varphi = \int_X s_n f \, d\mu$ for each integer $n \geq 1$. As $s_n \nearrow g$, $s_n f \nearrow g f$. Therefore, by the Lebesgue Monotone Convergence Theorem (Theorem 23), $\int_X s_n \, d\varphi \nearrow \int_X g \, d\varphi$ and that $\int_X s_n \, d\varphi = \int_X s_n f \, d\mu \nearrow \int_X g \, f \, d\mu$. Hence, $\int_X g \, d\varphi = \int_X g \, f \, d\mu$.

Integration of Complex Functions With Respect To Positive Measure

Definition 29.

Suppose $(X, \mathscr{M})$ is a measure space and $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X. Suppose $f : X \to \overline{\mathbb{R}}$ is a measurable function. We say f is *Lebesgue integrable* or *summable* on X if

$$\int_X |f| \, d\mu < \infty \ .$$

This makes sense since $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\} = -\min\{f, 0\}$ are measurable. As $|f| = f_+ + f_-$ and that when $f_+(x) = \infty$, $f_-(x) = 0$ and when $f_-(x) = \infty$ $f_+(x) = 0$ so that the sum $f_+ + f_-$ is always meaningful in $\overline{\mathbb{R}^+}$. Using this fact, we can show that $f_+ + f_- = |f|$ is measurable.

We now consider real valued measurable function $f : X \to \mathbb{R}$. Likewise, we say f is *Lebesgue integrable* or *summable* on X if

$$\int_X |f| \, d\mu < \infty \ .$$

Note that if we let $E_+ = f^{-1}([0, \infty))$ and $E_- = f^{-1}\big((-\infty, 0)\big)$, then E_+ and E_- are measurable and $f = \chi_{E_+} f + \chi_{E_-} f$.

Let $L^1(X, \mu)$ be the set of all Lebesgue integrable real valued measurable functions on X. Thus, if $f \in L^1(X, \mu)$, $|f| = f_+ + f_-$ is measurable and $\int_X (f_+ + f_-) \, d\mu < \infty$ so that $\int_X f_+ d\mu < \infty$ and $\int_X f_- d\mu < \infty$. Hence, a measurable real valued function f is Lebesgue integrable if and only if $\int_X f_+ d\mu < \infty$ and $\int_X f_- d\mu < \infty$. Define the Lebesgue integral of f, $\int_X f \, d\mu$, by

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu \ .$$

Obviously, this is well defined as $-\infty < \int_X f_+ \, d\mu - \int_X f_- \, d\mu < \infty$.

Proposition 30. $L^1(X, \mu)$ is a real vector space and the Lebesgue integration $\int_X \ : L^1(X, \mu) \to \mathbb{R}$ is a real linear functional.

Proof. If $f, g \in L^1(X,\mu)$ and $\alpha, \beta \in \mathbb{R}$, then by Corollary 4, $\alpha f + \beta g$ is measurable. Moreover, $|\alpha f + \beta g| \le |\alpha||f| + |\beta||g|$ so that by Properties 20 (1) and (3),

$$\int_X |\alpha f + \beta g|\, d\mu \le \int_X |\alpha||f|\, d\mu + \int_X |\beta||g|\, d\mu = |\alpha|\int_X |f|\, d\mu + |\beta|\int_X |g|\, d\mu < \infty .$$

Thus, $\alpha f + \beta g \in L^1(X,\mu)$. Hence, $L^1(X,\mu)$ is a real vector space.

We shall now show that Lebesgue integration is linear on $L^1(X,\mu)$.

Observe that $f + g = (f+g)_+ - (f+g)_- = f_+ + g_+ - f_- - g_-$ so that

$$(f+g)_+ + f_- + g_- = (f+g)_- + f_+ + g_+ .$$

Therefore,

$$\int_X \left((f+g)_+ + f_- + g_-\right) d\mu = \int_X (f+g)_+\, d\mu + \int_X f_- d\mu + \int_X g_- d\mu$$

$$= \int_X \left((f+g)_+ + f_- + g_-\right) d\mu = \int_X (f+g)_- d\mu + \int_X f_+ d\mu + \int_X g_+ d\mu .$$

Hence,

$$\int_X (f+g)_+\, d\mu - \int_X (f+g)_-\, d\mu = \int_X f_+ d\mu + \int_X g_+ d\mu - \int_X f_- d\mu + \int_X g_- d\mu$$

$$= \int_X f\, d\mu + \int_X g\, d\mu .$$

This means $\int_X (f+g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$.

Take any $\alpha \ge 0$, then $(\alpha f)_+ = \alpha f_+$ and $(\alpha f)_- = \alpha f_-$. Therefore,

$$\int_X (\alpha f) d\mu = \int_X (\alpha f)_+ d\mu - \int_X (\alpha f)_- d\mu = \alpha \int_X f_+ d\mu - \alpha \int_X f_- d\mu = \alpha \int_X f\, d\mu .$$

Suppose $\alpha < 0$. Then $-\alpha > 0$ and so $\int_X (-\alpha f) d\mu = -\alpha \int_X f\, d\mu$.

Now $\int_X -f\, d\mu = -\int_X f\, d\mu$ because $(-f)_+ = \max\{-f, 0\} = f_-$ and $(-f)_- = \max\{f, 0\} = f_+$. Therefore, $-\int_X (\alpha f) d\mu = \int_X (-\alpha f) d\mu = -\alpha \int_X f\, d\mu$ and so $\int_X (\alpha f) d\mu = \alpha \int_X f\, d\mu$. This shows that the Lebesgue integral is a real linear functional.

Now, we consider measurable complex function on *X*. Suppose $f : X \to \mathbb{C}$ is measurable. Then by Corollary 5, Re f and Im f are measurable. By Corollary 7, $|f|$ is measurable. We say a measurable complex function f is *Lebesgue integrable*, if $\int_X |f|\, d\mu < \infty$. Note that

$|\operatorname{Re} f|, |\operatorname{Im} f| \le |f|$ and $|f| \le |\operatorname{Re} f| + |\operatorname{Im} f|$. It follows that f is Lebesgue integrable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are Lebesgue integrable. We define

$$\int_X f\,d\mu = \int_X \operatorname{Re} f\,d\mu + i\int_X \operatorname{Im} f\,d\mu .$$

Let $L^1(X,\mathbb{C},\mu)$ be the collection of all Lebesgue integrable measurable complex functions on X.

Then we have,

Proposition 31. $L^1(X,\mathbb{C},\mu)$ is a complex vector space and the Lebesgue integration $\int_X : L^1(X,\mathbb{C},\mu) \to \mathbb{C}$ is a complex linear functional.

Proof.

If $f, g \in L^1(X,\mathbb{C},\mu)$ and $\alpha, \beta \in \mathbb{C}$, then by Corollary 4, $\alpha f + \beta g$ is measurable.

Now $\int_X |\alpha f + \beta g|\,d\mu \le \int_X |\alpha f|\,d\mu + \int_X |\beta g|\,d\mu = |\alpha|\int_X |f|\,d\mu + |\beta|\int_X |g|\,d\mu < \infty$. Hence, $\alpha f + \beta g$ is Lebesgue integrable. Therefore, $L^1(X,\mathbb{C},\mu)$ is a complex vector space.

For $f, g \in L^1(X,\mathbb{C},\mu)$,

$$\int_X (f+g)\,d\mu = \int_X \operatorname{Re}(f+g)\,d\mu + i\int_X \operatorname{Im}(f+g)\,d\mu$$

$$= \int_X (\operatorname{Re} f + \operatorname{Re} g)\,d\mu + i\int_X (\operatorname{Im} f + \operatorname{Im} g)\,d\mu$$

$$= \int_X \operatorname{Re} f\,d\mu + \int_X \operatorname{Re} g\,d\mu + i\left(\int_X \operatorname{Im} f\,d\mu + \int_X \operatorname{Im} g\,d\mu\right),$$ by Proposition 30,

$$= \int_X f\,d\mu + \int_X g\,d\mu .$$

Take any $\alpha \in \mathbb{C}$. Then for $f \in L^1(X,\mathbb{C},\mu)$,
$\alpha f = (\operatorname{Re}\alpha + i\operatorname{Im}\alpha)(\operatorname{Re} f + i\operatorname{Im} f) = \operatorname{Re}\alpha\operatorname{Re} f - \operatorname{Im}\alpha\operatorname{Im} f + i(\operatorname{Re}\alpha\operatorname{Im} f + \operatorname{Im}\alpha\operatorname{Re} f)$.

Therefore,

$$\int_X \alpha f\,d\mu = \int_X (\operatorname{Re}\alpha\operatorname{Re} f - \operatorname{Im}\alpha\operatorname{Im} f)\,d\mu + i\int_X (\operatorname{Re}\alpha\operatorname{Im} f + \operatorname{Im}\alpha\operatorname{Re} f)\,d\mu$$

$$= \operatorname{Re}\alpha\int_X \operatorname{Re} f\,d\mu - \operatorname{Im}\alpha\int_X \operatorname{Im} f\,d\mu + i\left(\operatorname{Re}\alpha\int_X \operatorname{Im} f\,d\mu + \operatorname{Im}\alpha\int_X \operatorname{Re} f\,d\mu\right),$$

by Proposition 30,

$= (\operatorname{Re}\alpha + i\operatorname{Im}\alpha)\left(\int_X \operatorname{Re} f\, d\mu + i\int_X \operatorname{Im} f\, d\mu\right)$

$= \alpha\int_X f\, d\mu$.

This proves that the Lebesgue integral is a complex linear functional on $L^1(X,\mathbb{C},\mu)$.

Proposition 32. If $f \in L^1(X,\mathbb{C},\mu)$ or $f \in L^1(X,\mu)$ then $\left|\int_X f\, d\mu\right| \le \int_X |f|\, d\mu$.

Proof. Suppose $f \in L^1(X,\mathbb{C},\mu)$. Let $\int_X f\, d\mu = re^{i\theta}$, where $r = \left|\int_X f\, d\mu\right|$ and $\theta = \arg\int_X f\, d\mu$.

Therefore, $\left|\int_X f\, d\mu\right| = r = e^{-i\theta}\int_X f\, d\mu = \int_X \left(e^{-i\theta} f\right) d\mu = \int_X \operatorname{Re}\left(e^{-i\theta} f\right) d\mu + i\int_X \operatorname{Im}\left(e^{-i\theta} f\right) d\mu$.

Since $e^{-i\theta}\int_X f\, d\mu = r$ is real, $\int_X \operatorname{Im}\left(e^{-i\theta} f\right) d\mu = 0$ and $\left|\int_X f\, d\mu\right| = \int_X \operatorname{Re}\left(e^{-i\theta} f\right) d\mu$.

It follows that $\left|\int_X f\, d\mu\right| \le \int_X |f|\, d\mu$ as $\operatorname{Re}\left(e^{-i\theta} f\right) \le |f|$.

Suppose $f \in L^1(X,\mu)$. If $\int_X f\, d\mu = 0$, then we have nothing to prove. Suppose $\int_X f\, d\mu \ge 0$, then $\left|\int_X f\, d\mu\right| = \int_X f\, d\mu \le \int_X |f|\, d\mu$. If $\int_X f\, d\mu < 0$, $\left|\int_X f\, d\mu\right| = -\int_X f\, d\mu \le \int_X |f|\, d\mu$. It follows that $\left|\int_X f\, d\mu\right| \le \int_X |f|\, d\mu$.

Now for $f \in L^1(X,\mathbb{C},\mu)$ or $f \in L^1(X,\mu)$, we can define $\|f\|_{\mu,1} = \int_X |f|\, d\mu$. Then $L^1(X,\mathbb{C},\mu)$ and $L^1(X,\mu)$ are almost a normed linear space. In order that this definition gives rise to a norm, we have to take equivalence classes of functions in $L^1(X,\mathbb{C},\mu)$ or $L^1(X,\mu)$. We say $f = g$ *almost everywhere*, if there exists a measurable subset *N* such that $\mu(N) = 0$ and $f(x) = g(x)$ for all *x* not in *N*. Thus, if we take the equivalence classes of almost everywhere equal functions, then the above definition $\|f\|_{\mu,1} = \int_X |f|\, d\mu$ gives a norm on equivalence classes of almost everywhere equal measurable functions in $L^1(X,\mathbb{C},\mu)$ and $L^1(X,\mu)$. This norm induced a metric on the equivalence classes and with this metric the equivalence classes of almost everywhere equal measurable functions in $L^1(X,\mathbb{C},\mu)$ and $L^1(X,\mu)$ are complete metric spaces, which are also called Banach spaces. (See Theorem 11, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*.)

Theorem 33. Lebesgue Dominated Convergence Theorem.

Suppose $(X, \mathscr{M})$ is a measure space and $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X.

Suppose $(f_n : X \to \mathbb{C})$ is a sequence of $\mathscr{M}$- measurable functions on X and $f_n \to f$ pointwise on X. Suppose further that there exists a Lebesgue integrable function $g : X \to [0, \infty]$ such that $|f_n| \le g$ for all integer $n \ge 1$. Then, $f_n, f \in L^1(X, \mathbb{C}, \mu)$ and $\int_X f_n \, d\mu \to \int_X f \, d\mu$. Even more is true, $\int_X |f_n - f| \, d\mu \to 0$.

Proof. Since $|f_n| \le g$ and $f_n \to f$ pointwise on X, $|f| \le g$. By Corollary 14 part (1), f is measurable. As g is Lebesgue integrable and $|f_n| \le g$, f_n is Lebesgue integrable for all integer $n \ge 1$. For the same reason, f is Lebesgue integrable. That is to say, $f_n, f \in L^1(X, \mathbb{C}, \mu)$.

Note that for each integer $n \ge 1$, $|f_n - f| \le |f_n| + |f| \le 2g$ and so $2g - |f_n - f| \ge 0$ for all integer $n \ge 1$. We may now apply Fatou's Lemma (Theorem 27). Note that $2g - |f_n - f| \to 2g$ pointwise on X. Therefore, $\liminf_{n\to\infty} \left(2g - |f_n - f|\right) = 2g$ and by Fatou's Lemma,

$$\int_X 2g \, d\mu \le \liminf_{n\to\infty} \int_X \left(2g - |f_n - f|\right) d\mu = \int_X 2g d\mu + \liminf_{n\to\infty} \int_X \left(-|f_n - f|\right) d\mu$$

But by Proposition 10, *All About Lim Sup And Lim Inf,*

$$\liminf_{n\to\infty} \int_X \left(-|f_n - f|\right) d\mu = -\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu .$$

Therefore,

$$\int_X 2g \, d\mu \le \int_X 2g d\mu - \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \ .$$

This implies that $\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \le 0$. But $\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \ge 0$. Hence,

$$\limsup_{n\to\infty} \int_X |f_n - f| \, d\mu = 0 .$$

It follows that $\liminf_{n\to\infty} \int_X |f_n - f| \, d\mu = \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$.

Therefore, $\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$.

By Proposition 32, $0 \le \left| \int_X (f_n - f) \, d\mu \right| \le \int_X |f_n - f| \, d\mu$. Therefore, by the Squeeze Theorem,

$\int_X (f_n - f)\, d\mu \to 0$. It follows that $\int_X f_n\, d\mu = \int_X f\, d\mu + \int_X (f_n - f)\, d\mu \to \int_X f\, d\mu$.

Remark.

We may replace the convergence of f_n by $f_n \to f$ pointwise *almost everywhere* on *X* and that $|f_n| \le g$ for all integer $n \ge 1$ except on a set of measure zero. We explain this below.

Suppose there exists a $\mathscr{M}$- measurable subset *A* such that $\mu(A) = 0$ and $f_n(x) \to f(x)$ for all *x* not in *A*. Suppose that there exists a $\mathscr{M}$- measurable subset *B* such that $\mu(B) = 0$ and $|f_n(x)| \le g(x)$ for integer $n \ge 1$ and for all *x* not in *B*. Let $N = A \cup B$ and $E = (A \cup B)^c$. Then $\mu(N) = 0$ and *E* is $\mathscr{M}$- measurable. It follows by Theorem 33, that f_n and f are Lebesgue integrable over *E* and hence over *X* in *some sense* since $E^c = N$ is of measure zero. We elaborate on this below. By Theorem 33, we have that $\int_E f_n\, d\mu \to \int_E f\, d\mu$ and so

$$\int_X f_n\, d\mu = \int_E f_n\, d\mu + \int_N f_n\, d\mu = \int_E f_n\, d\mu \to \int_E f\, d\mu .$$

As $\mu(N) = 0$, we may ignore the behaviour of the function over *N*, we may arbitrarily set the meaning of the integral over a set of measure zero to be zero, even though the function may not be measurable over the null set and hence may not actually be μ integrable over *X*. Thus, we may set $\int_X f\, d\mu = \int_E f\, d\mu + \int_N f\, d\mu = \int_E f\, d\mu$ and

$$\int_X |f_n - f|\, d\mu = \int_E |f_n - f|\, d\mu + \int_N |f_n - f|\, d\mu$$

so that $\int_X |f_n - f|\, d\mu = \int_E |f_n - f|\, d\mu + \int_N |f_n - f|\, d\mu = \int_E |f_n - f|\, d\mu \to 0$.

We may legitimately do this if we simply take the integral over the completion $\overline{\mu}$ of the measure μ.

Note that the restriction of f_n to *E*, $f_n|_E$ converges pointwise to a measurable function *h* on *E*. Therefore, since *E* is measurable, *h* is measurable on *E*. This means that for any open set, *U*, in $\mathbb{C}$ or $\overline{\mathbb{R}}$, $f^{-1}(U) \cap E = h^{-1}(U)$ is measurable but it is not necessary that $f^{-1}(U)$ is measurable as $f^{-1}(U) \cap E^c \subseteq A \cup B$ and $f^{-1}(U) \cap E^c$ need not be measurable. If the measure space (*X*, $\mathscr{M}$) is μ- complete, then $f^{-1}(U) \cap E^c$ is measurable, so that $f^{-1}(U)$ is measurable. Hence, if the measure space (*X*, $\mathscr{M}$) is μ- complete, then we may conclude that the almost everywhere limit function, f, is measurable. Note that here, we have used the meaning of the integral $\int_X f\, d\mu$ to be $\int_E f\, d\mu$ since E^c is of measure zero. Similarly, for the other integral, $\int_X |f_n - f|\,d\mu$ is to be understood as $\int_E |f_n - f|\,d\mu$ and of course, $\int_N f\,d\mu$ and $\int_N |f_n - f|\,d\mu$ are to be taken as zero.

To state the corresponding conclusion in Theorem 33 for $(f_n : X \to \mathbb{C})$ a sequence of $\mathcal{M}$-measurable functions on X converging almost everywhere pointwise to f on X, it is customary to assume that f is μ-measurable so that the integral, $\int_X f\,d\mu$, is defined.

If we are just interested in the limit of the integral, $\int_X f_n\,d\mu$, we may define a function $h : X \to \mathbb{C}$ by $h(x) = \begin{cases} \lim_{n\to\infty} f_n|_E(x), x \in E \\ 0, x \in E^c \end{cases}$. Then h is μ- measurable and $h = f$ almost everywhere on X and $\int_X f_n\,d\mu \to \int_X h\,d\mu$. We may not conclude that f is μ-measurable and integrable with respect to the measure μ. Of course, if the measure μ is complete, then f is μ-measurable and $\int_X f\,d\mu = \int_X h\,d\mu$.

In view of the above remark, we may state the following variation of the Dominated Convergence Theorem:

Suppose $(X, \mathcal{M})$ is a measure space and $\mu : \mathcal{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X.

Suppose $(f_n : X \to \mathbb{C})$ is a sequence of $\mathcal{M}$- measurable functions on X and $f_n \to f$ pointwise almost everywhere on X. Suppose further that there exists a Lebesgue integrable function $g : X \to [0, \infty]$ such that $|f_n(x)| \le g(x)$ for almost all x in X and for all integer $n \ge 1$. Then, $f_n \in L^1(X, \mathbb{C}, \mu)$ and there exists $h \in L^1(X, \mathbb{C}, \mu)$ such that $f = h$ almost everywhere on X, with respect to μ, $\int_X f_n\,d\mu \to \int_X h\,d\mu$ and $\int_X |f_n - h|\,d\mu \to 0$.

Corollary 34. Suppose $(X, \mathcal{M})$ is a measure space and $\mu : \mathcal{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X. Suppose $(f_n : X \to \mathbb{C})$ is a sequence of $\mathcal{M}$- measurable functions on X such that

$$\sum_{n=1}^{\infty} \int_X |f_n|\,d\mu < \infty.$$

Then $\sum_{n=1}^{\infty} f_n$ converges except perhaps on the points of some set contained in some set of μ-measure zero, N, and if we define f by $f(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x), \text{ if } x \in N^c \\ 0, \text{ if } x \in N \end{cases}$, then

$$\int_X f\,d\mu = \sum_{n=1}^{\infty}\int_X f_n d\mu .$$

Proof. By hypothesis, each f_n is Lebesgue integrable over *X*. Consider the function $g=\sum_{n=1}^{\infty}|f_n|$. Then *g* is a function into $\overline{\mathbb{R}^+}$. Since each $|f_n|$ is measurable, *g* is measurable.

By Theorem 26, $\int_X g\,d\mu = \int_X\left(\sum_{n=1}^{\infty}|f_n|\right)d\mu = \sum_{n=1}^{\infty}\int_X |f_n|\,d\mu < \infty$. This implies that g is Lebesgue integrable on *X*, i.e, $g\in L^1(X,\mathbb{C},\mu)$, and that if we let $G=\{x\in X : g(x)<\infty\}=g^{-1}([0,\infty))$, then *G* is $\mathscr{M}$- measurable and $\mu(G^c)=0$. Let $N=G^c$. This means $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely on *G*. Since $\mu(G^c)=0$, $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere on *X*. We define

$$f(x)=\begin{cases}\sum_{n=1}^{\infty} f_n(x), & \text{if } x\in G,\\ 0, & \text{if } x\in G^c = N\end{cases}.$$

Then f is measurable and $\sum_{k=1}^{n} f_k(x)$ converges pointwise to $f(x)$ for *x* in *G*. Now $\left|\sum_{k=1}^{n} f_k(x)\right| \le \sum_{k=1}^{n}|f_k(x)| \le g$ and g is Lebesgue integrabe implies that $\sum_{k=1}^{n} f_k(x)$ is Lebesgue integrable on X, hence on *G*. Therefore, by the Lebesgue Dominated Convergence Theorem (Theorem 33),

$$\int_G\left(\sum_{k=1}^{n} f_k\right)d\mu \to \int_G f\,d\mu .$$

Hence $\sum_{k=1}^{n}\int_G f_k\,d\mu \to \int_G f\,d\mu$. Thus $\sum_{k=1}^{\infty}\int_G f_k\,d\mu = \int_G f\,d\mu$. As $\mu(G^c)=0$, it follows that $\int_X f\,d\mu = \sum_{n=1}^{\infty}\int_X f_n d\mu$.

Set of Measure Zero And The Completion of A Measure

If *P* is some property of points of a measure space $(X, \mathscr{M})$ and μ is a positive measure on *X*, (for example, " $\sum_{n=1}^{\infty} f_n(x)$ converges ") and if $\{x : \textit{not } P(x)\}$ is contained in some set of μ-measure zero, then we say that the property *P* holds *almost everywhere with respect to* μ on

X, abbreviated *a. e.* [μ]. We may simply say that P holds almost everywhere on X when the measure μ is understood to have been given.

Examples.

1. In the previous Corollary, $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely almost everywhere on X.

2. If $f, g : X \to \mathbb{C}$ are two complex functions and if $\{x : f(x) \neq g(x)\}$ is contained in a set of measure zero, then f and g are equal *a.e.* [μ]. Note that the relation "equal *a.e.* [μ]" is an equivalence relation on the collection of all complex functions on X. Moreover, if $f = g$ *a.e.* [μ], then $\int_X f\, d\mu = \int_X g\, d\mu$, the integrals either both exist or both do not exist. The behaviour of functions on set of measure zero is not noticed by the integral.

3. It may happen (though rarely in practice) that $A \in \mathcal{M}$, $\mu(A) = 0$ and $B \subseteq A$ but $B \notin \mathcal{M}$. However, we would like $\mu(B) = 0$.

Example. $X = [0,1]$, $\mathcal{M} = \{ \emptyset, X, [0,1], (1,2] \}$. $\mu(\emptyset) = \mu\big((1,2]\big) = 0$, $\mu\big([0,1]\big) = \mu(X) = 1$. So $\mu\big((1,2]\big) = 0$ but no proper subset of $(1,2]$ is $\mathcal{M}$- measurable.

If the behaviour of (3) above does not happen, then we say the measure μ is *complete.*

The triplet $(X, \mathcal{M}, \mu)$, where $(X, \mathcal{M})$ is a measure space and $\mu : \mathcal{M} \to \overline{\mathbb{R}^+}$ is a positive measure on the σ-algebra, is also called a *measure space*, where we specifically specify the measure function μ.

Proposition 35. Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let

$$\mathcal{M}^* = \{ E \subseteq X : \text{there exists } A, B \in \mathcal{M}, \text{ such that } A \subseteq E \subseteq B \text{ and } \mu(B - A) = 0 \}.$$

Define for E in $\mathcal{M}^*$, $\mu^*(E) = \mu^*(A) = \mu^*(B) = \mu(A)$. Then $\mathcal{M}^*$ is a σ-algebra, μ^* is a (positive) measure on $\mathcal{M}^*$ and the measure space $(X, \mathcal{M}^*, \mu^*)$ is complete. We call the triplet $(X, \mathcal{M}^*, \mu^*)$ the μ - completion of $\mathcal{M}$.

Proof.

(1) $\mathcal{M}^*$ is a σ-algebra.

(i) Plainly, $X \in \mathcal{M}^*$.

(ii) If $E \in \mathcal{M}^*$, then there exists $A, B \in \mathcal{M}$, such that $A \subseteq E \subseteq B$ and $\mu(B - A) = 0$. Hence,

$B^c \subseteq E^c \subseteq A^c$ and $\mu\left(A^c - B^c\right) = \mu\left(B - A\right) = 0$. Therefore, $E^c \in \mathscr{M}^*$.

(iii) If $\{E_n\}_{n=1}^{\infty}$ is a countable collection of sets in $\mathscr{M}^*$, then for each integer $n \geq 1$, there exists A_n, $B_n \in \mathscr{M}$, such that $A_n \subseteq E_n \subseteq B_n$ and $\mu(B_n - A_n) = 0$. Hence,

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} B_n.$$

Since $\mathscr{M}$ is a σ-algebra, $\bigcup_{n=1}^{\infty} A_n$ and $\bigcup_{n=1}^{\infty} B_n$ are in $\mathscr{M}$. Note that

$$\mu\left(\bigcup_{n=1}^{\infty} B_n - \bigcup_{n=1}^{\infty} A_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty}\left(B_n - A_n\right)\right) \leq \sum_{n=1}^{\infty} \mu\left(B_n - A_n\right) = 0.$$

The last inequality is by the σ-sub-additive of μ.

Hence $\mu\left(\bigcup_{n=1}^{\infty} B_n - \bigcup_{n=1}^{\infty} A_n\right) = 0$ and so $\bigcup_{n=1}^{\infty} E_n \in \mathscr{M}^*$.

Therefore, $\mathscr{M}^*$ is a σ-algebra.

(2) μ^* is well defined on $\mathscr{M}^*$.

Let $E \in \mathscr{M}^*$.

Suppose we have A_i, $B_i \in \mathscr{M}$, such that $A_i \subseteq E \subseteq B_i$ and $\mu(B_i - A_i) = 0$ for $i = 1, 2$.

Then $A_1 - A_2 \subseteq B_2 - A$ and since $\mu(B_2 - A_2) = 0$, $\mu(A_1 - A_2) = 0$. Similarly, we get $\mu(A_2 - A_1) = 0$.

Therefore,

$$\mu(A_1) = \mu\left(\left(A_1 \cap A_2\right) \cup \left(A_1 - A_2\right)\right) = \mu\left(A_1 \cap A_2\right) + \mu\left(A_1 - A_2\right) = \mu\left(A_1 \cap A_2\right)$$

$$= \mu\left(A_1 \cap A_2\right) + \mu\left(A_2 - A_1\right) = \mu(A_2).$$

Therefore, $\mu^*(E)$ is independent of the choice of A, $B \in \mathscr{M}$, such that $A \subseteq E \subseteq B$ and $\mu(B - A) = 0$.

(3) μ^* is a positive measure on $\mathscr{M}^*$.

Plainly, $\mu^*(\emptyset) = 0$. We now show that μ^* is σ-additive.

Suppose $\{E_n\}_{n=1}^{\infty}$ is a countable collection of pairwise disjoint sets in $\mathscr{M}^*$. Then for each integer $n \geq 1$, there exists A_n, $B_n \in \mathscr{M}$, such that $A_n \subseteq E_n \subseteq B_n$ and $\mu(B_n - A_n) = 0$. Since the collection $\{E_n\}_{n=1}^{\infty}$ is pairwise disjoint, $\{A_n\}_{n=1}^{\infty}$ is a countable collection of disjoint sets in $\mathscr{M}$.

We have shown that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} B_n$ and $\mu\left(\bigcup_{n=1}^{\infty} B_n - \bigcup_{n=1}^{\infty} A_n\right) = 0$. Therefore,

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n), \text{ by the } \sigma\text{-additivity of } \mu,$$

$$= \sum_{n=1}^{\infty} \mu^*(E_n).$$

This proves that μ^* is σ-additive and so it is a positive measure.

If $\mathscr{M} = \mathscr{M}^*$, then we say the σ-algebra is *μ-complete.*

We have made use of the σ-sub-additivity of the measure μ.

We state the result below.

Lemma 36. Any positive measure, μ, on a σ-algebra $\mathscr{M}$ is σ-sub-additive.

Proof.

Suppose $\{C_n\}_{n=1}^{\infty}$ is a countable collection of sets in the σ-algebra $\mathscr{M}$.

Let $C = \bigcup_{n=1}^{\infty} C_n$, $H_1 = C_1$, $H_2 = C_2 - C_1$ and for integer $n \geq 2$, $H_n = C_n - \bigcup_{k=1}^{n-1} C_k$. Then for integer $n \geq 1$, $H_n \subseteq C_n$, $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} H_n$ and the collection $\{H_n\}_{n=1}^{\infty}$ is pairwise disjoint. Note that each $H_n \in \mathscr{M}$. Therefore, by σ-additivity or countable additivity,

$$\mu\left(\bigcup_{n=1}^{\infty} C_n\right) = \mu\left(\bigcup_{n=1}^{\infty} H_n\right) = \sum_{n=1}^{\infty} \mu(H_n) \leq \sum_{n=1}^{\infty} \mu(C_n).$$

This shows that μ is σ-sub-additive.

Proposition 37. Suppose $(X, \mathscr{M}, \mu)$ is a measure space with μ-completion, $(X, \mathscr{M}^*, \mu^*)$.

Let f be a $\mathscr{M}^*$ - measurable real or complex function on X. (f might not be $\mathscr{M}$ - measurable.) Then there exists a $\mathscr{M}$ - measurable function g such that $f = g$ *a.e.* $[\mu]$.

Proof.

(1) We prove first for the characteristic function of members of $\mathscr{M}^*$. Suppose $E \in \mathscr{M}^*$, and $A, B \in \mathscr{M}$, such that $A \subseteq E \subseteq B$ and $\mu(B - A) = 0$. For $f = \chi_E$, take $g = \chi_A$. Therefore, $f(x) = g(x)$ except possibility for x in $B - A$ which is of μ-measure zero.

Hence, $f = g$ *a.e.* $[\mu]$.

(2) Hence, for $\mathscr{M}^*$-measurable simple function, f, which is a finite linear combination of characteristic functions, there exists a $\mathscr{M}$-measurable simple function g such that $f = g$ *a.e.* $[\mu]$. We elaborate this as follows.

Suppose $s = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ is a $\mathscr{M}^*$-measurable simple functions. We may assume that E_i's are disjoint sets in $\mathscr{M}^*$. Then take $A_i, B_i \in \mathscr{M}$, such that $A_i \subseteq E_i \subseteq B_i$ and $\mu(B_i - A_i) = 0$, $i = 1,\ldots, n$. Let $g = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$. Then $s = g$ *a.e.* $[\mu]$ since $\mu\left(\bigcup_{i=1}^{n} (B_i - A_i) \right) = 0$.

(3) Suppose f is a non-negative $\mathscr{M}^*$ - measurable function from X into $\overline{\mathbb{R}^+}$. By Theorem16, there exists a monotone increasing sequence of non-negative $\mathscr{M}^*$ - measurable simple functions (s_n) converging pointwise to f. By (2) above, there exists a monotone increasing sequence of non-negative $\mathscr{M}$ - measurable simple functions (t_n) such that $s_n = t_n$ *a.e.* $[\mu]$. Therefore, (t_n) converges pointwise to a $\mathscr{M}$ - measurable function g. Since union of countable number of sets of μ-measure 0 is also of μ-measure 0, $g = f$ *a.e.* $[\mu]$.

(4) Suppose f is a real $\mathscr{M}^*$ - measurable function from X into $\mathbb{R}$. Then write $f = f_+ - f_-$. By part (3), there exists non-negative real $\mathscr{M}$ - measurable functions k and h such that

$k = f_+$ *a.e.* $[\mu]$ and $h = f_-$ *a.e.* $[\mu]$. Thus, $g = k - h = f$ *a.e.* $[\mu]$.

Suppose f is a complex $\mathscr{M}^*$ - measurable function. Then $f = \operatorname{Re} f + i \operatorname{Im} f$ and $\operatorname{Re} f$ and $\operatorname{Im} f$ are real $\mathscr{M}^*$ - measurable functions. By what we have just proven, there exists real $\mathscr{M}$ - measurable functions g_1 and g_2 such that $\operatorname{Re} f_+ = g_1$ *a.e.* $[\mu]$ and $\operatorname{Im} f_+ = g_2$ *a.e.* $[\mu]$. Then $g = g_1 + i\, g_2$ is a complex $\mathscr{M}$ - measurable function and $f = g$ *a.e.* $[\mu]$.

This completes the proof.

Remark. Thus, in view of the above proposition, we may ignore the behaviour of functions on sets of measure zero.

1. We may extend the definition of measurability of a function $f: X \to Y$ to mean "there exists a set $E \subseteq X$ such that $\mu(E^c) = 0$ and $f^{-1}(U) \cap E \in \mathscr{M}$, So f may be badly behaved and not even defined on E^c.

2. Corollary to Lebesgue Dominated Convergence Theorem. Suppose $(f_n : X \to \mathbb{C})$ is a sequence of complex $\mathscr{M}$- measurable functions on X such that $\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty$. Then $\sum_{n=1}^{\infty} f_n(x)$ converges for almost all x with respect to μ and

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu .$$

The function $\sum_{n=1}^{\infty} f_n(x)$ may not be defined on some set of μ-measure zero.

Proposition 38. Suppose $(X, \mathscr{M}, \mu)$ is a measure space and $\mu : \mathscr{M} \to \overline{\mathbb{R}^+}$ is a positive measure on X. Suppose $f \in L^1(X, \mathbb{C}, \mu)$ or $f \in L^1(X, \mathbb{R}, \mu) = L^1(X, \mu)$.

(1) If $E \in \mathscr{M}$ and $\int_E |f| \, d\mu = 0$, then $f = 0$ *a.e.* [μ] on E.

(2) If for all $E \in \mathscr{M}$, $\int_E f \, d\mu = 0$, then $f = 0$ *a.e.* [μ] on X.

Proof.

Proof of part (1)

(1) We shall prove first for non-negative function f in $L^1(X, \mu)$.

Take $E \in \mathscr{M}$. For each integer $n \geq 1$, let $A_n = \left\{ x \in E : f(x) \geq \frac{1}{n} \right\}$. Then $A_n \in \mathscr{M}$ and

$$\int_E |f| \, d\mu = \int_E f \, d\mu \geq \int_{A_n} f \, d\mu \geq \int_{A_n} \frac{1}{n} d\mu = \frac{1}{n} \mu(A_n) .$$

As $\int_E |f| \, d\mu = 0$, $\mu(A_n) = 0$. Since $\{x \in E : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n$,

$$\mu\left(\{x \in E : f(x) > 0\}\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n) = 0 .$$

Therefore, $\mu\left(\{x \in E : f(x) > 0\}\right) = 0$. Hence, $f = 0$ *a.e.* [μ] on E.

(2) Suppose f is in $L^1(X, \mu)$. Then f is real valued. Write f as $f_+ - f_-$ so that $|f| = f_+ + f_-$. Then f_+ and f_- are both Lebesgue integrable. Moreover $\int_E |f| \, d\mu = 0$

implies that $\int_E f_+ \, d\mu = 0$ and $\int_E f_- \, d\mu = 0$. It follows by (1), f_+, $f_- = 0$ *a.e.* [μ] on *E*. Hence, $f = f_+ - f_- = 0$ *a.e.* [μ] on *E*.

Suppose $f \in L^1(X, \mathbb{C}, \mu)$. Then $f = \text{Re} f + i \,\text{Im} f$ and $\text{Re} f$ and $\text{Im} f$ are Lebesgue integral. Moreover as $\int_E |\text{Re} f| \, d\mu, \int_E |\text{Im} f| \, d\mu \le \int_E |f| \, d\mu = 0$, $\int_E |\text{Re} f| \, d\mu = \int_E |\text{Im} f| \, d\mu = 0$.

It follows from (2) that $\text{Re} f$, $\text{Im} f = 0$ *a.e.* [μ] on *E*. Hence, $f = \text{Re} f + i \,\text{Im} f = 0$ *a.e.* [μ] on *E*.

Proof of part (2)

Suppose f is real valued, i.e., $f \in L^1(X, \mathbb{R}, \mu) = L^1(X, \mu)$. Write $f = f_+ - f_-$. Let $E = \{x \in X : f_+(x) > 0\}$. Then $\int_E f \, d\mu = \int_E f_+ \, d\mu = 0$ implies by part (1) that $f_+ = 0$ *a.e.* [μ] on *E*. It then follows that $f_+ = 0$ *a.e.* [μ] on *X*. Similarly, we show that $f_- = 0$ *a.e.* [μ] on *X*. Therefore, $f = f_+ - f_- = 0$ *a.e.* [μ] on *X*.

Suppose now $f \in L^1(X, \mathbb{C}, \mu)$. Then write $f = \text{Re} f + i \,\text{Im} f$. For all $E \in \mathcal{M}$, $\int_E f \, d\mu = 0$ implies that for all $E \in \mathcal{M}$, $\int_E \text{Re} f \, d\mu = \int_E \text{Im} f \, d\mu = 0$. Hence by what we have just proven, $\text{Re} f, \text{Im} f = 0$ *a.e.* [μ] on X. Therefore, $f = \text{Re} f + i \,\text{Im} f = 0$ *a.e.* [μ] on *X*.

The advantage of having a complete measure space is evident in the following proposition.

Proposition 39. Suppose (X, $\mathcal{M}$, μ) is a complete measure space. That is, $\mathcal{M}$ is a σ-algebra, $\mu : \mathcal{M} \to \overline{\mathbb{R}^+}$ is a positive measure on $\mathcal{M}$, and $\mathcal{M}$ is μ-complete. Let $E \subseteq X$ be a $\mathcal{M}$-measurable subset of *X*. Suppose $f, g : E \to \overline{\mathbb{R}}$ are any two extended real valued functions which are equal almost everywhere with respect to μ on *E*. Then f is $\mathcal{M}$-measurable on *E* if, and only if, g is $\mathcal{M}$-measurable on *E*.

Proof. By hypothesis, there exists $D \in \mathcal{M}$ such that $f = g$ on $E-D$ and $\mu(D) = 0$. Let $H = \{x \in E : f(x) \ne g(x)\}$. Then $H \subseteq D$. Since $\mathcal{M}$ is μ-complete and $\mu(D) = 0$, we have $H \in \mathcal{M}$ and $\mu(H) = 0$.

Suppose f is $\mathcal{M}$-measurable. Then by Proposition 10, for any $a \in \mathbb{R}$, $f^{-1}((a, \infty]) \in \mathcal{M}$. Consider $g^{-1}((a, \infty]) \cap H$. Since $\mu\left(g^{-1}((a, \infty]) \cap H\right) = 0$ and $\mathcal{M}$ is μ-complete, $g^{-1}((a, \infty]) \cap H \in \mathcal{M}$. Now $g^{-1}((a, \infty]) \cap E = \left(g^{-1}((a, \infty]) \cap H\right) \cup g^{-1}\left(((a, \infty]) \cap (E - H)\right)$.

Now, E and H are $\mathscr{M}$-measurable implies that $E–H$ is also $\mathscr{M}$-measurable. Observe that $g^{-1}\left((a,\infty]\right)\cap\left(E-H\right)=f^{-1}\left((a,\infty]\right)\cap\left(E-H\right)\in\mathscr{M}$. Therefore, $g^{-1}\left((a,\infty]\right)\cap E\in\mathscr{M}$. Hence, by Proposition 10, g is $\mathscr{M}$-measurable on E.

We can show similarly, that if g is $\mathscr{M}$-measurable, then f is $\mathscr{M}$-measurable.

Suppose $(X, \mathscr{T})$ is a topological space. Suppose $(X, \mathscr{M}, \mu)$ is a measure space, where $\mathscr{M}$ contains all the Borel subsets of X and μ is a positive measure on $\mathscr{M}$. Let $E\in\mathscr{M}$. An extended real valued function $f:E\to\overline{\mathbb{R}}$ is said to be *continuous a.e.* $[\mu]$ on E if, and only if, there exists a $\mathscr{M}$-measurable subset $D\subseteq E$ such that f is continuous on $E–D$ and $\mu(D) = 0$.

Proposition 40. Suppose $(X, \mathscr{T})$ is a topological space and $(X, \mathscr{M}, \mu)$ is a μ-complete measure space, where $\mathscr{M}$ contains $\mathscr{B}$, the collection of all the Borel subsets of X and μ is a positive measure on $\mathscr{M}$. Let E be a non-empty $\mathscr{M}$-measurable subset of X. If $f:E\to\overline{\mathbb{R}}$ is *continuous a.e.* $[\mu]$ on E, then f is $\mathscr{M}$-measurable.

Proof. By definition, there exists a $\mathscr{M}$-measurable subset $D\subseteq E$ such that f is continuous on $E–D$ and $\mu(D) = 0$. By Proposition 10 (3), it is sufficient to show that for any a in $\mathbb{R}$, $f^{-1}(a,\infty]$ is measurable, i.e., $f^{-1}(a,\infty]\in\mathscr{M}_E$.

Let $x\in f^{-1}\left((a,\infty]\right)\cap\left(E-D\right)$. Then $f(x)>a$. If $f(x) = \infty$, then by continuity at x, there exists an open set U_x containing x such that $f\left(U_x\cap(E-D)\right)\subseteq\left(a,\infty\right]$.

Hence, $x\in U_x\cap(E-D)\subseteq f^{-1}\left((a,\infty]\right)$. If $f(x) < \infty$, then let $\varepsilon=\dfrac{f(x)-a}{2}>0$. By the continuity of f at x, there exists an open set U_x containing x such that $f\left(U_x\cap(E-D)\right)\subseteq\left(f(x)-\varepsilon,f(x)+\varepsilon\right)$. Therefore,

$$x\in U_x\cap(E-D)\subseteq f^{-1}\left((f(x)-\varepsilon,f(x)+\varepsilon)\right)\subseteq f^{-1}\left((a,\infty]\right).$$

Take $V=\bigcup_{x\in f^{-1}\left((a,\infty]\right)\cap(E-D)}U_x$. Then $f^{-1}\left((a,\infty]\right)\cap\left(E-D\right)=V\cap\left(E-D\right)$. It follows that

$$f^{-1}\left((a,\infty]\right)=\left(V\cap\left(E-D\right)\right)\cup\left(f^{-1}\left((a,\infty]\right)\cap D\right).$$

Since V is open and $\mathscr{M}$ contains all the Borel subsets of X, $V\in\mathscr{M}$ and as $E-D\subseteq E$ is $\mathscr{M}$-measurable $V\cap\left(E-D\right)\in\mathscr{M}_E$. Since $f^{-1}\left((a,\infty]\right)\cap D\subseteq D$, $\mu(D) = 0$ and $\mathscr{M}$ is μ-complete, $f^{-1}\left((a,\infty]\right)\cap D\in\mathscr{M}_E$. It follows that $f^{-1}\left((a,\infty]\right)\in\mathscr{M}$. Thus, we have shown that for any $a\in\mathbb{R}$, $f^{-1}\left((a,\infty]\right)\in\mathscr{M}$ and so f is $\mathscr{M}$-measurable.

Suppose $(X, \mathscr{M}, \mu)$ is a measure space, where $\mathscr{M}$ is a σ-algebra and μ is a positive measure, Recall the space of Lebesgue integrable functions on X, $L^1(X,\mathbb{C},\mu)$ or $L^1(X,\mu)=L^1(X,\mathbb{R},\mu)$. We have shown in Proposition 30 and Proposition 31 that $L^1(X,\mu)=L^1(X,\mathbb{R},\mu)$ is a real vector space and $L^1(X,\mathbb{C},\mu)$ is a complex vector space. We define the non-negative function $\| \ \|_{1,\mu}$ on the respective vector space by

$$\|f\|_{1,\mu}=\int_X |f|\,d\mu \quad \text{for } f\in L^1(X,\mu) \text{ or } L^1(X,\mathbb{C},\mu).$$

Then this function satisfies

(i) for all f, $\|f\|_{1,\mu}\geq 0$; $f=0 \Rightarrow \|f\|_{1,\mu}=0$,

(ii) for any scalar λ and any f, $\|\lambda f\|_{1,\mu}=|\lambda|\|f\|_{1,\mu}$ and

(iii) for any f and g, $\|f+g\|_{1,\mu}\leq \|f\|_{1,\mu}+\|g\|_{1,\mu}$.

Any function satisfying the analogous conditions to (i), (ii) and (iii) on a vector space is called a *semi-norm.*

By Proposition 38, $\|f\|_{1,\mu}=0$ implies that $f = 0$ *a.e.* $[\mu]$.

Now let $N=\{f\in L^1(X,\mathbb{R},\mu): f=0 \text{ } a.e. \text{ } [\mu]\}$. Then N is a vector subspace of $L^1(X,\mu)$.

Define $\mathscr{L}^1(X,\mu) = L^1(X,\mu)/N$ the equivalence classes of almost everywhere equal functions in $L^1(X,\mu)$. That is, the equivalence relation on $L^1(X,\mu)$ is given by f is *equivalent* to g if $f = g$ *a.e.* $[\mu]$. $\mathscr{L}^1(X,\mu)$ is again a real vector space with the zero element given by the equivalence class of all functions $f = 0$ *a.e.* $[\mu]$. We can extend the definition of $\| \ \|_{1,\mu}$ to $\mathscr{L}^1(X,\mu)$ and it now satisfies,

(i)′ for all f in $\mathscr{L}^1(X,\mu)$, $\|f\|_{1,\mu}\geq 0$, $\|f\|_{1,\mu}=0 \Leftrightarrow f=0$,

(ii) for any scalar λ and any f in $\mathscr{L}^1(X,\mu)$, $\|\lambda f\|_{1,\mu}=|\lambda|\|f\|_{1,\mu}$ and

(iii) for any f and g in $\mathscr{L}^1(X,\mu)$, $\|f+g\|_{1,\mu}\leq \|f\|_{1,\mu}+\|g\|_{1,\mu}$.

This means that $\| \ \|_{1,\mu}$ is a norm on $\mathscr{L}^1(X,\mu)$. We can view $\mathscr{L}^1(X,\mu)$ as a metric space by giving it the metric associated with this norm, by $d(f,g)=\|f-g\|_{1,\mu}$. With this metric, $\mathscr{L}^1(X,\mu)$ is a complete metric space, which is called a Banach space. (See Theorem 11, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*.)

Similarly, if $\mathscr{L}^1(X, \mathbb{C}, \mu) = L^1(X, \mathbb{C}, \mu) / N$, where $N = \{ f \in L^1(X, \mathbb{C}, \mu) : f = 0 \; a.e. \; [\mu] \}$, with the norm given by, for $\tilde{f} \in \mathscr{L}^1(X, \mathbb{C}, \mu)$, $\|\tilde{f}\|_{1,\mu} = \int_X |f| \, d\mu$, where f is a representative of the equivalence class $\tilde{f}$, $\mathscr{L}^1(X, \mathbb{C}, \mu)$ is a normed vector space and with the metric associated with this norm, it is a Banach space.

For convenience, when there is no confusion, we often also denote $\mathscr{L}^1(X, \mathbb{C}, \mu)$ by $L^1(X, \mathbb{C}, \mu)$ and $\mathscr{L}^1(X, \mu)$ by $L^1(X, \mu)$.

This concludes the modest introduction to measure theory. For follow up on L^p Spaces, see my article, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*.