Positive Borel Measure and Riesz Representation Theorem

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Introduction.

The Riemann integral operates on some collection of functions, these functions are continuous or at most not very discontinuous, that is, discontinuous on at most a set of measure zero and be bounded with a domain usually bounded and if unbounded the function would be zero outside a bounded subset. It produces a number, a scalar. The functions on which it operates form a vector space; usually the vector space comes with some natural definition of a norm. Being a space of functions, the collection of real functions in the space comes with a natural partial order, $f \leq s$ if and only if $f(x) \leq s(x)$ for all x in the domain. The Riemann integral integrates the positive function to give a positive number C. More generally, if $f \leq g$, then $\int f \leq \int g$. The Riemann integration is linear.

Hence the classical Riemann integral is a particular real linear functional on $C_c(\mathbb{R})$, the real vector space of all continuous real valued function on $\mathbb{R}$ with compact support. Observe that from this definition of the Riemann integral we can derive the basic idea of the length of the interval.

$$\text{Length of } [a,b] = b - a = \int_{\mathbb{R}} \chi_{[a,b]} = \inf_{[a,b] \prec f} \int f = \sup_{g \prec (a,b)} \int g \ .$$

By Urysohn's Lemma, since $\mathbb{R}$ is a locally compact Hausdorff topological space, taking $K = [a,b]$ and U be any open interval (c, d) such that $(c,d) \supseteq [a,b]$, there exists a function $f \in C_c(\mathbb{R})$ such that $[a,b] \prec f \prec (c,d)$. This means that $\chi_{[a,b]} \leq f \leq \chi_{(c,d)}$. Taking $[\alpha, \beta] \subseteq (a,b)$, there exists $f \in C_c(\mathbb{R})$ such that $[\alpha, \beta] \prec f \prec (a,b)$, that is, $\chi_{[\alpha,\beta]} \leq f \leq \chi_{(a,b)}$. We may thus define the length of $[a, b]$ by taking it to be $\inf_{[a,b] \prec f} \int f$ or $\sup_{g \prec (a,b)} \int g$.

We then aim to construct a Lebesgue integral, which will integrate much more general functions, generalising this definition of a Riemann integral. We shall consider complex vector space and complex linear functional. Recall that a linear transformation from a real vector space to the real numbers $\mathbb{R}$ is called a real linear functional and that a linear transformation from a complex vector space to the complex numbers $\mathbb{C}$ is called a complex linear functional. We show approximately to any *positive* complex linear functional Λ on $C_c(X)$, the space of all continuous complex function on *X* with compact support, where *X* is a locally compact Hausdorff topological space, there corresponds a measure μ defined on some σ- algebra $\mathscr{M}$ containing the Borel sets of *X* such that

$$\Lambda(f) = \int_X f \, d\mu \ ,$$

for all *f* in $C_c(X)$. Since $C_c(X)$ is dense in $L^1(X,\mu)$ in the L^1 norm as the measure μ will have the additional properties that satisfies Theorem 23 of *Convex Functions, L^p Spaces, Space of |Continuous Functions, Lusin's Theorem*, we can then extend the linear functional Λ from $C_c(X)$ to $L^1(X,\mu)$.

A complex linear functional, $\Lambda : C_c(X) \to \mathbb{C}$ on $C_c(X)$, is said to be *positive*, if for any real valued function *f* in $C_c(X)$, $f \geq 0 \Rightarrow \Lambda(f) \geq 0$. Similarly, a real linear functional $\Phi : C_{c,\mathbb{R}}(X) \to \mathbb{R}$ on $C_{c,\mathbb{R}}(X)$, the space of all continuous real valued functions on *X* with compact support, is *positive*, if $f \in C_{c,\mathbb{R}}(X)$ and $f \geq 0 \Rightarrow \Phi(f) \geq 0$.

The following trivial example gives a simplistic view of this correspondence. Let $x_0 \in X$ be a fixed point. Define $\Lambda(f) = f(x_0)$ for any function *f*. This corresponds to the measure, 'unit mass' at x_0. That is, for $E \in \mathscr{M} = \sigma$- algebra of all subsets of *X*,

$$\mu(E) = \begin{cases} 1 \text{ if } x_0 \in E, \\ 0, \text{ otherwise} \end{cases}.$$

For this case, any function $f : X \to \mathbb{C}$ is measurable and $\int_X f \, d\mu = f(x_0)$.

We shall consider the function space of complex valued functions with compact support on a locally compact Hausdorff space *X* and positive complex linear functional on this function space. Then we specialize to the function space of real valued functions on *X* with compact support and positive real linear functional on this function space and bounded real linear functional on this function space.

Riesz Representation Theorem

We now state the association of a complex linear functional with a measure in more explicit terms.

Theorem 1. (Riesz Representation Theorem)

Let X be a locally compact Hausdorff topological space. Let $C_c(X) = \{f : X \to \mathbb{C}; f \text{ is continuous with compact support}\}$. Let $\Lambda : C_c(X) \to \mathbb{C}$ be a positive complex linear functional on $C_c(X)$, i.e., whenever $f \in C_c(X)$ and f is real valued with $f \geq 0$, then $\Lambda(f) \geq 0$. Then we have the following:

(a) There exists a σ- algebra $\mathscr{M}$ on X, containing all the Borel sets of X and there exists a unique positive measure, μ, on $\mathscr{M}$ such that

$$\Lambda(f) = \int_X f \, d\mu \text{ for all } f \in C_c(X).$$

(b) For all compact $K \subseteq X$, $\mu(K) < \infty$.

(c) For all $E \in \mathscr{M}$, $\mu(E) = \inf\{\mu(V) : V \supseteq E \text{ and } V \text{ is open}\}$. (Outer regularity)

(d) For all $E \in \mathscr{M}$, such that either E is open or $\mu(E) < \infty$,

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}.$$ (Inner regularity)

(e) $\mathscr{M}$ is μ-complete, i.e., for all $N \in \mathscr{M}$ such that $\mu(N) = 0$, for $E \subseteq N$, $E \in \mathscr{M}$.

Proof.

We proceed with the proof in the following order. Firstly, we prove that the measure μ is unique. Then we show the existence of the measure μ. The remaining of the proof deals with the conclusions (b) (c) (d) and (e) of the theorem.

A technical result that we need is partition of unity. We shall state and prove this technical result at the end of this note.

Uniqueness of μ.

Suppose μ_1 and μ_2 are two measures on $\mathscr{M}$ satisfying the conclusion of the theorem. Note that the value of the measure, μ, is entirely determined by the value of μ on compact subsets of X by part (d). Thus, it is sufficient to show that $\mu_1(K)=\mu_2(K)$ for any compact subset K of X.

Take any compact subset K of X. By part (b), $\mu_1(K), \mu_2(K) < \infty$. Therefore, given any $\varepsilon > 0$, by part (c), there exists an open set V containing K such that

$$\mu_1(V) < \mu_1(K) + \varepsilon .$$

Now we appeal to Urysohn's Lemma (Lemma 22, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*). Since X is a locally compact Hausdorff topological space, and $K \subseteq V$, with K compact and V open, by Urysohn's Lemma, there exists a continuous function $f \in C_c(X)$ such that $K \prec f \prec V$. This means that $\chi_K \le f \le \chi_V$. Note that

$$\mu_2(K) = \int_X \chi_K \, d\mu_2 \le \int_X f \, d\mu_2 = \Lambda(f) = \int_X f \, d\mu_1 \le \int_X \chi_V \, d\mu_1 = \mu_1(V) < \mu_1(K) + \varepsilon .$$

Since ε is arbitrary, it follows that $\mu_2(K) \le \mu_1(K)$.

Similarly, by reversing the role of μ_1 and μ_2, we can show that $\mu_1(K) \le \mu_2(K)$. Hence $\mu_1(K)=\mu_2(K)$ for any compact subset K of X. Thus, the uniqueness of the measure μ is established.

Now we shall define μ first on open set, then on any subset of X. Subsequently we shall define the σ-algebra $\mathscr{M}$.

Let V be an open set of X. Define $\mu(V)$ by

$$\mu(V) = \sup\{\Lambda(f) : f \in C_c(X) \text{ and } f \prec V\} .$$

For any subset $E \subseteq X$, define

$$\mu(E) = \inf\{\mu(V) : E \subseteq V \text{ and } V \text{ is open in } X\} .$$

Let $\mathscr{M}_F = \{E \subseteq X : \mu(E) < \infty \text{ and } \mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}\}$ and

$\mathscr{M} = \{ E \subseteq X : E \cap K \in \mathscr{M}_F$ for all compact $K \subseteq X \}$.

Observe that (a) $\Rightarrow$ (b).

Take any compact subset K of X. Take any open set $U \supseteq K$. By Urysohn's Lemma, there exists a continuous function with compact support $f \in C_c(X)$ such that $K \prec f \prec U$. That is, $\chi_K \le f \le \chi_U$. Therefore,

$$\mu(K) = \int_X \chi_K d\mu \le \int_X f\, d\mu = \Lambda(f) < \infty .$$

Now suppose U and V are open subsets of X and $V \subseteq U$, then $\mu(V) \le \mu(U)$. This is because $\{f : f \in C_c(X) \text{ and } f \prec V\} \subseteq \{f : f \in C_c(X) \text{ and } f \prec U\}$ so that

$$\mu(V) = \sup\{\Lambda(f) : f \in C_c(X) \text{ and } f \prec V\} \le \sup\{\Lambda(f) : f \in C_c(X) \text{ and } f \prec U\} = \mu(U) .$$

Therefore, if E is open, $\mu(E) = \inf\{\mu(U) : E \subseteq U \text{ and } U \text{ is open in } X\}$. Thus, our definition of $\mu(E)$ for any subset E of X is consistent with the open sets in X.

We shall prove that μ is countably additive on $\mathscr{M}$ and that $\mathscr{M}$ is a σ- algebra in stages.

We note the following properties of the positive (real or complex) linear functional Λ and the function, μ, which is defined on all subsets of X.

(1) Λ is *monotone*, i.e., for f and $g \in C_c(X)$ and f and g are real valued, $f \le g \Rightarrow \Lambda(f) \le \Lambda(g)$. This is because by linearity, $\Lambda(g) = \Lambda(f) + \Lambda(g - f) \ge \Lambda(f)$ as $\Lambda(g - f) \ge 0$.

(2) μ is *monotone*, i.e., for any subsets A and B of X, $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$.

If $A \subseteq B$, then $\{V : B \subseteq V \text{ and } V \text{ is open in } X\} \subseteq \{V : A \subseteq V \text{ and } V \text{ is open in } X\}$. Therefore,

$$\mu(B) = \inf\{\mu(V) : B \subseteq V \text{ and } V \text{ is open in } X\}$$

$$\ge \inf\{\mu(V) : A \subseteq V \text{ and } V \text{ is open in } X\} = \mu(A) .$$

We can prove part (e) easily.

Proof of part (e)

Suppose $\mu(E) = 0$. Plainly, by the monotonicity of μ, $E \in \mathscr{M}_F$ and that for any $N \subseteq E$, $\mu(N) = 0$. Obviously for any compact subset K of X, $\mu(E \cap K) = 0$. It follows that $E \in \mathscr{M}$ and that for any $N \subseteq E$, $E \in \mathscr{M}$.

Part (c) of the theorem plainly holds by the definition of μ.

Therefore, we only need to prove parts (a) and (d). That is, we need to prove that μ is a positive measure on $\mathscr{M}$, $\mathscr{M}$ is a σ-algebra, $\Lambda(f)=\int_X f\,d\mu$ for all $f\in C_c(X)$ and μ satisfies part (d).

Note that μ is defined on all subsets of X. We need to show that μ is countably additive on $\mathscr{M}$. We have the following consequence of the definition of μ on all subsets of X, which will contribute to part of the proof of the countable additivity of μ on $\mathscr{M}$.

(1) For any family $\{E_i\}_{1\le i<\infty}$ of subsets of X, $\mu\left(\bigcup_{n=1}^{\infty}E_n\right)\le\sum_{n=1}^{\infty}\mu(E_n)$.

To prove (1), we begin by considering open sets in X. If V_1 and V_2 are two open sets in X, then $\mu(V_1\cup V_2)\le\mu(V_1)+\mu(V_2)$. We shall prove this as follows. Recall that $\mu(V_1\cup V_2)=\sup\{\Lambda(g):g\in C_c(X)\text{ and }g\prec V_1\cup V_2\}$. Suppose $g\in C_c(X)$ and $g\prec V_1\cup V_2$. Then support $g\subseteq V_1\cup V_2$. Since support g is compact and plainly, $\{V_1,V_2\}$ is an open cover for support g, we can take a partition of unity $\{h_1,h_2\}$ on support g subordinate to the covering $\{V_1,V_2\}$, such that $h_i\in C_c(X)$, $0\le h_i\le 1$, $h_i\prec V_i$, $h_i\left((V_i)^c\right)=0$, $i=1, 2$ and $h_1+h_2=1$ on support g. Note that support $h_i\subseteq V_i$, $i=1, 2$. Hence, we get $h_ig\prec V_i$ for $i=1, 2$ and $h_1g+h_2g=g$. Therefore,

$\Lambda(g)=\Lambda(h_1g)+\Lambda(h_2g)\le\mu(V_1)+\mu(V_2)$. This is true for any $g\in C_c(X)$ with $g\prec V_1\cup V_2$. Hence,

$$\mu(V_1\cup V_2)=\sup\{\Lambda(g):g\prec V_1\cup V_2\text{ and }g\in C_c(X)\}\le\mu(V_1)+\mu(V_2).$$

It then follows by induction that for a finite family of open sets, $\{V_i\}_{1\le i\le n}$, $\mu\left(\bigcup_{i=1}^{n}V_i\right)\le\sum_{i=1}^{n}\mu(V_i)$. With this proven, we shall apply this to arbitrary family of subsets $\{E_i\}_{1\le i<\infty}$.

If there exists an integer i such that $\mu(E_i)=\infty$, then trivially, $\mu\left(\bigcup_{n=1}^{\infty}E_n\right)\le\sum_{n=1}^{\infty}\mu(E_n)$. So, we now assume that $\mu(E_i)<\infty$ for all integer $i\ge 1$. By the definition of $\mu(E_i)$, given $\varepsilon>0$, there exists open set V_i such that $E_i\subseteq V_i$

$$\mu(V_i) < \mu(E_i) + \frac{\varepsilon}{2^i} \ .$$

Let $V = \bigcup_{i=1}^{\infty} V_i$. Then V is an open subset of X. Take any $f \in C_c(X)$ such that $f \prec V$. Since support f is compact and support $f \subseteq V$, a set of finite number of the open subsets $\{V_i\}_{1\le i<\infty}$ covers support f. Hence, there exists a positive integer n such that support $f \subseteq \bigcup_{i=1}^{n} V_i$. Therefore, $f \prec \bigcup_{i=1}^{n} V_i$. Hence,

$$\Lambda(f) \le \mu\left(\bigcup_{i=1}^{n} V_i\right) \le \sum_{i=1}^{n} \mu(V_i) \le \sum_{i=1}^{n} \mu(E_i) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i} \le \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon .$$

It follows that $\mu(V) \le \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$. Since $\bigcup_{i=1}^{\infty} E_i \subseteq V$, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \mu(V) \le \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$. As ε is arbitrary, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu(E_i)$.

(2) Every compact subset of X belongs to $\mathscr{M}_F$.

Take any compact subset K of X. Take any $f \in C_c(X)$ with $K \prec f$. Let $V = \left\{ x \in X : f(x) > \frac{1}{2} \right\}$. Since f is continuous, V is open in X. Take any $g \in C_c(X)$ with $g \prec V$. Then for $x \in V$, $g(x) \le 1 < 2f(x)$. Since $g(x) = 0$ for

$x \in V^c$, $g(x) \le 2f(x)$ for all x in X, i.e., $g \le 2f$. Hence, $\Lambda(g) \le \Lambda(2f) < \infty$ for any $g \in C_c(X)$ with $g \prec V$. It follows that

$$\mu(V) = \sup\{\Lambda(g) : g \prec V \text{ and } g \in C_c(X)\} \le \Lambda(2f) < \infty .$$

Since $K \subseteq V$, it follows that $\mu(K) \le \mu(V) < \infty$. Plainly, $\sup\{\mu(L) : L \subseteq K \text{ and } L \text{ is compact}\} = \mu(K)$ and so K is in $\mathscr{M}_F$. Incidentally, this also shows that for any compact subset K, $\mu(K) < \infty$.

(3) Every open subset V of X with $\mu(V) < \infty$ belongs to $\mathscr{M}_F$.

Take any open subset V of X with $\mu(V) < \infty$. By definition of μ on open subset, given $\varepsilon > 0$, there exists a continuous function f with compact support such that $f \prec V$ and $\mu(V) - \varepsilon < \Lambda(f) \le \mu(V)$. Let K = support f. Then $K \subseteq V$ and so $\mu(K) \le \mu(V)$. Suppose now W is any open set containing K. Then $f \prec W$. By the definition of $\mu(W)$, $\Lambda(f) \le \mu(W)$. Therefore, $\Lambda(f)$ is a lower bound for $\{\mu(V) : K \subseteq V \text{ and } V \text{ is open in } X\}$ and so

$$\Lambda(f) \le \mu(K) = \inf\{\mu(V) : K \subseteq V \text{ and } V \text{ is open in } X\}.$$

It follows that $\mu(V) - \varepsilon < \Lambda(f) \le \mu(K) \le \mu(V)$. This means $\mu(V) = \sup\{\mu(K) : K \subseteq V \text{ and } K \text{ is compact}\}$. Hence, $V \in \mathscr{M}_F$.

(4) μ is countably additive on $\mathscr{M}_F$. That is, suppose E_1, E_2, ….., are in $\mathscr{M}_F$ and are pairwise disjoint, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$. Moreover, if $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) < \infty$, then $\bigcup_{i=1}^{\infty} E_i \in \mathscr{M}_F$.

We shall prove this in stages. First on compact subsets since compact subsets are contained in $\mathscr{M}_F$ by (2).

Suppose K_1, K_2 are disjoint compact subsets of X. We shall show that $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$. Since X is Hausdorff and $K_1 \cap K_2 = \varnothing$, for a fix $y \in K_2$, for each $x \in K_1$, there exists open sets, U_x, V_x with $x \in U_x$, $y \in V_x$ such that $U_x \cap V_x = \varnothing$. Hence, $\{U_x : x \in K_1\}$ is an open cover for K_1. Therefore, since K_1 is compact, $\{U_x : x \in K_1\}$ has a finite subcover, $K_1 \subseteq U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$. Let $U_y = U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$. Then $K_1 \subseteq U_y$ and $U_y \cap \bigcap_{i=1}^{n} V_{x_i} = \varnothing$. Let $V_y = \bigcap_{i=1}^{n} V_{x_i}$. Then U_y and V_y are open with $K_1 \subseteq U_y$ and $y \in V_y$ and $U_y \cap V_y = \varnothing$. It follows that $\{V_y : y \in K_2\}$ is an open cover for K_2. As K_2 is compact, it has a finite subcover $V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_j} \supseteq K_2$. Let $V_2 = V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_j} \supseteq K_2$ and $V_1 = \bigcap_{i=1}^{j} U_{y_i} \supseteq K_1$. Plainly, $V_1 \cap V_2 = \varnothing$. As $K_1 \cup K_2$ is compact, $K_1 \cup K_2$ is in $\mathscr{M}_F$. As $\mu(K_1 \cup K_2) = \inf\{\mu(V) : K_1 \cup K_2 \subseteq V \text{ and } V \text{ is open in } X\}$, given $\varepsilon > 0$, there exists open set $W \supseteq K_1 \cup K_2$ such that

$$\mu(K_1 \cup K_2) \le \mu(W) < \mu(K_1 \cup K_2) + \varepsilon.$$

Note that $W \cap V_1$ and $W \cap V_2$ are open in X and are disjoint. Plainly, $\mu(W \cap V_i) < \mu(W) < \infty$ for $i = 1, 2$. Therefore, by definition of μ on open set, there exists $f_i \in C_c(X)$ such that $f_i \prec W \cap V_i$ and $\Lambda(f_i) > \mu(W \cap V_i) - \varepsilon$ for $i = 1, 2$.

Note that support $f_i \subseteq W \cap V_i$, for $i = 1, 2$, and so support f_1 and support f_2 are disjoint. It follows that $f_1 + f_2 \prec W$.

Now, $\mu(K_1) + \mu(K_1) \le \mu(W \cap V_1) + \mu(W \cap V_2)$ as $K_1 \subseteq W \cap V_2$ and $K_2 \subseteq W \cap V_2$,

$$\le \Lambda(f_1) + \varepsilon + \Lambda(f_2) + \varepsilon = \Lambda(f_1 + f_2) + 2\varepsilon$$

$$\le \mu(W) + 2\varepsilon \ , \quad \text{by definition of } \mu(W),$$

$$< \mu\left(K_1 \cup K_2\right) + 3\varepsilon .$$

Since ε is arbitrary, $\mu\left(K_1\right) + \mu\left(K_2\right) \le \mu\left(K_1 \cup K_2\right)$. We have already proved as in (1) that $\mu\left(K_1 \cup K_2\right) \le \mu\left(K_1\right) + \mu\left(K_2\right)$ and so $\mu\left(K_1 \cup K_2\right) = \mu\left(K_1\right) + \mu\left(K_2\right)$. By a simple mathematical induction, if $K_1, K_2, \ldots.., K_n$ are compact subsets of X and are pairwise disjoint, then $\mu\left(\bigcup_{i=1}^{n} K_i\right) = \sum_{i=1}^{n} \mu\left(K_i\right)$

Now suppose $E_1, E_2, \ldots..,$ are in $\mathscr{M}_F$ and are pairwise disjoint. Let $E = \bigcup_{i=1}^{\infty} E_i$.
Suppose $\mu(E) = \infty$. Then it follows by the inequality in part (1),
$\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \mu(E_n)$ that $\sum_{i=1}^{\infty} \mu\left(E_i\right) = \infty$. Hence, trivially $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) = \infty$.

Suppose now $\mu(E) < \infty$. Since each $E_i \in \mathscr{M}_F$,
$\mu(E_i) = \sup\left\{\mu(K) : K \subseteq E_i \text{ and } K \text{ is compact}\right\}$. Given $\varepsilon > 0$, there exists compact subset $K_i \subseteq E_i$ such that

$$\mu(E_i) \ge \mu(K_i) > \mu(E_i) - \frac{\varepsilon}{2^i} \ .$$

For each integer $n \ge 1$, let $H_n = \bigcup_{i=1}^{n} K_i$. Then $H_n \subseteq \bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i=1}^{\infty} E_i = E$. Therefore,
$\mu\left(E\right) \ge \mu\left(H_n\right) = \mu\left(\bigcup_{i=1}^{n} K_i\right) = \sum_{i=1}^{n} \mu\left(K_i\right)$, since $K_1, K_2, \ldots, K_n$ are pairwise disjoint,

$$> \sum_{i=1}^{n} \mu(E_i) - \varepsilon \sum_{i=1}^{n} \frac{1}{2^i} > \sum_{i=1}^{n} \mu(E_i) - \varepsilon .$$

It follows that $\mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i) - \varepsilon$. Since ε is arbitrary, $\mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i)$. Hence this together with part (1) gives $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$. We now show that if $\mu(E) < \infty$, then $E \in \mathscr{M}_F$.

If $\mu(E) < \infty$, given any $\varepsilon > 0$, there exists an integer N such that $n \geq N$ implies that $\mu(E) \leq \sum_{i=1}^{n} \mu(E_i) + \varepsilon \leq \sum_{i=1}^{n} \mu(K_i) + 2\varepsilon = \mu(H_n) + 2\varepsilon$. Here H_n is compact and $H_n \subseteq E$. It follows that $\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}$. Therefore, $E \in \mathscr{M}_F$.

(5) For all $E \in \mathscr{M}_F$, given $\varepsilon > 0$, there exists compact subset K of X and open subset V with $K \subseteq E \subseteq V$ such that $\mu(V - K) < \varepsilon$.

For $E \in \mathscr{M}_F$, $\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}$. Hence given $\varepsilon > 0$, there exists compact subset $K \subseteq E$ such that

$$\mu(E) \geq \mu(K) > \mu(E) - \frac{\varepsilon}{2}.$$

Since $\mu(E) = \inf\{\mu(V) : E \subseteq V \text{ and } V \text{ is open in } X\}$, there exists open set V such that $E \subseteq V$ and

$$\mu(V) < \mu(E) + \frac{\varepsilon}{2}.$$

Hence, $\mu(V) - \frac{\varepsilon}{2} < \mu(E) < \mu(K) + \frac{\varepsilon}{2}$. Now by (2), $K \in \mathscr{M}_F$. Since X is Hausdorff, K is closed in X. Therefore, $V - K$ is open in X. $\mu(V - K) \leq \mu(V) < \mu(E) + \varepsilon < \infty$. It follows from (3) that $V - K \in \mathscr{M}_F$. By part (4), $\mu(V) = \mu(K) + \mu(V - K)$ and so $\mu(V - K) = \mu(V) - \mu(K) < \varepsilon$.

(6) If $A_1, A_2 \in \mathscr{M}_F$, then $A_1 - A_2$, $A_1 \cup A_2$ and $A_1 \cap A_2 \in \mathscr{M}_F$.

By (5), given ε > 0, there exist compact K_i, open V_i such that $K_i \subseteq A_i \subseteq V_i$ and $\mu(V_i - K_i) < \varepsilon$ for $i = 1, 2$.

Then $A_1 - A_2 \subseteq V_1 - K_2 \subseteq (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$. Therefore,

$$\mu\left(A_1 - A_2\right) \le \mu\left(V_1 - K_1\right) + \mu(K_1 - V_2) + \mu(V_2 - K_2)$$

$$< 2\varepsilon + \mu(K_1 - V_2) \le 2\varepsilon + \mu(K_1) < \infty .$$

Note that $K_1 - V_2$ is compact, $K_1 - V_2 \subseteq A_1 - A_2$ and $\mu(K_1 - V_2) > \mu(A_1 - A_2) - 2\varepsilon$. This shows that given any $\varepsilon > 0$, there exists a compact set L such that $L \subseteq A_1 - A_2$ and $\mu(A_1 - A_2) \ge \mu(L) > \mu(A_1 - A_2) - \varepsilon$. Hence,

$$\mu(A_1 - A_2) = \sup\left\{\mu\left(L\right) : L \subseteq A_1 - A_2 \text{ and } L \text{ is compact}\right\}.$$

Therefore, $A_1 - A_2 \in \mathscr{M}_F$.

Now, $A_1 \cup A_2 = \left(A_1 - A_2\right) \cup A_2$ and as $\mu\left(A_1 \cup A_2\right) \le \mu\left(A_1\right) + \mu\left(A_2\right) < \infty$, $A_1 - A_2, A_2 \in \mathscr{M}_F$ and $A_1 - A_2$, A_2 are disjoint and so by part (4), $A_1 \cup A_2 = \left(A_1 - A_2\right) \cup A_2 \in \mathscr{M}_F$.

Next, $A_1 \cap A_2 = A_1 - \left(A_1 - A_2\right) \in \mathscr{M}_F$, since $A_1 - A_2$ and $A_1 \in \mathscr{M}_F$.

(7) $\mathscr{M}$ is a σ-algebra containing all Borel sets of X.

Recall that $A \in \mathscr{M}$ if $A \cap K \in \mathscr{M}_F$ for all compact subset K of X. Take $A \in \mathscr{M}$. We shall show that the complement $A^c \in \mathscr{M}$. Now $A^c \cap K = K - A \cap K \in \mathscr{M}_F$ by part (6) since K and $A \cap K \in \mathscr{M}_F$. Hence, $A^c \in \mathscr{M}$. Suppose $\{A_i\}$ is a countable collection of members of $\mathscr{M}$. If $\{A_i\}_{i=1}^{n}$ is a finite collection, then by part (6) for any compact K, $\left(\bigcup_{i=1}^{n} A_i\right) \cap K = \bigcup_{i=1}^{n} A_i \cap K \in \mathscr{M}_F$ and so $\bigcup_{i=1}^{n} A_i \in \mathscr{M}$. So, we now assume that $\{A_i\}_{i=1}^{\infty}$ is a collection of infinitely countable number of members of $\mathscr{M}$. We shall show that $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{M}$. Let $B_1 = A_1 \cap K$. Inductively, define $B_n = A_n \cap K - B_1 \cup B_2 \cup \cdots \cup B_{n-1}$. Then $A \cap K = \bigcup_{i=1}^{\infty} B_i$. Since $A_n \cap K \in \mathscr{M}_F$ for all integer $n \ge 1$, it follows from part (6) that $B_i \in \mathscr{M}_F$ for all integer $i \ge 1$. Moreover, the collection $\{B_i\}_{i=1}^{n}$ are pairwise disjoint. By part (4),

since $\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(A \cap K) < \infty$, $\bigcup_{i=1}^{\infty} B_i = A \cap K \in \mathscr{M}_F$. Hence, $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{M}$.

Next we shall show that if $C \subseteq X$ is closed in X, then $C \in \mathscr{M}$. In particular, $X \in \mathscr{M}$.

If C is closed, then $C \cap K$ is compact for any compact subset K of X and so $C \cap K$

$\in \mathscr{M}_F$. Thus $C \in \mathscr{M}$. Hence, $X \in \mathscr{M}$ and $\mathscr{M}$ is a σ-algebra containing all closed subsets of X, hence all open subsets of X. So, it contains all Borel sets of X.

(8) $\mathscr{M}_F = \{ E \in \mathscr{M} : \mu(E) < \infty \}$.

Suppose $E \in \mathscr{M}_F$. Then by (6), since by (2) any compact $K \in \mathscr{M}_F$, $E \cap K \in \mathscr{M}_F$. Hence, $E \in \mathscr{M}$. That is, $\mathscr{M}_F \subseteq \mathscr{M}$ and $\mathscr{M}_F \subseteq \{ E \in \mathscr{M} : \mu(E) < \infty \}$.

Conversely, suppose $E \in \mathscr{M}$ and $\mu(E) < \infty$. As $\mu(E) = \inf\{\mu(V) : E \subseteq V \text{ and } V \text{ is open in } X\}$, there exists an open set V in X such that $E \subseteq V$ and $\mu(V) < \infty$. Since V is open, by (3), $V \in \mathscr{M}_F$. Hence, $\mu(V) = \sup\{\mu(K) : K \subseteq V \text{ and } K \text{ is compact}\}$. Therefore, given any $\varepsilon > 0$, there exists a compact set $K \subseteq V$ such that $\mu(K) > \mu(V) - \varepsilon$ so that $\mu(V - K) < \varepsilon$. Since by definition of $\mathscr{M}$, $E \cap K \in \mathscr{M}_F$, there exists compact $H \subseteq E \cap K$ such that $\mu(H) > \mu(E \cap K) - \varepsilon$. Since $E \subseteq (E \cap K) \cup (V - K)$,

$$\mu(E) \leq \mu(E \cap K) + \mu(V - K) < \mu(H) + 2\varepsilon .$$

As H is compact and $H \subseteq E$, this shows that

$$\mu(E) = \sup\{\mu(H) : H \subseteq E \text{ and } H \text{ is compact}\} .$$

Therefore, $E \in \mathscr{M}_F$. Hence, $\{ E \in \mathscr{M} : \mu(E) < \infty \} \subseteq \mathscr{M}_F$. Thus, $\{ E \in \mathscr{M} : \mu(E) < \infty \} = \mathscr{M}_F$.

Remark. Thus, part (d) holds if $\mu(E) < \infty$. If E is open and $\mu(E) < \infty$, then the conclusion obviously holds too. We are left with the case E is open and $\mu(E) = \infty$. Now $\mu(E) = \sup\{\Lambda(f) : f \in C_c(X) \text{ and } f \prec E\} = \infty$ implies that given any $M > 0$, there exists $f \in C_c(X)$ and $f \prec E$ such that $\Lambda(f) > M$. Let $K =$ support f and so $K \subseteq E$. Since $\mu(K) < \infty$, there exists open set V containing K such that

$$\mu(V) < \mu(K) + \varepsilon$$

Let $U = V \cap E$. Then U is open and support $f = K \subseteq U$ so that $f \prec U$ and $\mu(U) < \mu(K) + \varepsilon$. Therefore, $\mu(U) \geq \Lambda(f) > M$. It follows that $\mu(K) > \mu(U) - \varepsilon > M - \varepsilon$. We can now conclude that

$\sup\{\mu(K): K \subseteq E \text{ and } K \text{ is compact}\} = \infty$ and so
$\mu(E) = \sup\{\mu(K): K \subseteq E \text{ and } K \text{ is compact}\} = \infty$.

(9) μ is a measure on $\mathscr{M}$.

We have proved that μ is countably additive on $\mathscr{M}_F$ and that $\mathscr{M}$ is a σ-algebra.

Suppose E_1, E_2, … , are in $\mathscr{M}$ and are pairwise disjoint.

If for some integer i, $\mu(E_i) = \infty$, then plainly, $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \infty$. We now assume $\mu(E_i) < \infty$ for all integer $i \geq 1$. By part (8), $E_i \in \mathscr{M}_F$ for all integer $i \geq 1$. It follows by part (4) that $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$. Hence, μ is countably additive on $\mathscr{M}$ and so μ is a measure on $\mathscr{M}$.

(10) For all $f \in C_c(X)$, $\Lambda(f) = \int_X f \, d\mu$.

We note that it is sufficient to prove this for real f. For complex f we may write $f = \operatorname{Re} f + i \operatorname{Im} f$. Then the real part of f, $\operatorname{Re} f$, and the imaginary part of f, $\operatorname{Im} f$, are continuous real functions with compact support. Then,

$$\Lambda(f) = \Lambda(\operatorname{Re} f + i \operatorname{Im} f) = \Lambda(\operatorname{Re} f) + i\Lambda(\operatorname{Im} f) = \int_X \operatorname{Re} f \, d\mu + i\int_X \operatorname{Im} f \, d\mu = \int_X f \, d\mu .$$

Let f be a continuous real valued function with compact support in $C_c(X)$. Let K = support f and so K is compact. Since f is continuous, $f(K)$ is compact and is a compact subset of $\mathbb{C}$ and resides in the real line. $f(K)$ is closed and bounded on the real line. Therefore, $f(X)$ is a bounded subset on the real line. Thus, we may assume that $f(X) \subseteq [a,b]$. Given $\varepsilon > 0$, partition $[a,b]$ as follows

$$a < y_1 < y_2 < \cdots < y_n = b \quad \text{with } y_i - y_{i-1} < \varepsilon \text{ for } 2 \leq i \leq n .$$

and add a point $y_0 < a$ so that $y_1 - y_0 < \varepsilon$.

Let $E_i = \{x \in X : y_{i-1} < f(x) \leq y_i\} \cap K$ for $1 \leq i \leq n$. That is, $E_i = f^{-1}((y_{i-1}, y_i]) \cap K$. Since f is continuous and so is Borel measurable, it follows that each E_i is a Borel set. Moreover $\{E_i\}$ are pairwise disjoint and covers K. Since K is compact, $\mu(E_i) \leq \mu(K) < \infty$ by part (2). Therefore, by part (8), $E_i \in \mathscr{M}_F$ for $1 \leq i \leq n$. By the definition of $\mu(E_i)$, given $\varepsilon > 0$, there exists open set $W_i \supseteq E_i$ such that

$\mu(W_i) < \mu(E_i) + \frac{\varepsilon}{n}$. Let $D_i = \{z \in \mathbb{C} : |z - \frac{1}{2}(y_{i-1} + y_i + \varepsilon)| < \frac{1}{2}(y_i + \varepsilon - y_{i-1})\}$. Then D_i is an open disk and so $U_i = f^{-1}(D_i)$ is open and $y_{i-1} < f(x) < y_i + \varepsilon$ for all x in U_i. Then $U_i \supseteq E_i$. Let $V_i = W_i \cap U_i$ and we have $\mu(V_i) \le \mu(W_i) < \mu(E_i) + \frac{\varepsilon}{n}$ and $f(x) < y_i + \varepsilon$ for all x in V_i. Note that $\bigcup_{i=1}^{n} V_i \supseteq \bigcup_{i=1}^{n} E_i = K$. Take a partition of unity $\{h_i\}_{1\le i\le n}$ on K subordinate to the covering $\{V_i\}_{1\le i\le n}$ such that, for $1 \le i \le n$, $0 \le h_i \le 1$, $h_i \prec V_i$ and $h_1 + \cdots + h_n = 1$ on K. Then we have

$\sum_{i=1}^{n} h_i f = f$ since $\sum_{i=1}^{n} h_i = 1$ on K, and for $1 \le i \le n$,

$h_i(x) f(x) \le h_i(x)(y_i + \varepsilon)$ since $h_i \prec V_i$ and $f(x) < y_i + \varepsilon$ for all x in V_i and

$y_i - \varepsilon < f(x) < y_i + \varepsilon$ for all x in E_i .

By linearity, $\Lambda(f) = \sum_{i=1}^{n} \Lambda(h_i f)$. As Λ is a positive functional and $h_i f \le h_i(y_i + \varepsilon)$, $\Lambda(h_i f) \le \Lambda((y_i + \varepsilon) h_i) = (y_i + \varepsilon)\Lambda(h_i)$ for $1 \le i \le n$. Therefore,

$$\Lambda(f) = \sum_{i=1}^{n} \Lambda(h_i f) \le \sum_{i=1}^{n} (y_i + \varepsilon)\Lambda(h_i).$$

Since $h_i \prec V_i$ for $1 \le i \le n$, by definition of $\mu(V_i)$, $\Lambda(h_i) \le \mu(V_i)$ for $1 \le i \le n$. For $1 \le i \le n$, $a < y_i \le b$ so that $y_i + \varepsilon + |a| > 0$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} (y_i + \varepsilon)\Lambda(h_i) &= \sum_{i=1}^{n} (y_i + \varepsilon + |a|)\Lambda(h_i) - \sum_{i=1}^{n} |a|\Lambda(h_i) \\
&= \sum_{i=1}^{n} (y_i + \varepsilon + |a|)\Lambda(h_i) - |a|\Lambda\left(\sum_{i=1}^{n} h_i\right) \\
&\le \sum_{i=1}^{n} (y_i + \varepsilon + |a|)\mu(V_i) - |a|\Lambda\left(\sum_{i=1}^{n} h_i\right) \\
&\le \sum_{i=1}^{n} (y_i + \varepsilon + |a|)\mu(E_i) + \sum_{i=1}^{n} (y_i + \varepsilon + |a|)\frac{\varepsilon}{n} - |a|\Lambda\left(\sum_{i=1}^{n} h_i\right) \\
&= \sum_{i=1}^{n} (y_i - \varepsilon)\mu(E_i) + (2\varepsilon + |a|)\sum_{i=1}^{n} \mu(E_i) + \sum_{i=1}^{n} (y_i + \varepsilon + |a|)\frac{\varepsilon}{n} - |a|\Lambda\left(\sum_{i=1}^{n} h_i\right)
\end{aligned}$$

$$\le \sum_{i=1}^{n} (y_i - \varepsilon)\mu(E_i) + (2\varepsilon + |a|)\mu(K) + (b + \varepsilon + |a|)\varepsilon - |a|\Lambda\left(\sum_{i=1}^{n} h_i\right)$$

$$\le \sum_{i=1}^{n} \int_{E_i} f\,d\mu + \varepsilon\left(2\mu(K) + (b + \varepsilon + |a|)\right) + |a|\mu(K) - |a|\Lambda\left(\sum_{i=1}^{n} h_i\right)$$

$$= \int_X f\,d\mu + \varepsilon\left(2\mu(K) + (b + \varepsilon + |a|)\right) + |a|\mu(K) - |a|\Lambda\left(\sum_{i=1}^{n} h_i\right).$$

Now we claim that $\mu(K) \le \Lambda\left(\sum_{i=1}^{n} h_i\right)$. Note that $0 \le \sum_{i=1}^{n} h_i = h \le 1$ and $K \prec h$. Take $0 < k < 1$. Let $V_k = \{x \in X : h(x) > k\}$. Since h is continuous, V_k is open. For any $g \in C_k(X)$ such that $g \prec V_k$, $g(x) \le 1 < \frac{1}{k}h(x)$ for x in V_k. Since $g(x) = 0$ for $x \in V_k^{\ c}$, $g(x) \le \frac{1}{k}h(x)$ for all x in X. Therefore, $\Lambda(g) \le \Lambda\left(\frac{1}{k}h\right) = \frac{1}{k}\Lambda(h)$. Hence, for any $g \in C_c(X)$ such that $g \prec V_k$, $\Lambda(g) \le \frac{1}{k}\Lambda(h)$ and so $\mu(V_k) = \sup\{\Lambda(g) : g \prec V_k, g \in C_c(x)\} \le \frac{1}{k}\Lambda(h)$. As $h = 1$ on K, for any $0 < k < 1$, $K \subseteq V_k$. Therefore, $\mu(K) \le \mu(V_k) \le \frac{1}{k}\Lambda(h)$. Letting $k \to 1$, we deduce that $\mu(K) \le \Lambda(h)$.

Hence,

$$\Lambda(f) \le \int_X f\,d\mu + \varepsilon\left(2\mu(K) + (b + \varepsilon + |a|)\right) + |a|\mu(K) - |a|\Lambda\left(\sum_{i=1}^{n} h_i\right)$$

$$\le \int_X f\,d\mu + \varepsilon\left(2\mu(K) + (b + \varepsilon + |a|)\right).$$

Since ε is arbitrary, $\Lambda(f) \le \int_X f\,d\mu$.

As Λ is linear, $-\Lambda(f) = \Lambda(-f) \le \int_X (-f)\,d\mu = -\int_X f\,d\mu$ and so $\Lambda(f) \ge \int_X f\,d\mu$, Thus, $\Lambda(f) = \int_X f\,d\mu$.

We say a positive Borel measure μ is *regular*, if the conclusion (c) and (d) holds for any Borel measurable set E without any condition.

Remark.

1. Take the space of all continuous real-valued functions with compact support on a locally compact Hausdorff topological space X,

$$C_{c,\mathbb{R}}(X) = \{f : X \to \mathbb{R}; f \text{ is continuous with compact support}\}.$$

Take $\Lambda : C_{c,\mathbb{R}}(X) \to \mathbb{R}$ to be a positive real linear functional on $C_{c,\mathbb{R}}(X)$. Then the above proof applies equally well to this positive linear functional to give the same conclusion (a) to (d).

2. If X is a compact Hausdorff space and $\Lambda : C_c(X) \to \mathbb{C}$ is a positive linear functional, then by the Riesz Representation Theorem (Theorem 1), there exists a σ-algebra $\mathscr{M}$ on X, containing all the Borel sets of X and a unique positive measure, μ, on $\mathscr{M}$ such that $\Lambda(f) = \int_X f \, d\mu$. Since X is compact, $\mu(X) < \infty$ and so μ is a finite positive regular Borel measure, meaning part (c) and (d) of Riesz Theorem hold without any condition. Moreover, $C_c(X) = C(X)$, the space of continuous function with the sup norm, $\|f\|_u = \sup\{|f(x)| : x \in X\}$, is a Banach space. Furthermore, $|\Lambda(f)| = \left|\int_X f \, d\mu\right| \le \int_X |f| \, d\mu \le \|f\|_u \, \mu(X)$. This means Λ is bounded and so is continuous and $\|\Lambda\| \le \mu(X)$. Actually, as $\mu(X) = \Lambda(1) < \infty$,

$$\|\Lambda\| = \sup\{|\Lambda(f)| : \|f\|_u = 1, f \in C_c(X) = C(X)\} = \mu(X) < \infty.$$

Thus, $\Lambda : C_c(X) \to \mathbb{C}$ is a bounded complex linear functional. In this case, we have a one-one map from the collection of positive complex linear functionals to the collection of finite positive regular Borel measures with the norm $\|\mu\| = \mu(X)$. Furthermore, this map preserves norm.

Suppose X is a compact Hausdorff topological space. Then

$$C_{c,\mathbb{R}}(X) = \{f : X \to \mathbb{R}; f \text{ is continuous with compact support}\}$$

$$= C_{\mathbb{R}}(X) = \{f : X \to \mathbb{R}; f \text{ is continuous}\}.$$

Suppose $\Lambda : C_{\mathbb{R}}(X) \to \mathbb{R}$ is a positive real linear functional. As remark before, the Riesz Representation Theorem for positive real linear functional on the space of real valued functions on the locally compact topological space X

follows from Theorem 1 as the proof is exactly the same. That is, we have conclusion (a) to (d) of Theorem 1. Then the Riesz Representation Theorem applies to give a positive Borel mesure, μ, such that $\Lambda(f) = \int_X f\, d\mu$ for all $f \in C_{\mathbb{R}}(X)$. Note that for a positive real linear functional Λ, for any f in $C_{\mathbb{R}}(X)$, $\Lambda(f), \Lambda(-f) = -\Lambda(f) \le \Lambda(|f|)$. Thus, if $|f| \le 1$, $|\Lambda(f)| \le \Lambda(|f|) \le \Lambda(1)$ and since $\|\Lambda\| = \sup\{|\Lambda(f)| : \|f\|_u = 1 \text{ and } f \in C_{\mathbb{R}}(X)\}$, $\|\Lambda\| = \Lambda(1) < \infty$. This means any positive real linear functional is a bounded real linear functional. Thus, $\mu(X) = \int_X 1\, d\mu = \Lambda(1) < \infty$ and so μ is a finite, positive and hence regular Borel measure by part (d) since $\mu(E) \le \mu(X) < \infty$ for all E in $\mathscr{M}$. Thus, any positive real linear functional $\Lambda : C_{\mathbb{R}}(X) \to \mathbb{R}$ is represented by a unique finite positive regular Borel measure.

If we consider just bounded real linear functional $\Phi : C_{\mathbb{R}}(X) \to \mathbb{R}$, the situation is somewhat different. We may not apply Theorem 1 directly. But if we can write Φ as the difference of two positive real linear functionals, we may proceed to apply Riesz Representation Theorem (Theorem 1). We may decompose a bounded real linear functional Φ on $C_{\mathbb{R}}(X)$ for any compact Hausdorff topological space X as the difference of two positive real linear functionals.

Proposition 2. Suppose X is a compact Hausdorff topological space and $C_{\mathbb{R}}(X) = \{f : X \to \mathbb{R}; f \text{ is continuous}\}$. Suppose $\Phi : C_{\mathbb{R}}(X) \to \mathbb{R}$ is a bounded real linear functional. Then we can decompose Φ as $\Phi = \Phi^+ - \Phi^-$ such that Φ^+ and Φ^- are positive real linear functionals and $\|\Phi\| = \|\Phi^+\| + \|\Phi^-\| = \Phi^+(1) + \Phi^-(1)$.

Proof.

Let $C_{\mathbb{R}}{}^+(X)$ denote the set of non-negative functions in $C_{\mathbb{R}}(X)$.

Define for f in $C_{\mathbb{R}}{}^+(X)$,

$\Phi^+(f) = \sup\{\Phi(h) : h \in C_{\mathbb{R}}(X) \text{ and } 0 \le h \le f\} = \sup\{\Phi(h) : h \in C_{\mathbb{R}}{}^+(X) \text{ and } 0 \le h \le f\}$.

This is well defined since Φ is bounded so that the supremum above exists.

Since $\Phi(0)=0$, $\Phi^+(f)\ge 0$ for all $f\in C_{\mathbb{R}}{}^+(X)$. Plainly, $\Phi^+(f)\ge\Phi(f)$ for all $f\in C_{\mathbb{R}}{}^+(X)$. Obviously, for $k > 0$, $\Phi^+(k\,f)=k\Phi^+(f)$.

We need to show that $\Phi^+(f_1+f_2)=\Phi^+(f_1)+\Phi^+(f_2)$ for $f_1,f_2\in C_{\mathbb{R}}{}^+(X)$.

By definition of $\Phi^+(f_i)$, given ε > 0, there exists $h_i\in C_{\mathbb{R}}{}^+(X)$ such that $0\le h_i\le f_i$ and $\Phi^+(f_i)-\varepsilon<\Phi(h_i)$ for i = 1, 2. Then we have, as $0\le h_1+h_2\le f_1+f_2$,

$$\Phi^+(f_1)+\Phi^+(f_2)<\Phi(h_1)+\Phi(h_2)+2\varepsilon=\Phi(h_1+h_2)+2\varepsilon\le\Phi^+(f_1+f_2)+2\varepsilon .$$

Since ε is arbitrary, $\Phi^+(f_1)+\Phi^+(f_2)\le\Phi^+(f_1+f_2)$.

Take $h\in C_{\mathbb{R}}{}^+(X)$ with $0\le h\le f_1+f_2$. Let $V=\{x: f_1(x)+f_2(x)>0\}$. Then V is open in X.

Let $h_i(x)=\begin{cases}\dfrac{f_i(x)h(x)}{f_1(x)+f_2(x)}, & x\in V\\ 0, \; x\in V^c\end{cases}$.

We claim that h_i is non-negative, continuous with compact support, $0\le h_i\le f_i$ for i = 1, 2.

Plainly, $h_i(x)\ge 0$ for all $x\in X$, $h_i(x)\le f_i(x)$ for $x\in V$ and $h_i(x)=f_i(x)=0$ for $x\in V^c$ for i=1, 2.

Since $\dfrac{f_i(x)}{f_1(x)+f_2(x)}$ and $h(x)$ is continuous on the open set V, $\dfrac{f_i(x)h(x)}{f_1(x)+f_2(x)}$ is continuous on V so that h_i is continuous on V for i = 1, 2. Now we show that h_1 is continuous at any point $x_0\in V^c$. For such a $x_0\in V^c$, $h_1(x_0)=0$ and also $h(x_0)=0$. Since h is continuous at x_0, given any open interval $I=(-\delta,\delta)$, $\delta>0$, containing $h(x_0)=0$, there exists an open set U containing $x_0\in V^c$ such that $h(U)\subseteq I$. Now for $x\in U$, $|h_1(x)|\le|h(x)|<\delta$ implies that $h_1(x)\in I$ and so $h_1(U)\subseteq I$. Hence, h_1 is continuous at x_0. Therefore, h_1 is continuous on X. Similarly, we can show that h_2 is continuous on X. Therefore, $h_i\in C_{\mathbb{R}}{}^+(X)$ for i = 1,2. Then

$h_1(x)+h_2(x)=h(x)$ for all x in X and $\Phi(h)=\Phi(h_1)+\Phi(h_2)\le\Phi^+(f_1)+\Phi^+(f_2)$.

This means that for all $h\in C_{\mathbb{R}}{}^+(X)$ with $0\le h\le f_1+f_2$, $\Phi(h)\le\Phi^+(f_1)+\Phi^+(f_2)$. Therefore, $\Phi^+(f_1+f_2)\le\Phi^+(f_1)+\Phi^+(f_2)$. Thus, $\Phi^+(f_1+f_2)=\Phi^+(f_1)+\Phi^+(f_2)$.

We now extend this definition of Φ^+ to all of $f \in C_{\mathbb{R}}(X)$. For $f \in C_{\mathbb{R}}(X)$, $|f(x)|$ is bounded above, say by a positive constant, N. Then $f + N \geq 0$. We define $\Phi^+(f) = \Phi^+(f+N) - \Phi^+(N)$. This is well defined. For suppose $f + M \geq 0$, then $\Phi^+(f+N+M) = \Phi^+(f+N) + \Phi^+(M) = \Phi^+(f+M) + \Phi^+(N)$ so that

$$\Phi^+(f+N) - \Phi^+(N) = \Phi^+(f+M) - \Phi^+(M).$$

It is clear that Φ^+ is linear on $C_{\mathbb{R}}(X)$. Suppose $f_1, f_2 \in C_{\mathbb{R}}(X)$ and $f_1 + N \geq 0, f_2 + M \geq 0$. Then $\Phi^+(f_1 + f_2) = \Phi^+(f_1 + f_2 + M + N) - \Phi^+(M+N)$

$$= \Phi^+(f_1 + N) + \Phi^+(f_2 + M) - \Phi^+(M) - \Phi^+(N)$$

$$= \Phi^+(f_1) + \Phi^+(f_2).$$

Plainly, $\Phi^+(0) = 0$ and for $c \geq 0$, $\Phi^+(c\,f) = c\,\Phi^+(f)$ for all $f \in C_{\mathbb{R}}(X)$.

In particular, for $f \in C_{\mathbb{R}}(X)$, $\Phi^+(f) + \Phi^+(-f) = \Phi^+(f + (-f)) = \Phi^+(0) = 0$ so that $\Phi^+(-f) = -\Phi^+(f)$. Thus, Φ^+ is a linear functional on $C_{\mathbb{R}}(X)$. Since $\Phi^+(f) \geq 0$ for $f \geq 0$, Φ^+ is a positive linear functional on $C_{\mathbb{R}}(X)$. Note that by definition of Φ^+ for $f \geq 0$, $\Phi(f) \leq \Phi^+(f)$. Define $\Phi^-(f) = \Phi^+(f) - \Phi(f)$ for $f \in C_{\mathbb{R}}(X)$.

Then for $f \geq 0$, $\Phi^-(f) = \Phi^+(f) - \Phi(f) \geq 0$, it follows that Φ^- is also a positive linear functional on $C_{\mathbb{R}}(X)$ and $\Phi = \Phi^+ - \Phi^-$.

Note that

$$|\Phi(f)| = |\Phi^+(f) - \Phi^-(f)| \leq |\Phi^+(f)| + |\Phi^-(f)|$$

$$\leq \|\Phi^+\| \|f\|_u + \|\Phi^-\| \|f\|_u = \left(\|\Phi^+\| + \|\Phi^-\|\right) \|f\|_u.$$

Therefore, $\|\Phi\| \leq \|\Phi^+\| + \|\Phi^-\|$. Note that since X is compact, for positive linear functionals, Φ^+ and Φ^-, $\|\Phi^+\| = \Phi^+(1)$ and $\|\Phi^-\| = \Phi^-(1)$. (If Λ is a positive linear functional, then for any f in $C_{\mathbb{R}}(X)$, $\Lambda(f), \Lambda(-f) \leq \Lambda(|f|)$. Thus, if $|f| \leq 1$, $|\Lambda(f)| \leq \Lambda(|f|) \leq \Lambda(1)$ and since $\|\Lambda\| = \sup\{|\Lambda(f)| : \|f\|_u = 1 \text{ and } f \in C_{\mathbb{R}}(X)\}$ and $1 \in C_{\mathbb{R}}(X)$, $\|\Lambda\| = \Lambda(1)$.)

Recall that $\Phi^+(1) = \sup\{\Phi(h) : h \in C_{\mathbb{R}}(X) \text{ and } 0 \leq h \leq 1\}$.

Take any $h \in C_{\mathbb{R}}(X)$ with $0 \le h \le 1$. Then $-1 \le 2h-1 \le 1$. Therefore, by definition of $\|\Phi\|$, $|\Phi(2h-1)| \le \|\Phi\|\|2h-1\|_u \le \|\Phi\|$ so that $\Phi(2h-1) \le |\Phi(2h-1| \le \|\Phi\|\|2h-1\|_u \le \|\Phi\|$. This means, $2\Phi(h)-\Phi(1) = \Phi(2h-1) \le \|\Phi\|$ for all $h \in C_{\mathbb{R}}(X)$ such that $0 \le h \le 1$. Therefore, by definition of $\Phi^+(1)$, $2\Phi^+(1)-\Phi(1) \le \|\Phi\|$, that is to say,

$\Phi^+(1)+\Phi^-(1) = 2\Phi^+(1)-\Phi(1) \le \|\Phi\|$. Consequently, $\Phi^+(1)+\Phi^-(1) = \|\Phi\|$.

This completes the proof of Proposition 2.

Suppose X is a compact Hausdorff topological space.

Suppose $\Phi : C_{\mathbb{R}}(X) \to \mathbb{R}$ is a bounded real linear functional. Then by Proposition 2, we can decompose Φ as $\Phi = \Phi^+ - \Phi^-$ such that Φ^+ and Φ^- are positive real linear functional and $\|\Phi\| = \|\Phi^+\| + \|\Phi^-\| = \Phi^+(1) + \Phi^-(1)$. By the Riesz Representation Theorem (Theorem 1), there are unique positive regular *finite* Borel measures, μ_1 and μ_2, on $\mathscr{M}$ such that $\Phi^+(f) = \int_X f\, d\mu_1$ and $\Phi^-(f) = \int_X f\, d\mu_2$. Thus, $\Phi(f) = \int_X f\, d\mu_1 - \int_X f\, d\mu_2 = \int_X f\, d(\mu_1 - \mu_2)$. Let $\lambda = \mu_1 - \mu_2$. Then λ is a real measure.

Then for all $f \in C_{\mathbb{R}}(X)$,

$$\Phi(f) = \int_X f\, d\lambda \text{ and}$$

$$|\Phi(f)| = \left|\int_X f\, d(\mu_1-\mu_2)\right| = \left|\int_X f^+\, d(\mu_1-\mu_2) - \int_X f^-\, d(\mu_1-\mu_2)\right|$$

$$\le \left|\int_X f^+ d\lambda\right| + \left|\int_X f^- d\lambda\right| \le \int_X f^+ d|\lambda| + \int_X f^- d|\lambda| = \int_X |f|\, d|\lambda|$$

$$\le \|f\|_u \int_X d|\lambda| = \|f\|_u |\lambda|(X).$$

Note here that for a real measure, μ, the variation measure of μ, is defined to be $|\mu|: \mathscr{M} \to \overline{\mathbb{R}^+}$ given by

$$|\mu|(E) = \sup_{\text{All partitions } \{E_i\} \text{ of } E} \sum_i |\mu(E_i)|.$$

Note that $|\mu|$ is a measure. (See Theorem 1, *Complex Measure, Dual Space of L^p Space, Radon-Nikodym Theorem and Riesz Representation Theorem*.)

Hence, $\|\Phi\| \le |\lambda|(X)$. But $|\lambda|(X) \le \mu_1(X) + \mu_2(X) = \Phi^+(1) + \Phi^-(1) = \|\Phi\|$ and so $\|\Phi\| = |\lambda|(X) = \mu_1(X) + \mu_2(X)$.

Note that it is easy to see that if μ_1 and μ_2 are positive *finite* regular Borel measure, then $|\lambda| = |\mu_1 - \mu_2|$ is a positive regular measure. (See Proposition 21, *Complex Measure, Dual Space of L^p Space, Radon-Nikodym Theorem and Riesz Representation Theorem*.) For a signed measure or real measure μ, we say μ is *regular* if $|\mu|$ is regular and μ is *finite* if $|\mu|$ is finite. As $|\lambda|(E) = |\mu_1 - \mu_2|(E) \le |\mu_1|(E) + |\mu_2|(E)$ for all $E \in \mathscr{M}$, if μ_1 and μ_2 are finite positive measures, $|\lambda|$ is finite and so $\lambda = \mu_1 - \mu_2$ is a finite real measure.

We shall now show that λ is unique.

Suppose there exists finite regular real Borel measures, λ_1 and λ_2 such that

$\Phi(f) = \int_X f\, d\lambda_1 = \int_X f\, d\lambda_2$. Let $\mu = \lambda_1 - \lambda_2$, then $\int_X f\, d\mu = 0$. μ is a finite regular real Borel measure. Then by the Jordan decomposition of measure (see Theorem 13, *Complex Measure, Dual Space of L^p Space, Radon-Nikodym Theorem and Riesz Representation Theorem*),

$$\mu = \mu^+ - \mu^- \text{ , where } \mu^+ \text{ and } \mu^- \text{ are positive measures.}$$

As $\int_X f\, d\mu = 0$ for all $f \in C_{\mathbb{R}}(X)$, $\int_X f\, d\mu^+ = \int_X f\, d\mu^-$ for all $f \in C_{\mathbb{R}}(X)$ and both define the same positive real linear functional. Therefore, by the uniqueness part of Theorem 1 (Riesz Representation Theorem), $\mu^+ = \mu^-$, consequently $\mu = 0$ and so $\lambda_1 = \lambda_2$.

Hence, we conclude that the real dual space of $C_{\mathbb{R}}(X)$, that is, the space of all bounded real linear functional on $C_{\mathbb{R}}(X)$ is isometrically isomorphic (i.e., via a norm preserving map) with the space of all regular real (signed) finite Borel measures, expressible as the difference of two finite positive regular measures, on the σ-algebra $\mathscr{M}$ on X, with norm given by $\|\mu\| = |\mu|(X)$.

Thus, we have proved:

Theorem 3. Suppose X is a compact Hausdorff topological space and $C_{\mathbb{R}}(X) = \{f : X \to \mathbb{R}; f \text{ is continuous}\}$. Suppose $\Phi : C_{\mathbb{R}}(X) \to \mathbb{R}$ is a bounded real linear functional. Then there exists a σ-algebra $\mathscr{M}$ on X, containing all the Borel sets of X and a unique regular finite real Borel measure (signed measure), λ, on $\mathscr{M}$, expressible as the difference of two finite positive regular Borel measures, such that $\Phi(f)=\int_X f\,d\lambda$ and $\|\Phi\|=|\lambda|(X)$. Let M be the collection of all regular finite real Borel measures, expressible as the difference of two finite positive regular Borel measures, with a norm on M given by $\|\mu\|=|\mu|(X)$ for μ in M. Then the association $\Gamma:C_{\mathbb{R}}(X)^*\to M$, where $C_{\mathbb{R}}(X)^*$ is the real dual space of $C_{\mathbb{R}}(X)$, given by $\Gamma(\Phi)=\lambda$, where $\Phi(f)=\int_X f\,d\lambda$, is a linear isometric isomorphism preserving norm.

Note that given a finite regular real Borel measure, λ, we can decompose $\lambda=\lambda^+-\lambda^-$ by the Jordan Decomposition Theorem (Theorem 13, *Complex Measure, Dual Space of L^p Space, Radon-Nikodym Theorem and Riesz Representation Theorem*) where λ^+ and λ^- are finite positive Borel measure. Then for any $f\in C_{\mathbb{R}}(X)$, we may define

$$\int_X f\,d\lambda=\int_X f\,d\lambda^+-\int_X f\,d\lambda^- .$$

Plainly, if we define $\Lambda(f)=\int_X f\,d\lambda$, Λ is a real linear functional and is clearly bounded, since

$$\begin{aligned}|\Lambda(f)|&=\left|\int_X f\,d\lambda\right|=\left|\int_X f\,d\lambda^+-\int_X f\,d\lambda^-\right|\le\left|\int_X f\,d\lambda^+\right|+\left|\int_X f\,d\lambda^-\right|\\&\le\|f\|_u\left(\lambda^+(X)+\lambda^-(X)\right)=\|f\|_u|\lambda|(X)\end{aligned}$$

so that $\|\Lambda\|\le|\lambda|(X)<\infty$.

If λ is expressible as the difference of two finite regular positive Borel measures, i.e., $\lambda=\alpha-\beta$, where α and β are finite regular positive Borel measures, then $\lambda=\alpha-\beta=\lambda^+-\lambda^-$ so that $\alpha+\lambda^-=\beta+\lambda^+$. Then $\alpha+2\lambda^-=\beta+\lambda^++\lambda^-=\beta+\|\lambda\|$ is regular. It follows that λ^- is regular and hence λ^+ is also regular. Thus, if λ is expressible as the difference of two finite regular positive Borel measures, then the Jordan decomposition gives $\lambda=\lambda^+-\lambda^-$ and λ^+ and λ^- are finite regular positive Borel measures.

Theorem 1 does not give a regular representing measure. We now impose additional condition on X so that the regularity of the representing measure is a consequence.

Theorem 4. Let X be a locally compact Hausdorff topological space. Suppose X is also σ-compact, i.e., X is a countable union of compact subspaces. Let $C_c(X) = \{f : X \to \mathbb{C}; f \text{ is continuous with compact support}\}$. Let $\Lambda : C_c(X) \to \mathbb{C}$ be a positive complex linear functional on $C_c(X)$, i.e., whenever $f \in C_c(X)$ and f is real valued with $f \geq 0$, then $\Lambda(f) \geq 0$. Then in addition to the conclusions (a) to (e) of Theorem 1, we have the following.

(f) For all $E \in \mathscr{M}$ and for all $\varepsilon > 0$, there exist closed set F in X and an open set V in X such that $F \subseteq E \subseteq V$ and $\mu(V - F) < \varepsilon$.

(g) μ is a regular Borel measure on X, i.e., condition (c) and (d) hold without any condition for any $E \in \mathscr{M}$.

(h) For all $E \in \mathscr{M}$, there exists a F_σ set A and a G_δ set B such that $A \subseteq E \subseteq B$ and $\mu(B - A) = 0$. That is to say, each measurable set differs from a F_σ set or a G_δ set by a null set.

A F_σ set is a set, which is a countable union of closed set. A G_δ set is a countable intersection of open sets.

Proof. Since X is σ-compact, let $X = \bigcup_{i=1}^{\infty} K_i$, where each K_i is a compact sub space of X.

(f) Take $E \in \mathscr{M}$. Then $\mu(K_n \cap E) \leq \mu(K_n) < \infty$ by part (b). Thus, by part (c), given any $\varepsilon > 0$, there exists a set V_n open in X such that $V_n \supseteq K_n \cap E$ and

$$\mu(K_n \cap E) \leq \mu(V_n) < \mu(K_n \cap E) + \frac{1}{2^{n+1}}\varepsilon .$$

Therefore,

$$\mu(V_n - K_n \cap E) < \frac{1}{2^{n+1}}\varepsilon . \quad \text{----------------------- (1)}$$

Let $V = \bigcup_{n=1}^{\infty} V_n$. Then V is open and $V \supseteq E$. Note that

$$V - E = \bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{\infty} K_n \cap E \subseteq \bigcup_{n=1}^{\infty} \left(V_n - K_n \cap E\right).$$

Therefore,

$$\mu\left(V - E\right) \leq \mu\left(\bigcup_{n=1}^{\infty}\left(V_n - K_n \cap E\right)\right) \leq \sum_{n=1}^{\infty} \mu\left(V_n - K_n \cap E\right) < \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{\varepsilon}{2}.$$

Now, the complement of E, $E^c \in \mathscr{M}$. By the same argument as above, there exists an open subset U in X such that $E^c \subseteq U$ and

$$\mu\left(U - E^c\right) < \frac{\varepsilon}{2}.$$

Let $F = U^c$. Then F is closed and

$$\mu(V - F) = \mu\left(\left(V - E\right) \cup \left(F^c - E^c\right)\right) = \mu\left(\left(V - E\right) \cup \left(U - E^c\right)\right)$$

$$\leq \mu\left(V - E\right) + \mu\left(U - E^c\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

(g)

Since (c) holds for any $E \in \mathscr{M}$ by definition of μ, we need to show that (d) holds for all $E \in \mathscr{M}$. We already knew that (d) holds for any open E or any E with $\mu(E) < \infty$.

By (f), we can choose a closed set $F \subseteq E$ such that $\mu(E - F)$ is arbitrarily small. Therefore, it is sufficient to show that (d) holds for all closed set F in X.

If F is closed, then for each integer $n \geq 1$, $F \cap K_n$ is compact and so $F \cap K_n \in \mathscr{M}$. Note that $F = \bigcup_{i=1}^{\infty}\left(F \cap K_i\right)$. By the usual property of measure,

$$\mu\left(\bigcup_{i=1}^{n}\left(F \cap K_i\right)\right) \to \mu(F) \text{ as } n \to \infty .$$

Since $\bigcup_{i=1}^{n}\left(F \cap K_i\right)$ is compact and $\bigcup_{i=1}^{n}\left(F \cap K_i\right) \subseteq F$, this implies that (d) holds for F.

Now we show how this implies that (d) holds for any $E \in \mathscr{M}$.

Choose closed $F \subseteq E$ such that $\mu(E-F) < \frac{\varepsilon}{2}$. There exists an integer N such that

$$\mu(F) - \mu\left(\bigcup_{i=1}^{n}(F \cap K_i)\right) < \frac{\varepsilon}{2} \text{ for integer } n \geq N.$$

Then for integer $n \geq N$,

$$\mu\left(E - \bigcup_{i=1}^{n}(F \cap K_i)\right) \leq \mu(E-F) + \mu\left(F - \bigcup_{i=1}^{n}(F \cap K_i)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

As noted, $\bigcup_{i=1}^{n}(F \cap K_i)$ is compact and so this shows that (d) holds for any $E \in \mathscr{M}$.

(h)

Take $E \in \mathscr{M}$. Apply (f) successively with $\varepsilon = \frac{1}{n}$ to give closed set F_n and open set V_n such that $F_n \subseteq E \subseteq V_n$ with $\mu(V_n - F_n) < \frac{1}{n}$. Let $A = \bigcup_{n=1}^{\infty} F_n$ and $B = \bigcap_{n=1}^{\infty} V_n$. Then A is a F_σ set and B is a G_δ set such that $A \subseteq E \subseteq B$. Note that

$$B - A = \bigcap_{n=1}^{\infty} V_n - \bigcup_{n=1}^{\infty} F_n \subseteq V_n - F_n \text{ for each integer } n \geq 1.$$

Therefore, $\mu(B-A) \leq \mu(V_n - F_n) < \frac{1}{n}$ for all integer $n \geq 1$. Hence, $\mu(B-A) = 0$.

This completes the proof.

Remark.

Compare this with Theorem 3. The positive Borel measure representing a positive real linear functional on the space $C_{\mathbb{R}}(X)$, when X is a compact Hausdorff topological space is regular. There is an example of positive non-regular Borel measure arising from a positive complex linear functional on the space $C_c(X)$, where X is a locally compact Hausdorff topological space. So, Theorem 4 is not vacuous.

Theorem 4 says that the positive Borel measure representing a positive complex linear functional on the space $C_c(X)$, when X is a locally compact and σ-compact Hausdorff topological space is regular. The next result is about a condition on a locally compact Hausdorff topological space X that implies any positive Borel measure on X is regular. This condition is enjoyed by the metric space $\mathbb{R}^n$.

This condition is stated as every open subset of X is σ-compact. In particular, X is σ-compact.

Theorem 5. Let X be a locally compact Hausdorff topological space, in which every open subset is σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for any compact subset K in X. Then λ is regular.

Remark.

1. $\mathbb{R}^n$ satisfies the hypothesis of Theorem 5.

2. Not every locally compact Hausdorff topological space must necessarily have its open sets σ-compact.

Proof of Theorem 5.

Take any $f \in C_c(X)$. Since f is continuous f is Borel measurable. Define $\Lambda(f) = \int_X f d\lambda$. Since f has compact support K,

$$\left|\Lambda(f)\right| = \left|\int_X f\, d\lambda\right| \leq \left\|f\right\|_u \lambda(K) < \infty ,$$

as $\left\|f\right\|_u < \infty$ and $\lambda(K) < \infty$ because K is compact. If f is real valued and $f \geq$ o, then $\int_X f\, d\lambda \geq 0$. It follows that $\Lambda : C_c(X) \to \mathbb{C}$ is a positive complex linear functional. By Theorem 4, we get a positive regular Borel measure μ on $\mathscr{M}$ such that $\Lambda(f) = \int_X f d\mu$. We shall next show that for any open set V in X, $\lambda(V) = \mu(V)$. Since V is σ-compact, $V = \bigcup_{i=1}^{\infty} H_i$ and H_i is compact for each integer $i \geq 1$. Since H_1 is compact and $H_1 \subseteq V$, by Urysohn's Lemma, there exists $f_1 \in C_c(X)$ such that $H_1 \prec f_1 \prec V$. If $f_1, f_2, \cdots, f_n$ have been chosen so that K_i = support f_i , choose f_{n+1} by Urysohn's Lemma such that

$$K_1 \cup K_2 \cup \cdots \cup K_n \cup H_1 \cup H_2 \cup \cdots \cup H_n \prec f_{n+1} \prec V .$$

Observe that $K_1 \cup K_2 \cup \cdots \cup K_n \cup H_1 \cup H_2 \cup \cdots \cup H_n$ is compact and is contained in V. Note that support $f_n = K_n \subseteq V$, $0 \le f_n(x) \le 1$ for all x in X,

$f_n(x) = 1$ for all x in $K_1 \cup K_2 \cup \cdots \cup K_{n-1} \cup H_1 \cup H_2 \cup \cdots \cup H_{n-1}$.

Therefore,

$f_n(x) \le f_{n+1}(x)$ for x in $K_1 \cup K_2 \cup \cdots \cup K_n \cup H_1 \cup H_2 \cup \cdots \cup H_n$

and as $f_n(x) = 0$ for $x \notin K_n$, $f_n(x) \le f_{n+1}(x)$ for all x in X. It follows that $\{f_n\}$ is a monotone increasing sequence of real valued functions in $C_c(X)$. As $V = \bigcup_{i=1}^{\infty} H_i$, $f_n \nearrow \chi_V$ pointwise on X. Therefore,

$\lambda(V) = \int_X \chi_V d\lambda = \lim_{n\to\infty} \int_X f_n d\lambda$, by the Lebesgue Monotone Convergence Theorem,

$= \lim_{n\to\infty} \Lambda(f_n) = \lim_{n\to\infty} \int_X f_n d\mu$, by Theorem 4,

$= \int_X \chi_V d\mu$, by the Lebesgue Monotone Convergence Theorem,

$= \mu(V)$.

For any Borel set E in $\mathscr{M}$, by Theorem 4 part (f), there exist closed F and open V such that $F \subseteq E \subseteq V$ and

$\mu(V - F) < \varepsilon$. ---------------------- (*)

Hence, since $V - F$ is open, by what we have just proved,

$\lambda(V - F) = \mu(V - F) < \varepsilon$. ---------------------- (**)

Therefore, $\lambda(V - E) \le \lambda(V - F) = \mu(V - F) < \varepsilon$. This implies λ is outer regular.

By (*), we can choose a closed set $F \subseteq E$ such that $\lambda(E - F) \le \lambda(V - F) < \varepsilon$ is arbitrarily small. Therefore, it is sufficient to show that (d) holds for all closed set F in X.

Since X is σ-compact, let $X = \bigcup_{i=1}^{\infty} L_i$, where L_i is a compact subspace for each integer $i \ge 1$.

If F is closed, then for each integer $n \geq 1$, $F \cap L_n$ is compact and so $\lambda(F \cap L_n) < \infty$. Note that $F = \bigcup_{i=1}^{\infty}(F \cap L_i)$. By the usual property of measure,

$$\lambda\left(\bigcup_{i=1}^{n}(F \cap L_i)\right) \to \lambda(F) \text{ as } n \to \infty.$$

Since $\bigcup_{i=1}^{n}(F \cap L_i)$ is compact and $\bigcup_{i=1}^{n}(F \cap L_i) \subseteq F$, this implies that (d) holds for closed set F.

Now we show how this implies that (d) holds for any Borel measurable $E \in \mathscr{M}$.

Choose closed $F \subseteq E$ such that $\lambda(E - F) \leq \lambda(V - F) < \dfrac{\varepsilon}{2}$. There exists an integer N such that $\lambda(F) - \lambda\left(\bigcup_{i=1}^{n}(F \cap L_i)\right) < \dfrac{\varepsilon}{2}$ for integer $n \geq N$.

Then for integer $n \geq N$,

$$\lambda\left(E - \bigcup_{i=1}^{n}(F \cap L_i)\right) \leq \lambda(E - F) + \lambda\left(F - \bigcup_{i=1}^{n}(F \cap L_i)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As noted, $\bigcup_{i=1}^{n}(F \cap L_i)$ is compact and so this shows that (d) holds for any Borel measurable $E \in \mathscr{M}$.

Now we can define the Lebesgue measure on $\mathbb{R}^k$ via the Riemann integral.

Theorem 6. There is a regular complete positive measure, m, defined on a σ-algebra $\mathscr{M}$ on $\mathbb{R}^k$ satisfying:

(a) If I is any k-cell in $\mathbb{R}^k$, then $m(I) = vol(I)$.

Here a k-cell $I = \{x \in \mathbb{R}^k : \alpha_i < x_i < \beta_i, 1 \leq i \leq k$ or anything gotten by replacing any $<$ by $\leq\}$ and $vol(I)$ is the usual volume of the k-cell I.

(b) $\mathscr{M}$ contains all Borel subsets of $\mathbb{R}^k$. More precisely,

$$\mathscr{M} = \{E \subseteq \mathbb{R}^k : \text{ there exists an } F_\sigma \text{ set } A \text{ and a } G_\delta \text{ set } B, A \subseteq E \subseteq B, m(B - A) = 0\}.$$

(c) The measure m is translation invariant, i.e., for all E in $\mathscr{M}$,

$$m(E) = m(E + x) \text{ for any } x \in \mathbb{R}^k .$$

(d) The measure, m, is unique upto multiplication in the following sense:

If μ is another positive translation invariant Borel measure on $\mathbb{R}^k$ such that $\mu(K) < \infty$ for all compact K, then there exists a constant c such that $\mu(E) = c\, m(E)$ for all Borel set $E \subseteq \mathbb{R}^k$.

The sets in $\mathscr{M}$ are called Lebesgue measurable sets and m the (k-dimensional) Lebesgue measure on $\mathbb{R}^k$.

Proof.

Define $\Lambda(f) = \int_{\mathbb{R}^k} f$ for $f \in C_c\left(\mathbb{R}^k\right)$, the Riemann integral of the continuous function f with compact support in $\mathbb{R}^k$. By the generalized Heine Borel Theorem, support f is closed and bounded and so the Riemann integral is finite. This means $|\Lambda(f)| < \infty$ for all $f \in C_c\left(\mathbb{R}^k\right)$. Plainly, for all $f \in C_c\left(\mathbb{R}^k\right)$, $f \geq 0 \Rightarrow \Lambda(f) \geq 0$. By Theorem 4 (Riesz Representation Theorem), there exists a σ-algebra $\mathscr{M}$, containing all the Borel subsets on $\mathbb{R}^k$ and a positive regular complete Borel measure, m, associated with Λ such that $\Lambda(f) = \int_{\mathbb{R}^k} f\, dm$ for all $f \in C_c\left(\mathbb{R}^k\right)$.

(a) We may suppose that the k-cell I is open. Adding or taking away bits of boundary of I, does not altered its volume. So, we may just prove (a) for open k-cell.

Since I is open, $m(I) = \sup\{\Lambda(f) : f \prec I\}$. For any $f \prec I$, $0 \leq f \leq 1$, support $f \subseteq I$ and so $f \leq \chi_I$ so that $\Lambda(f) \leq \Lambda(\chi_I) = vol(I)$. For any ε > 0, we can shrink I to I_0 an open k-cell such that $I_0 \subseteq \overline{I_0} \subseteq I$ and $vol(I) - \varepsilon < vol(I_0) < vol(I)$. By the Urysohn's Lemma (Lemma 22, *Convex Function, L^p Spaces, Space of Continuous Functions, Lusin's Theorem*), there exists $f \in C_c(\mathbb{R}^k)$ such that $\overline{I_0} \prec f \prec I$. Thus,

$$vol\left(\overline{I_0}\right) = \int_{\mathbb{R}^k} \chi_{\overline{I_0}} \leq \int_{\mathbb{R}^k} f = \Lambda\left(f\right) = \int_{\mathbb{R}^k} f\, dm \leq \int_{\mathbb{R}^k} \chi_I = vol(I) .$$

Hence $\Lambda\left(f\right) \geq vol\left(\overline{I_0}\right) = vol\left(I_0\right) > vol(I) - \varepsilon$. This means $m(I) = vol(I)$.

(b) That $\mathscr{M}$ contains all Borel subsets of $\mathbb{R}^k$ follows from Theorem 4 (or Riesz Representation Theorem, Theorem 1). The other statement follows from Theorem 4 (h) and the fact that *m* is complete by Theorem 1.

(c) Since $vol(I+x) = vol(I)$ for any *k*-cell *I* and any *x* in $\mathbb{R}^k$, $m(I+x) = m(I)$ for any *k*-cell *I* and any *x* in $\mathbb{R}^k$. Next, if *V* is open, we can write $V = \bigcup_{i=1}^{\infty} I_i$ a countable disjoint union of *k*-cells. So, by the countable additivity of *m*,

$$m(V) = \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} vol(I_i) = \sum_{i=1}^{\infty} vol(I_i + x) = \sum_{i=1}^{\infty} m(I_i + x) = m(V+x) .$$

Finally if $E \in \mathscr{M}$, then

$$m(E) = \inf\{m(V) : E \subseteq V \text{ and } V \text{ is open}\} = \inf\{m(V+x) : E \subseteq V \text{ and } V \text{ is open}\}$$

$$= \inf\{m(V+x) : E+x \subseteq V+x \text{ and } V \text{ is open}\} = \inf\{m(V) : E+x \subseteq V \text{ and } V \text{ is open}\}$$

$$= m(E+x) .$$

(d) Suppose μ is a positive translation invariant Borel measure on $\mathbb{R}^k$ such that $\mu(K) < \infty$ for all compact *K* in $\mathbb{R}^k$. Then by Theorem 5, μ is regular.

Let I^1 be a *k*-cell of side of length 1. Let $\mu(I^1) = c$. We can write I^1 as a disjoint union of 2^{nk} cells of side $\frac{1}{2^n}$, $I^1 = \bigcup_{i=1}^{2^{nk}} I_i$. Choose one of these *k*-cells and call it $\tilde{I}$. Then, since μ is translation invariant,

$$\mu(I^1) = \mu\left(\bigcup_{i=1}^{2^{nk}} I_i\right) = 2^{nk}\mu\left(\tilde{I}\right) = c = c\,m(I^1), \text{ as } m(I^1) = 1,$$

$$= c2^{nk}\, m\left(\tilde{I}\right).$$

Therefore, $\mu\left(\tilde{I}\right) = c\,m\left(\tilde{I}\right)$.

Now, any open subset *V* of $\mathbb{R}^k$ can be written as a disjoint countable union of *k*-cells, where *k*-cells are open balls with centre *x* of radius *r* defined by

$$B_r(x) = \left\{y : \max_{1\le i\le k} |y_i - x_i| < r\right\} .$$

Each of these open balls can be written as a countable disjoint union of k-cells of side $\frac{1}{2^n}$ of the type like $\tilde{I}$ above. Hence, $\mu(V) = c\, m(V)$. Finally as μ and m are regular Borel measures, for any Borel set $E \subseteq \mathbb{R}^k$,

$$\mu(E) = \inf\left\{\mu(V) : E \subseteq V \text{ and } V \text{is open}\right\} = \inf\left\{c\, m(V) : E \subseteq V \text{ and } V \text{is open}\right\}$$

$$= c \inf\left\{ m(V) : E \subseteq V \text{ and } V \text{is open}\right\} = c\, m(E).$$

Remark.

We have thus obtained the Lebesgue measure *m* on $\mathbb{R}^k$ so that the Lebsgue integral $\int_{\mathbb{R}^k} f\, dm$ is equal to the Riemann integral $\int_{\mathbb{R}^k} f(x) dx$ for all $f \in C_c\left(\mathbb{R}^k\right)$.

Actually, more generally, when f and $|f|$ are Riemann integrable,

$$\int_{\mathbb{R}^k} f\, dm = \int_{\mathbb{R}^k} f(x)\, dx.$$

For $k = 1$, we examine the Lebesgue measure *m* so obtained by Theorem 6 more explicitly.

A *step function s* on [*a*, *b*] is a function that assumes finite values on the open subintervals of [*a*, *b*] defined by some partition of [*a*, *b*]. More precisely, there is a partition $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ and a set of constants, $\xi_1, \xi_2, \cdots, \xi_n$ such that $s(x) = \xi_i$ for $x_{i-1} < x < x_i$ for $1 \le i \le n$. Plainly, a step function is a simple μ-measurable function.

Suppose $f : [a,b] \to \mathbb{R}$ is bounded.

Let $S^*([a,b])$ be the set of all step functions on [*a*, *b*]. The *lower Riemann integral* of f is defined to be

$$R\underline{\int_a^b} f = \sup\left\{\int_a^b \varphi : \varphi \le f, \varphi \in S^*([a,b])\right\}$$

and the *upper Riemann integral* of f is defined to be

$$R\overline{\int_a^b} f = \inf\left\{\int_a^b \varphi : f \le \varphi, \varphi \in S^*([a,b])\right\}.$$

As *f* is bounded, the upper and lower Riemann integrals exist. The bounded function *f* is said to be Riemann integrable if

$$R\underline{\int_a^b} f = R\overline{\int_a^b} f .$$

Now a step function in $S^*([a,b])$ is a linear combination of characteristic functions of subintervals. So, by Theorem 6 part (a), for $\varphi \in S^*([a,b])$, $\int_a^b \varphi = \int_{[a,b]} \varphi dm$. Now let $S([a,b])$ be the set of real-valued simple functions on [a, b]. Then $S^*([a,b]) \subseteq S([a,b])$. Moreover,

$$R\overline{\int_a^b} f = \inf\left\{\int_a^b \varphi : f \le \varphi, \varphi \in S^*([a,b])\right\} = \inf\left\{\int_{[a,b]} \varphi dm : f \le \varphi, \varphi \in S^*([a,b])\right\}$$

$$\ge \inf\left\{\int_a^b \varphi dm : f \le \varphi, \varphi \in S([a,b])\right\}.$$

and $$R\underline{\int_a^b} f = \sup\left\{\int_a^b \varphi : \varphi \le f, \varphi \in S^*([a,b])\right\} = \sup\left\{\int_{[a,b]} \varphi dm : \varphi \le f, \varphi \in S^*([a,b])\right\}$$

$$\le \sup\left\{\int_{[a,b]} \varphi dm : \varphi \le f, \varphi \in S([a,b])\right\}.$$

If we define the *lower Lebesgue integral* of f to be

$$\underline{\int_{[a,b]}} f dm = \sup\left\{\int_{[a,b]} \varphi dm : \varphi \le f, \varphi \in S([a,b])\right\}$$

and the *upper Lebesgue integral* of f to be
$\overline{\int_{[a,b]}} f dm = \inf\left\{\int_a^b \varphi dm : f \le \varphi, \varphi \in S([a,b])\right\}$, then we have

$$R\underline{\int_a^b} f \le \underline{\int_{[a,b]}} f dm \le \overline{\int_{[a,b]}} f dm \le R\overline{\int_a^b} f .$$

So, if f is Riemann integrable on [a, b], then $\underline{\int_{[a,b]}} f dm = \overline{\int_{[a,b]}} f dm$. It can be shown that if $\underline{\int_{[a,b]}} f = \overline{\int_{[a,b]}} f$, then $f : [a,b] \to \mathbb{R}$ is m-measurable. Therefore, f is bounded and Lebesgue measurable and so f is Lebesgue integrable and the Lebesgue integral, $\int_{[a,b]} f\, dm = \underline{\int_{[a,b]}} f dm = \overline{\int_{[a,b]}} f dm = R\underline{\int_a^b} f = R\overline{\int_a^b} f$. Thus, if f is Riemann integrable, then f is Lebesgue integrable and the Riemann integral and the Lebesgue integral are the same.

We have made use of the following two results from Lebesgue integration. For the sake of clarity, we shall state and prove them.

Theorem 7. Suppose E is a Lebesgue measurable subset of $\mathbb{R}$ and $m(E) < \infty$. Suppose $f : E \to \mathbb{R}$ is bounded. Then f is measurable, if and only if, the lower and upper Lebesgue integral of f are the same. The *lower Lebesgue integral* of f is defined by $\underline{\int_E} f dm = \sup\left\{\int_E \varphi dm : \varphi \le f, \varphi \in S(E)\right\}$ and the *upper Lebesgue integral* by $\overline{\int_E} f dm = \inf\left\{\int_E \varphi dm : f \le \varphi, \varphi \in S(E)\right\}$, where $S(E)$ is the set of real-valued simple measurable functions on E.

Proof.

Suppose f is measurable. As f is bounded, we assume $\alpha \le f < \beta$. Let $\delta_n = \frac{\beta - \alpha}{n}$, for integer $n \ge 1$. Define $E_{n.i} = f^{-1}[\alpha + (i-1)\delta_n, \alpha + i\delta_n)$ for $1 \le i \le n$, $n = 1, 2, \ldots$.

Then $E_{n,i}$ are measurable and for each integer $n \ge 1$,

$$m(E) = m\left(\bigcup_{i=1}^{n} E_{n,i}\right), \text{ where } \bigcup_{i=1}^{n} E_{n,i} \text{ is a disjoint union,}$$

$$= \sum_{i=1}^{n} m\left(E_{n,i}\right) < \infty .$$

Let $\phi_n = \sum_{i=1}^{n} \left(\alpha + (i-1)\delta_n\right)\chi_{E_{n,i}}$ and $\psi_n = \sum_{i=1}^{n} \left(\alpha + i\,\delta_n\right)\chi_{E_{n,i}}$ for each integer $n \ge 1$. Thus, ϕ_n, ψ_n are simple measurable functions on E such that

$$\phi_n(x) \le f(x) < \psi_n(x) \quad \text{for all } x \text{ in } E.$$

Therefore,

$$\underline{\int_E} f dm \ge \int_E \phi_n\, dm = \sum_{i=1}^{n} \left(\alpha + (i-1)\delta_n\right) m(E_{n,i})$$

and

$$\overline{\int_E} f dm \le \int_E \psi_n dm = \sum_{i=1}^{n} \left(\alpha + i\delta_n\right) m\left(E_{n,i}\right).$$

Hence, $\overline{\int_E} f dm - \underline{\int_E} f dm \le \sum_{i=1}^{n} \delta_n m(E_{n,i}) = \delta_n m(E)$.

But $\delta_n \to 0$ as $n \to \infty$ and so $\overline{\int_E} f dm \le \underline{\int_E} f dm$. Since $\underline{\int_E} f dm \le \overline{\int_E} f dm$, it follows that $\underline{\int_E} f dm = \overline{\int_E} f dm$.

Conversely, suppose $\underline{\int_E} f dm = \overline{\int_E} f dm$. We shall show that f is measurable or m-measurable.

Let $L(f) = \{\phi \in S(E) : \phi \le f\}$ and $U(f) = \{\psi \in S(E) : f \le \psi\}$. Then $\underline{\int_E} f dm = \sup\left\{\int_E \phi dm : \phi \in L(f)\right\}$ and $\overline{\int_E} f dm = \inf\left\{\int_E \psi dm : \psi \in U(f)\right\}$. Since f is bounded and $m(E) < \infty$, $\underline{\int_E} f dm = \overline{\int_E} f dm < \infty$. Thus, for any integer $n \ge 1$, there exists $\phi_n \in L(f)$ and $\psi_n \in U(f)$ such that

$$\int_E \phi_n dm > \underline{\int_E} f dm - \frac{1}{n} \quad \text{and} \quad \int_E \psi_n dm < \overline{\int_E} f dm + \frac{1}{n}.$$

Hence, $\int_E (\psi_n - \phi_n) dm = \int_E \psi_n dm - \int_E \phi_n dm \le \dfrac{2}{n}$. This holds for all integer $n \ge 1$.

Define $\phi, \psi : E \to \mathbb{R}$, by $\phi = \sup\{\phi_n\}_{n=1}^{\infty}$ and $\psi = \inf\{\psi_n\}_{n=1}^{\infty}$. Then both ϕ and ψ are measurable since each ϕ_n and ψ_n are measurable for all integer $n \ge 1$.

Plainly, $\phi_n \le \phi \le f \le \psi \le \psi_n$.

Let $D_k = \left\{x \in E : \psi(x) - \phi(x) > \dfrac{1}{k}\right\}$.

Obviously, $D_k \subseteq \left\{x \in E : \psi_n(x) - \phi_n(x) > \dfrac{1}{k}\right\} = D_{k,n}$ for all integer $n \ge 1$.

Hence, $\dfrac{1}{k}\chi_{D_{k,n}} \le \psi_n - \phi_n$ and so $\int_E \dfrac{1}{k}\chi_{D_{k,n}} dm \le \int_E (\psi_n - \phi_n) dm$. It follows that

$$\frac{1}{k} m\left(D_{k,n}\right) \le \int_E (\psi_n - \phi_n) dm \le \frac{2}{n}.$$

Therefore, $m\left(D_k\right) \le m\left(D_{k,n}\right) \le \dfrac{2k}{n}$ for all integer $n \ge 1$. It follows that $m\left(D_k\right) = 0$.

Let $D = \{x \in E : \psi(x) - \phi(x) > 0\}$. Then $D = \bigcup_{k=1}^{\infty} D_k$ and $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_k \subseteq \cdots \subseteq D$. Therefore, $m(D) = \lim_{k \to \infty} m\left(D_k\right) = 0$. This means $\psi = \phi$ almost everywhere with respect to the Lebesgue measure *m*. As $\phi \leq f \leq \psi$, $\phi = f = \psi$ on $E - D$ and $f = \psi$ almost everywhere with respect to the Lebesgue measure *m*. Since *E* is measurable and the Lebesgue measure is complete, $E - D$ is measurable Therefore, *f* is measurable on $E - D$ since ψ is measurable. Hence, *f* is measurable.

Theorem 8. Suppose *E* is a Lebesgue measurable subset of $\mathbb{R}$ and $m(E) < \infty$. Suppose $f : E \to \mathbb{R}$ is a bounded measurable function. Then f is Lebesgue integrable and

$$\int_E f dm = \int_E f^+ dm - \int_E f^- dm = \underline{\int_E} f dm = \overline{\int_E} f dm .$$

Proof.

Since $f : E \to \mathbb{R}$ is measurable, $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$ are measurable. Thus, $f = f^+ - f^-$ and $|f| = f^+ + f^-$ is measurable. Note that , f^+, f^- and $|f|$ are bounded non-negative functions. Then by definition,

$$\int_E f^+ dm = \sup\left\{\int_E s\, dm : 0 \leq s \leq f^+, s \in S(E)\right\}$$

and $\int_E f^- dm = \sup\left\{\int_E s\, dm : 0 \leq s \leq f^-, s \in S(E)\right\}$.

Since both $\left\{\int_E s\, dm : 0 \leq s \leq f^+, s \in S(E)\right\}$ and $\left\{\int_E s\, dm : 0 \leq s \leq f^-, s \in S(E)\right\}$ are bounded above by $Km(E)$ for some constant *K* such that $|f(x)| < K$ for all *x* in *E*, $\int_E f^+ dm$ and $\int_E f^- dm$ exist and are finite and so $\int_E |f| dm = \int_E f^+ dm + \int_E f^- dm < \infty$. Thus, by definition, *f* is Lebesgue integrable on *E* and $\int_E f\, dm = \int_E f^+ dm - \int_E f^- dm$.

Since f^+ is measurable,

$$\underline{\int_E} f^+ dm = \sup\left\{\int_E \phi dm : \phi \in L(f^+)\right\} = \sup\left\{\int_E \phi dm : \phi \in S(E) : \phi \leq f^+\right\}$$

$$= \sup\left\{\int_E \phi dm : \phi \in S(E) : 0 \leq \phi \leq f^+\right\} = \int_E f^+ dm .$$

Similarly, we have $\underline{\int_E} f^- dm = \int_E f^- dm$. By Theorem 7, $\underline{\int_E} f^- dm = \int_E f^- dm = \overline{\int_E} f^- dm$ and $\underline{\int_E} f^+ dm = \int_E f^+ dm = \overline{\int_E} f^+ dm$.

$$\int_E f^+ dm - \int_E f^- dm = \overline{\int_E} f^+ dm - \underline{\int_E} f^- dm$$

$$= \inf\left\{\int_E \psi dm : f^+ \le \psi, \psi \in S(E)\right\} - \sup\left\{\int_E \phi dm : \phi \in S(E), \phi \le f^-\right\}$$

$$= \inf\left\{\int_E \psi dm : f^+ \le \psi, \psi \in S(E)\right\} + \inf\left\{-\int_E \phi dm : \phi \in S(E), \phi \le f^-\right\}$$

$$= \inf\left\{\int_E \psi dm : f^+ \le \psi, \psi \in S(E)\right\} + \inf\left\{\int_E -\phi dm : -\phi \in S(E), -\phi \ge -f^-\right\}$$

$$= \inf\left\{\int_E \psi dm : f^+ \le \psi, \psi \in S(E)\right\} + \inf\left\{\int_E \phi dm : -f^- \le \phi, \phi \in S(E),\right\}$$

$$= \inf\left\{\int_E \psi dm + \int_E \phi dm : f^+ \le \psi, -f^- \le \phi,\ \psi, \phi \in S(E)\right\}$$

$$= \inf\left\{\int_E (\psi + \phi) dm : f^+ \le \psi, -f^- \le \phi,\ \psi, \phi \in S(E)\right\}$$

$$\ge \inf\left\{\int_E \psi dm : f = f^+ - f^- \le \psi, \psi \in S(E)\right\} = \overline{\int_E} f dm .$$

Similarly,

$$\int_E f^+ dm - \int_E f^- dm = \underline{\int_E} f^+ dm - \overline{\int_E} f^- dm$$

$$= \sup\left\{\int_E \psi dm : \psi \le f^+, \psi \in S(E)\right\} - \inf\left\{\int_E \phi dm : \phi \in S(E), f^- \le \phi\right\}$$

$$= \sup\left\{\int_E \psi dm : \psi \le f^+, \psi \in S(E)\right\} + \sup\left\{-\int_E \phi dm : \phi \in S(E), f^- \le \phi\right\}$$

$$= \sup\left\{\int_E \psi dm : \psi \le f^+, \psi \in S(E)\right\} + \sup\left\{\int_E -\phi dm : -\phi \in S(E), -f^- \ge -\phi\right\}$$

$$= \sup\left\{\int_E \psi dm : \psi \le f^+, \psi \in S(E)\right\} + \sup\left\{\int_E \phi dm : \phi \in S(E), \phi \le -f^-\right\}$$

$$= \sup\left\{\int_E \psi dm + \int_E \phi dm : \psi \le f^+, \phi \le -f^-,\ \psi, \phi \in S(E)\right\}$$

$$= \sup\left\{\int_E (\psi + \phi) dm : \psi \le f^+, \phi \le -f^-,\ \psi, \phi \in S(E)\right\}$$

$$\le \sup\left\{\int_E \psi\, dm : \psi \le f^+ - f^- = f, \psi \in S(E)\right\} = \underline{\int_E} f\, dm .$$

Thus, we have $\overline{\int_E} f\, dm \le \int_E f^+ dm - \int_E f^- dm \le \underline{\int_E} f\, dm$. Now, as *f* is measurable and $m(E) < \infty$, by Theorem 7, $\overline{\int_E} f\, dm = \underline{\int_E} f\, dm$ and so it follows that

$$\overline{\int_E} f\, dm = \int_E f^+ dm - \int_E f^- dm = \underline{\int_E} f\, dm .$$

Finally, we come to the last result, a very useful technical theorem.

Theorem 9. Partition of Unity.

Let *X* be a locally compact Hausdorff topological space and *K* a compact subspace of *X*. Suppose $U_1, U_2, \cdots, U_n$ is a finite covering of *K* by open sets. Then there exists $h_i \in C_c(X)$, the space of continuous functions on *X* with compact support, such that $h_i \prec U_i$, $1 \le i \le n$ and $h_1 + h_2 + \cdots + h_n = 1$ on *K*.

That is, $0 \le h_i \le 1$ and $h_i = 0$ on U_i^c. This collection of functions $\{h_i\}$ is called a *partition of unity on K subordinate to the covering* $\{U_1, U_2, \cdots, U_n\}$ of *K*.

Proof.

Suppose $x \in K$. Then since $K \subseteq \bigcup_{i=1}^{n} U_i$, there exists some *i* such that $x \in U_i$.
Since *X* is locally compact and Hausdorff, by (9) of Topological Ideas in *Convex Function, L^p spaces, Spaces of Continuous Functions, Lusin's Theorem*, there exists a relatively compact neighbourhood W_x , such that W_x is open, $x \in W_x \subseteq \overline{W_x} \subseteq U_i$ and $\overline{W_x}$ is compact. Then the collection $\{W_x : x \in K\}$ is an open covering of *K*. Since *K* is compact, it has a finite sub-covering say $\{W_{x_1}, W_{x_2}, \cdots, W_{x_N}\}$. Let $H_i = \bigcup_{W_{x_j} \subseteq U_i} \overline{W_{x_j}}$. Then H_i is a finite union of compact sets and so is compact. Note that $H_i \subseteq U_i$ for $1 \le i \le n$. By Urysohn's Lemma (Lemma 22 of *Convex Function, L^p spaces, Spaces of Continuous Functions, Lusin's Theorem*), there exists a function $g_i \in C_c(X)$ such that

$$H_i \prec g_i \prec U_i \text{ for } 1 \le i \le n.$$

Let $h_1 = g_1$, $h_2 = (1-g_1)g_2$, … , $h_n = (1-g_1)(1-g_2)\cdots(1-g_{n-1})g_n$. Since $g_i \prec U_i$, $h_i \prec U_i$ for $1 \le i \le n$. Observe that $\bigcup_{i=1}^{N} \overline{W_{x_i}} \subseteq \bigcup_{i=1}^{n} H_i$ as follows.

Note that for each W_{x_k}, $\overline{W_{x_k}} \subseteq U_j$ for some $1 \le j \le n$. Hence, $\overline{W_{x_k}} \subseteq H_j \subseteq \bigcup_{i=1}^{n} H_i$. Therefore, $\bigcup_{i=1}^{N} \overline{W_{x_i}} \subseteq \bigcup_{i=1}^{n} H_i$.

Now take any $x \in K$ and so $x \in W_{x_j}$ for some $1 \le j \le N$ and hence, $x \in H_i$ for some $1 \le i \le n$. It follows that $g_i(x) = 1$. Now

$$h_1 + h_2 + \cdots + h_n = 1 - (1-g_1)(1-g_2)\cdots(1-g_n)\,. \quad \text{-----------------------} \quad (*)$$

We can show this by induction. (*) is plainly true for n=1 and for n=2. If (*) is true for $n-1$, then

$$h_1 + h_2 + \cdots + h_{n-1} + h_n = 1 - (1-g_1)(1-g_2)\cdots(1-g_{n-1}) + g_n(1-g_1)(1-g_2)\cdots(1-g_{n-1})$$

$$= 1 - (1-g_n)(1-g_1)(1-g_2)\cdots(1-g_{n-1})\,.$$

For any $x \in K$, $(1-g_1(x))(1-g_2(x))\cdots(1-g_n(x)) = 0$ and so $h_1 + h_2 + \cdots + h_n = 1$ on K.