On Kestelman Change of Variable Theorem for Riemann Integral

By Ng Tze Beng

Kestelman gave the most general form of the change of variable theorem for Riemann integral. We present here a proof of this theorem involving a result about the chain rule for composition and the properties of absolute continuity.

Theorem 1 (Kestelman). Suppose *h* is Riemann integrable on the closed and bounded interval [*a*, *b*] and *H*: [*a*, *b*] → **R** is an indefinite integral of *h*, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for *x* in [*a*, *b*]. Suppose f is a bounded function on *H*([*a*, *b*]). Then f is Riemann integrable if and only if $f \circ H(x)h(x)$ is Riemann integrable. Moreover, whenever f or $f \circ H(x)h(x)$ is Riemann integrable, we have the change of variable formula for Riemann integral,

$$\int_a^b f(H(x))h(x)dx = \int_{H(a)}^{H(b)} f(x)dx \ .$$

The first assertion in Theorem 1 of the simple connection between the functions in the above integrands is stated below in the following Theorem.

Theorem 2. Suppose *h* is Riemann integrable on the closed and bounded interval [*a*, *b*] and *H*: [*a*, *b*] → **R** is an indefinite integral of *h*, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for *x* in [*a*, *b*]. Suppose f is a bounded real valued function on *H*([*a*, *b*]) = [*c*, *d*]. Then f is Riemann integrable on [*c*, *d*], if and only if, $f(H(x))h(x)$ is Riemann integrable on [*a*, *b*].

Poof.
Since *h* is Riemann integrable on [*a*, *b*], h is continuous almost everywhere on [*a*, *b*]. Therefore, there exists a subset *A* of [*a*, *b*] of measure zero and *h* is continuous on [*a*, *b*] − *A*. Thus, by the Fundamental Theorem of Calculus, for *x* in [*a*, *b*] − *A*, *H* is differentiable at x and $H'(x) = h(x)$.
Suppose f: [*c*, *d*] → **R** is Riemann integrable. Then f is continuous almost everywhere on [*c*, *d*]. Hence, there exists a subset *E* in [*c*, *d*] of measure zero such that *f* is continuous on [*c*, *d*]−*E*.
Suppose $x \in [a,b] - A$ and $h(x) = 0$. Since *h* is continuous at *x* and f is bounded so that $f \circ H$ is also bounded, $\lim_{y \to x} f(H(x))h(x) = \lim_{y \to x} h(x) = 0$. Hence $f(H(t))h(t)$ is continuous at *x*. Let $L = \{x \in [a,b] - A : H'(x) = h(x) \neq 0\}$. Let $B = H^{-1}(E)$.
By Theorem 2 of *Change of Variables Theorems*, since $m\left(B \cap ([a,b] - A)\right) = 0$ because $m(B) = 0$, $H'(x) = h(x) = 0$ almost everywhere on $B \cap ([a,b] - A)$. It follows that $f(H(t))h(t)$ is continuous almost everywhere on $B \cap ([a,b] - A)$. Hence $f(H(t))h(t)$ is continuous almost everywhere on $[a,b] - A$ and so is continuous almost everywhere on [*a*, *b*]. This means that $f(H(t))h(t)$ is Riemann integrable on [*a*, *b*].

Suppose f is bounded and $f(H(t))h(t)$ is Riemann integrable on [a, b]. We shall show that f is continuous almost everywhere on [c, d] and so is Riemann integrable. To do this we use the following proposition.

Proposition 3. Suppose h is Riemann integrable on the closed and bounded interval [a, b] and H: [a, b] $\to$ **R** is an indefinite integral of h, i.e., $H(x) = H(a) + \int_a^x h(t)dt$ for x in [a, b]. Suppose A is a subset of [a, b] of measure zero such that h is continuous on [a, b] − A and $H'(x) = h(x)$. Then for x in [a, b] − A, $f(H(t))h(t)$ is continuous at x, if and only if, $h(x) = 0$ or f is continuous at $H(x)$.

Proof. We have already shown that if $x \in [a, b] - A$ and $h(x) = 0$, then $f(H(t))h(t)$ is continuous at x. Plainly if f is continuous at $H(x)$, then $f(H(t))$ is continuous at x since H is continuous at x and so $f(H(x))h(x)$ is continuous at x.
Suppose $f(H(t))h(t)$ is continuous at $x \in [a,b] - A$ and $h(x) \neq 0$. Then plainly, $f(H(t))$ is continuous at x. We shall show that f is continuous at $H(x)$. Note that
$\lim_{t \to x} \frac{H(t) - H(x)}{t - x} = H'(x) = h(x) \neq 0$.
We may assume without loss of generality that x is in the interior of [a, b].
Suppose $h(x) > 0$. Then there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq (a,b)$ and

$$|t - x| < \delta \Rightarrow |h(t) - h(x)| < \frac{1}{2}h(x) \Rightarrow h(t) > \frac{1}{2}h(x) > 0 .$$

Therefore, for all $t_1 < t_2$ and $t_1, t_2 \in (x - \delta, x + \delta)$,

$$H(t_2) - H(t_1) = \int_{t_1}^{t_2} h(t)dt \geq \int_{t_1}^{t_2} \frac{h(x)}{2} dt = \frac{h(x)}{2}(t_2 - t_1) > 0 .$$

Hence, H is a continuous and strictly increasing function on $(x - \delta, x + \delta)$. This means that the restriction of H to the interval $(x - \delta, x + \delta)$ has a strictly increasing continuous inverse g. Since g is injective and is continuous at $H(x)$ and $f \circ H(t)$ is continuous at x so that $\lim_{t \to x} f \circ H(t) = f(H(x))$ and $g(H(x)) = x$,

$$\lim_{y \to H(x)} f(y) = \lim_{y \to H(x)} \left(f \circ H\right) \circ g(y) = f(H(x)) .$$

This means f is continuous at $H(x)$.
We deduce similarly that if $h(x) < 0$, f is continuous at $H(x)$.
This concludes the proof of Proposition 3.

Completion of the proof of Theorem 2.

Now suppose $f(H(t))h(t)$ is Riemann integrable on [a, b] and so $f(H(t))h(t)$ is continuous almost everywhere on [a, b] − A.

Now for x in [a, b] − A, by Proposition 3, $f(H(t))h(t)$ is not continuous at x if and only if $h(x) \neq 0$ and f is not continuous at $H(x)$.
Let $C = \{x \in [a,b] - A : f \text{ is not ontinuous at } H(x)\}$ and

$$D = \{x \in [a,b] - A : h(x) \neq 0\} = \{x \in [a,b] - A : H'(x) = h(x) \neq 0\} = L .$$

Thus, $f(H(t))h(t)$ is not continuous at x if and only if $x \in C \cap D$.
Therefore, since $f(H(t))h(t)$ is continuous almost everywhere on $[a, b] - A$, $m(C \cap D) = 0$.
Since H is absolutely continuous on $[a, b]$, $m\left(H(C \cap D)\right) = 0$.
Let $\widetilde{D} = \{x \in [a,b] - A : h(x) = 0\}$. Then $[a,b] - A = D \cup \widetilde{D}$ and

$$C = C \cap \left(D \cup \widetilde{D}\right) = \left(C \cap D\right) \cup \left(C \cap \widetilde{D}\right).$$

By Theorem 3 of *Functions Having Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*,

$$m(H(\widetilde{D})) = m\left(H\left(\{x \in [a,b] - A : H'(x) = h(x) = 0\}\right)\right) = 0.$$

Thus, since $H(C) \subseteq H\left(C \cap D\right) \cup H\left(C \cap \widetilde{D}\right)$, $m\left(H(C)\right) = 0$. Since A is of measure zero and H is absolutely continuous, $m\left(H(A)\right) = 0$. As H maps $[a, b]$ onto $[c, d]$, it follows that

$$E = \{y \in [C,d] : f \text{ is not continuous at } y\} \subseteq H(A) \cup H(C).$$

Hence, $m(E) = 0$. Thus, f is continuous almost everywhere on $[c, d]$ and so is Riemann integrable on $[c, d]$.

Proof of the Second part of Theorem 1.

By Theorem 2, we may assume that f is Riemann integrable on $[c, d]$. Let $F : [c,d] \to \mathbb{R}$ be an indefinite integral of f. Then F is an absolutely continuous function satisfying a Lipschitz condition. Since $H : [a,b] \to [c,d]$ is absolutely continuous, $F \circ H$ is absolutely continuous on $[a, b]$. Therefore, $F \circ H$ has finite derivative almost everywhere on $[a, b]$. Since F is absolutely continuous, F is an N-function, therefore by the following Chain Rule (see Theorem 4 below),

$$\left(F \circ H\right)'(x) = \left(f \circ H\right)(x) H'(x) = \left(f \circ H\right)(x) h(x)$$

almost everywhere on $[a, b]$. By Theorem 2, $f(H(x))h(x)$ is Riemann integrable on $[a, b]$.
Therefore, $\left(F \circ H\right)'$ is Riemann integrable on $[a, b]$. Hence, by the Fundamental Theorem of Calculus, the Riemann integral,

$$\int_a^b \left(F \circ H\right)'(t)\,dt = F \circ H(b) - F \circ H(a) = F(H(b)) - F(H(a)).$$

We may also deduce this as follows.
Since $F \circ H$ is absolutely continuous on $[a, b]$, $\left(F \circ H\right)'$ is Lebesgue integrable on $[a, b]$ and the Lebesgue integral of the derivative,

$$\text{Lebesgue} \int_a^b \left(F \circ H\right)'(t)\,dt = F \circ H(b) - F \circ H(a) = F(H(b)) - F(H(a)).$$

But $\left(F \circ H\right)'$ is Riemann integrable on $[a, b]$ so that the Lebesgue integral is equal to the Riemann integral and so

$$\int_a^b \left(F \circ H\right)'(t)\,dt = F(H(b)) - F(H(a)).$$

Since F is an indefinite Riemann integral of f, $\int_a^b \left(F \circ H\right)'(t)\,dt = \int_{H(a)}^{H(b)} f(t)\,dt$.

Hence, $\int_{H(a)}^{H(b)} f(x)dx = \int_a^b \left(F \circ H\right)'(x) = \int_a^b \left(f \circ H\right)(x)h(x)dx = \int_a^b \left(f \circ H\right)(x)h(x)dx$

Theorem 4. Suppose F has finite derivatives almost everywhere on $[c, d]$ and g and $F \circ g$ have finite derivatives almost everywhere on $[a, b]$. It is assumed that the range of g is contained in $[c, d]$. Suppose F is an N-function, i.e., F maps sets of measure zero to sets of measure zero. Then $(F \circ g)' = (f \circ g)\, g'$ almost everywhere on $[a, b]$, where $F' = f$ almost everywhere on $[c, d]$, that is to say, the chain rule holds almost everywhere on $[a, b]$.

Theorem 4 is Theorem 3 of *Change of Variables Theorems* and the proof can be found there.