Comment on A 2019 PSLE Math Question

By Ng Tze Beng

PSLE Question

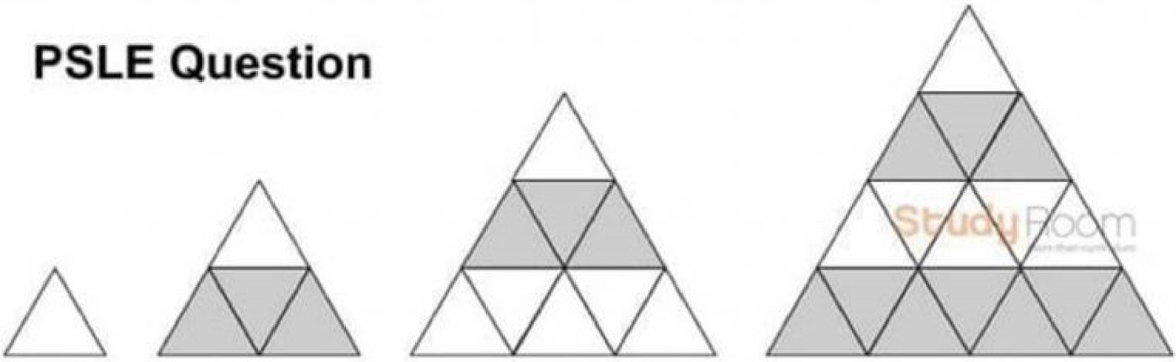


Fig 1 Fig 2 Fig 3 Fig 4

Figure	1	2	3	4	5
White triangles	1	1	6	6	15
Grey triangles	0	3	3	10	10
Total	1	4	9	16	25

a) Fill in the graph above (1m)
b) Find the total number of grey and white triangles for Figure 250 (1m)
c) Find the percentage of grey triangles in Figure 250 (3m)

This question aims at the ability to observe number pattern to *conjecture* a general formula and use the formula for computation. It does not require a proof of the formula nor the knowledge of the formula for the sum of Arithmetic Progression.

If one has prior knowledge of the formula for the sum of n terms of an arithmetic progression, then it is fairly easy to give the answer. (AP is in the H2 Maths syllabus under the item sequences and series.)

This is a very good question to lead the students to (i) examine the pattern of numbers to conjecture a formula, (ii) find a means to prove the formula and (iii) appreciate the path from conjecture to proof. One wishes that perhaps this kind of questions could have been the subject of discussion in the school lessons.

Firstly, we can examine the geometric pattern. The number of the triangles in the bottom row in each figure is obtained by adding two triangles from the row above. So, we can confirm that the sequence of number of triangles in the rows of each Figure form an AP with first term 1 and common difference 2.

By looking at the filled table values, one can *conjecture* that the total number of the triangles in Fig n is n^2.

The sum of AP to the n-th term is given by

$$S_n = \frac{n}{2}(2u_1 + (n-1)d),$$

where u_1 is the first term and d is the common difference. So, putting $u_1 = 1$ and $d = 2$ we get n^2.

Without knowledge of the sum formula, by examining the number pattern of the total number of triangles in the table, we may *conjecture* that the total number of triangles in Fig n, is n^2, as each entry under column n is n^2 for n =1 to 5. To *conjecture* the number of grey triangles in Fig 2n, we would need more entries of the table to observe the pattern of the number of grey triangles.

We observe that the number of triangles in any grey row for Fig 2n with $n \geq 1$ is obtained from the number of triangles from the grey row above by adding 4 triangles. The first term is 3 and there are n terms with common difference 4. Hence, it is an AP. Using the formula for the sum of AP, we can compute that the number of grey triangles in Fig 2n is given by

$$\frac{n}{2}(6 + (n-1)4) = n(2n+1) \ .$$

Figure	1	2	3	4	5	6	7	8	9
White triangles	1	1	6	6	15	15	28	28	45
Grey Triangles	0	3	3	10	10	21	21	36	36
Total	1	4	9	16	25	36	49	64	81

By examining the pattern, 3 ,10, 21, 36, ……, to *conjecture* or *guess* the n-th term formula is not easy. It just so happens that this is an easy AP and if one wants to make a connection with the digit n, one can write

3 = 1*3, 10 = 2*5, 21 = 3*7, 36 = 4*9, …...

And note that these numbers are in column, 2, 4, 6, 8, ……

So, for the first factor, we may associate it with $n/2$, where n = 2,4,6,8, … and for the second factor with n+1 and thus *conjecture* that the number of grey triangles in Fig *2n* is $n(2n+1)$.

Alternatively, form the difference of the even entries as below:

n	2		4		6		8		10	
Grey triangles	3		10		21		36		55	
Difference		7		11		15		19		

This clearly shows that the number of grey triangles is the sum of an AP with first term 3 and common difference 4. This is obtained purely by examining the number pattern, whereas we have deduced this before from the geometric pattern.

Thus, the total number of triangles in Fig 250 is 250*250 = 62,500 and the total number of grey triangles in Fig 250 is 125*251= 31,375 and from these numbers, you can compute the percentage of grey triangles in Fig 250.

The key lesson from the question is that there is a formula to deal with the number of triangles and so the *validity of the formula* is important so as not to conclude from the special case to the general. One hopes that this will motivate finding a proof of the formula. Computation in the case when n = 250 is just a self-verification of the formula, knowing especially that indeed one can just add up the numbers to double check, with the faith that the conjectured formula is correct.

Students, who have been schooled in *conjecturing* or *guessing* a formula from number pattern, probably will not find this question difficult. On the other hand, students who have never been exposed to *conjecturing* or *guessing* formula from number pattern may find this extremely difficult and will have to start to guess or think on the fly how to get a formula for the number of grey triangles. To get the formula for the total number of triangles by *guessing* is easy and if they stare at the number for the grey triangles long enough, some may eventually get the answer too. The concern is how long will it take under the examination setting, especially when it is one of many questions. This kind of *conjecturing* and *guessing* is of some value but what we should be after is the derivation of the formula in a mathematically correct way or a mathematically meaningful computation.

The question could have been made a little more palatable, simply by asking for, say the values for Figure 40, instead of Fig 250. A smaller number is less intimidating and serves the same purpose. A student may observe that the triangles in Figure n form a sequence, 1,3,5,7,9, … , $2n$-1 and that the grey triangles in Figure $2n$ form a sequence, 3,7,11,15, …., $4n$-1. It is possible that some students instead of conjecturing or guessing a formula, obtain the answer by adding the same sequence back to back. See table below.

n	1	2	3	4	--------	40
Number of triangles in row n	1	3	5	7	---------	80-1
Number of triangles in row n in reverse order	80-1	80-3	80-5	80-7	---------	1
Sum	80	80	80	80	80	80

Thus, it can be affirmed that the total number of triangles in Fig. 40 is $\frac{1}{2}40\times 80 = 40\times 40$.

Similarly,

n	1	2	3	4	--------	20
Number of triangles in row $2n$	3	7	11	15	------------	80-1
Number of triangles in row $2n$ in reverse order	80-1	80-5	80-9	80-13	-----------	3
sum	80+2	80+2	80+2	80+2	-------	80+2

Thus, the number of grey triangles in Figure 40 is $\frac{1}{2}20\times(80+2) = 20\times 41$.

This is a much better answer than *conjecturing* or *guessing*. Firstly, it is good and honest computation and secondly therein lies the analogous path towards a general formula for any AP.

The realization or the conclusion that the forward sequence increases by the common difference and the backward sequence starting from the n-th term decreases by the same common difference would render the sum of the first term with the n-th term, the second term with the n-1 term, the third with the n-2 term and so on to be the same. With a smaller value of n, coming to this realization, albeit with a small set of data, is a mark of cognitive achievement and application of this very simple principle: if you have two quantities, say a and b, if you increase a by an amount and decrease b by the same amount, then the sum is always the same. By asking for the result for Figure $2n$ with smaller n, we may get much better answers than what is expected. It is possible that different approach to the computation can be devised. I would say that the working says more about the student then just the numerical answer.

Below are some samples of pattern and number pattern questions (Question 11 to 17). Look at question 15. You will find a similarity with the triangle question above. Try these questions.

For Exercises 11–14, find the nth term in the sequence.

11.

1	2	3	4	5	6	...	n	...	20
-1	1	3	5	7	9	...	-?-	...	-?-

12.

1	2	3	4	5	6	...	n	...	20
0	3	10	21	36	55	...	-?-	...	-?-

13.

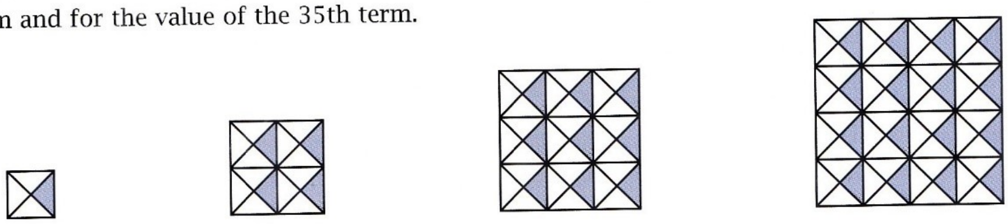
1	2	3	4	5	6	...	n	...	20
-1	-1	0	2	5	9	...	-?-	...	-?-

14.

1	2	3	4	5	6	...	n	...	20
0	4	11	21	34	50	...	-?-	...	-?-

For Exercises 15–17, copy and complete the table. Make a conjecture for the value of the nth term and for the value of the 35th term.

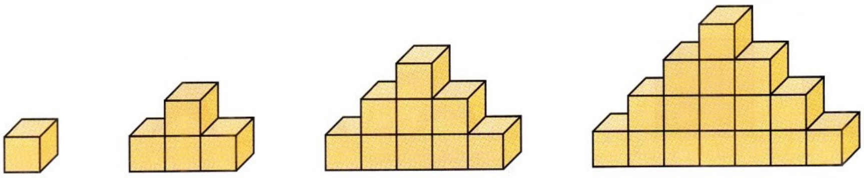
15.



Triangles in a square array

Squares/side	1	2	3	4	5	6	...	n	...	35
Shaded Δs	1	4	-?-	-?-	-?-	-?-	...	-?-	...	-?-
Unshaded Δs	3	12	-?-	-?-	-?-	-?-	...	-?-	...	-?-

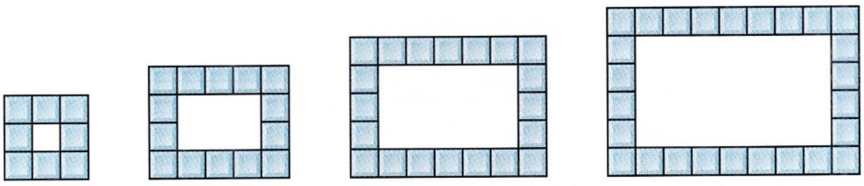
16.*



Building block pattern

Height	1	2	3	4	5	6	...	n	...	35
Total blocks	1	4	-?-	-?-	-?-	-?-	...	-?-	...	-?-

17.*



Rectangular donut pattern

Donut	1	2	3	4	5	...	n	...	35
Number of squares	8	14	-?-	-?-	-?-	...	-?-	...	-?-

October 2019