National University of Singapore

Department of Mathematics

Semester 1 (2006/2007) **MA3110 Mathematical Analysis II** Tutorial 8

1. Suppose $f : [a, b] \to \mathbf{R}$ is a Lispchitz function, i.e., there exists a constant C such that $|f(x) - f(y)| \le C|x - y|$ for all x and y in $[a, b]$. Prove that f is integrable.

2. Suppose that S is a non-empty bounded subset of $\mathbf{R}$. For a real number k, define $kS = \{ ks : s \in S\}$. Prove that
   (i) $\sup kS = k \sup S$ and $\inf kS = k \inf S$ if $k \ge 0$
   (ii) $\sup kS = k \inf S$ and $\inf kS = k \sup S$ if $k < 0$.

3. Suppose $f : [0, 1] \to \mathbf{R}$ is defined by $f(x) = \begin{cases} \frac{1}{2}x^2, & x \text{ rational} \\ -\frac{1}{2}x^2, & x \text{ irrational} \end{cases}$. Prove that f is not integrable.

4. Suppose $f : [a, b] \to \mathbf{R}$ is continuous and such that $\int_a^b f = 0$. Prove that there is a point c in $[a, b]$ such that $f(c) = 0$.

5. Suppose $f : [1, 2] \to \mathbf{R}$ is defined by $f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{n}, & x \text{ rational and } x = \frac{m}{n} \text{ in its lowest term} \end{cases}$. Prove that f is integrable.

6. Suppose $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ are integrable. Prove the following Cauchy Schwarz inequality:

$$\int_a^b fg \le \sqrt{\int_a^b f^2}\sqrt{\int_a^b g^2}.$$

[Hint: For each number λ, define $p(\lambda) = \int_a^b (f - \lambda g)^2$. Then $p(\lambda)$ is a quadratic function always ≥ 0.]

7. Suppose $f : [a, b] \to \mathbf{R}$ is bounded and continuous except at one point x_0 in the interior (a, b). Prove that f is integrable.

8. Suppose $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ are integrable. Prove that

$$\int_a^b |f + g| \le \int_a^b |f| + \int_a^b |g|.$$