Kestelman gave the most general form of the change of variable theorem for Riemann integral. We present here a proof of this theorem involving a result about the chain rule for composition and the properties of absolute continuity.

**Theorem 1 (Kestelman).** Suppose $h$ is Riemann integrable on the closed and bounded interval $[a, b]$ and $H: [a, b] \to \mathbb{R}$ is an indefinite integral of $h$, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for $x$ in $[a, b]$. Suppose $f$ is a bounded function on $H([a, b])$. Then $f$ is Riemann integrable if and only if $f \circ H(x)h(x)$ is Riemann integrable. Moreover, whenever $f$ or $f \circ H(x)h(x)$ is Riemann integrable, we have the change of variable formula for Riemann integral,

$$ \int_a^b f(H(x))h(x)dx = \int_{H(a)}^{H(b)} f(x)dx. $$

The first assertion in Theorem 1 of the simple connection between the functions in the above integrands is stated below in the following Theorem.

**Theorem 2.** Suppose $h$ is Riemann integrable on the closed and bounded interval $[a, b]$ and $H: [a, b] \to \mathbb{R}$ is an indefinite integral of $h$, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for $x$ in $[a, b]$. Suppose $f$ is a bounded real valued function on $H([a, b]) = [c, d]$. Then $f$ is Riemann integrable on $[c, d]$, if and only if, $f(H(x))h(x)$ is Riemann integrable on $[a, b]$.

**Proof.** Since $h$ is Riemann integrable on $[a, b]$, $h$ is continuous almost everywhere on $[a, b]$. Therefore, there exists a subset $A$ of $[a, b]$ of measure zero and $h$ is continuous on $[a, b] - A$. Thus, by the Fundamental Theorem of Calculus, for $x$ in $[a, b] - A$, $H$ is differentiable at $x$ and $H'(x) = h(x)$.

Suppose $f: [c, d] \to \mathbb{R}$ is Riemann integrable. Then $f$ is continuous almost everywhere on $[c, d]$. Hence, there exists a subset $E$ in $[c, d]$ of measure zero such that $f$ is continuous on $[c, d] - E$.

Suppose $x \in [a, b] - A$ and $h(x) = 0$. Since $h$ is continuous at $x$ and $f$ is bounded so that $f \circ H$ is also bounded, $\lim_{y \to x} f(H(y))h(y) = \lim_{y \to x} h(y) = 0$. Hence $f(H(t))h(t)$ is continuous at $x$. Let $L = \{ x \in [a, b] - A : H'(x) = h(x) \neq 0 \}$.

By Theorem 2 of *Change of Variables Theorems*, since $m(B \cap ([a, b] - A)) = 0$ because $m(B) = 0$, $H'(x) = h(x) = 0$ almost everywhere on $B \cap ([a, b] - A)$. It follows that $f(H(t))h(t)$ is continuous almost everywhere on $B \cap ([a, b] - A)$. Hence $f(H(t))h(t)$ is continuous almost everywhere on $[a, b]$ and so is continuous almost everywhere on $[a, b]$. This means that $f(H(t))h(t)$ is Riemann integrable on $[a, b]$. 
Suppose $f$ is bounded and $f(H(t))h(t)$ is Riemann integrable on $[a, b]$. We shall show that $f$ is continuous almost everywhere on $[c, d]$ and so is Riemann integrable. To do this we use the following proposition.

**Proposition 3.** Suppose $h$ is Riemann integrable on the closed and bounded interval $[a, b]$ and $H: [a, b] \to \mathbb{R}$ is an indefinite integral of $h$, i.e., $H(x) = H(a) + \int_a^x h(t)\,dt$ for $x$ in $[a, b]$.

Suppose $A$ is a subset of $[a, b]$ of measure zero such that $h$ is continuous on $[a, b] - A$ and $H'(x) = h(x)$. Then for $x$ in $[a, b] - A$, $f(H(t))h(t)$ is continuous at $x$, if and only if, $h(x) = 0$ or $f$ is continuous at $H(x)$.

**Proof.** We have already shown that if $x \in [a, b] - A$ and $h(x) = 0$, then $f(H(t))h(t)$ is continuous at $x$. Plainly if $f$ is continuous at $H(x)$, then $f(H(t))$ is continuous at $x$ since $H$ is continuous at $x$ and so $f(H(x))h(x)$ is continuous at $x$.

Suppose $f(H(t))h(t)$ is continuous at $x \in [a, b] - A$ and $h(x) \neq 0$. Then plainly, $f(H(t))$ is continuous at $x$. We shall show that $f$ is continuous at $H(x)$. Note that

$$\lim_{t \to x} \frac{H(t) - H(x)}{t - x} = H'(x) = h(x) \neq 0.$$ 

We may assume without loss of generality that $x$ is in the interior of $[a, b]$. Suppose $h(x) > 0$. Then there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq (a, b)$ and

$$|t - x| < \delta \Rightarrow |h(t) - h(x)| < \frac{1}{2} h(x) \Rightarrow h(t) > \frac{1}{2} h(x) > 0.$$ 

Therefore, for all $t_1 < t_2$ and $t_1, t_2 \in (x - \delta, x + \delta)$,

$$H(t_2) - H(t_1) = \int_{t_1}^{t_2} h(t)\,dt \geq \int_{t_1}^{t_2} \frac{h(x)}{2}\,dt = \frac{h(x)}{2} (t_2 - t_1) > 0.$$ 

Hence, $H$ is a continuous and strictly increasing function on $(x - \delta, x + \delta)$. This means that the restriction of $H$ to the interval $(x - \delta, x + \delta)$ has a strictly increasing continuous inverse $g$. Since $g$ is injective and is continuous at $H(x)$ and $f \circ H(t)$ is continuous at $x$ so that

$$\lim_{t \to x} f \circ H(t) = f(H(x))$$

and

$$g(H(x)) = x,$$

$$\lim_{y \to H(x)} f(y) = \lim_{y \to g(H(x))} (f \circ H) \circ g(y) = f(H(x)).$$

This means $f$ is continuous at $H(x)$.

We deduce similarly that if $h(x) < 0$, $f$ is continuous at $H(x)$.

This concludes the proof of Proposition 3.

**Completion of the proof of Theorem 2.**

Now suppose $f(H(t))h(t)$ is Riemann integrable on $[a, b]$ and so $f(H(t))h(t)$ is continuous almost everywhere on $[a, b] - A$.

Now for $x$ in $[a, b] - A$, by Proposition 3, $f(H(t))h(t)$ is not continuous at $x$ if and only if $h(x) \neq 0$ and $f$ is not continuous at $H(x)$.

Let $C = \{x \in [a, b] - A: f$ is not continuous at $H(x)\}$ and

$$D = \{x \in [a, b] - A: h(x) \neq 0\} = \{x \in [a, b] - A: H'(x) = h(x) \neq 0\} = L.$$
Thus, \( f(H(t))h(t) \) is not continuous at \( x \) if and only if \( x \in C \cap D \).

Therefore, since \( f(H(t))h(t) \) is continuous almost everywhere on \([a, b] - A, m(C \cap D) = 0\).

Since \( H \) is absolutely continuous on \([a, b], m(H(C \cap D)) = 0\).

Let \( \widetilde{D} = \{ x \in [a, b] - A : h(x) = 0 \} \). Then \([a, b] - A = D \cup \widetilde{D} \) and

\[
C = C \cap (D \cup \widetilde{D}) = (C \cap D) \cup (C \cap \widetilde{D})
\]

By Theorem 3 of Functions Having Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin’s Theorem,

\[
m(H(\widetilde{D})) = m \left( H \left( \left\{ x \in [a, b] - A : H'(x) = h(x) = 0 \right\} \right) \right) = 0.
\]

Thus, since \( H(C) \subseteq H(C \cap D) \cup H \left( C \cap \widetilde{D} \right), \ m \left( H(C) \right) = 0 \). Since \( A \) is of measure zero and \( H \) is absolutely continuous, \( m \left( H(A) \right) = 0 \). As \( H \) maps \([a, b]\) onto \([c, d]\), it follows that

\[
E = \{ y \in [c, d] : f \text{ is not continuous at } y \} \subseteq H(A) \cup H(C).
\]

Hence, \( m(E) = 0 \). Thus, \( f \) is continuous almost everywhere on \([c, d]\) and so is Riemann integrable on \([c, d]\).

**Proof of the Second part of Theorem 1.**

By Theorem 2, we may assume that \( f \) is Riemann integrable on \([c, d]\). Let \( F : [c, d] \to \mathbb{R} \) be an indefinite integral of \( f \). Then \( F \) is an absolutely continuous function satisfying a Lipschitz condition. Since \( H : [a, b] \to [c, d] \) is absolutely continuous, \( F \circ H \) is absolutely continuous on \([a, b]\). Therefore, \( F \circ H \) has finite derivative almost everywhere on \([a, b]\). Since \( F \) is absolutely continuous, \( F \) is an \( N \)-function, therefore by the following Chain Rule (see Theorem 4 below),

\[
(F \circ H)'(x) = (F \circ H)(x)H'(x) = (F \circ H)(x)h(x)
\]

almost everywhere on \([a, b]\). By Theorem 2, \( f(H(x))h(x) \) is Riemann integrable on \([a, b]\).

Therefore, \( (F \circ H)' \) is Riemann integrable on \([a, b]\). Hence, by the Fundamental Theorem of Calculus, the Riemann integral,

\[
\int_a^b (F \circ H)'(t)dt = F \circ H(b) - F \circ H(a) = F(H(b)) - F(H(a)).
\]

We may also deduce this as follows.

Since \( F \circ H \) is absolutely continuous on \([a, b]\), \( (F \circ H)' \) is Lebesgue integrable on \([a, b]\) and the Lebesgue integral of the derivative,

\[
\text{Lebesgue } \int_a^b (F \circ H)'(t)dt = F \circ H(b) - F \circ H(a) = F(H(b)) - F(H(a)).
\]

But \( (F \circ H)' \) is Riemann integrable on \([a, b]\) so that the Lebesgue integral is equal to the Riemann integral and so

\[
\int_a^b (F \circ H)'(t)dt = F(H(b)) - F(H(a)).
\]

Since \( F \) is an indefinite Riemann integral of \( f \), \( \int_a^b (F \circ H)'(t)dt = \int_{H(a)}^{H(b)} f(t)dt \).
Hence, \[ \int_{H(a)}^{H(b)} f(x)dx = \int_{a}^{b} (F \circ H)'(x) = \int_{a}^{b} (f \circ H)(x)h(x)dx = \int_{a}^{b} (f \circ H)(x)h(x)dx \]

**Theorem 4.** Suppose \( F \) has finite derivatives almost everywhere on \([c, d]\) and \( g \) and \( F \circ g \) have finite derivatives almost everywhere on \([a, b]\). It is assumed that the range of \( g \) is contained in \([c, d]\). Suppose \( F \) is an \( N \)-function, i.e., \( F \) maps sets of measure zero to sets of measure zero. Then \((F \circ g)' = (f \circ g)'\) almost everywhere on \([a, b]\), where \( F' = f \) almost everywhere on \([c, d]\), that is to say, the chain rule holds almost everywhere on \([a, b]\).

Theorem 4 is Theorem 3 of *Change of Variables Theorems* and the proof can be found there.