On The Primitive Of Product Of Two Functions

By Ng Tze Beng

Recently I came across an interesting result of Daniel Lesnic, which one can prove using results not beyond that of the ideas of derivative and the Mean Value Theorem. I state the result as follows.

**Theorem 1.** Suppose $a < b$ and $[a, b]$ is a closed and bounded interval. If $f : [a, b] \to \mathbb{R}$ is a continuous function of bounded variation and $g : [a, b] \to \mathbb{R}$ is a function that has a primitive, then the product $f \cdot g$ has a primitive on $[a, b]$.

**Remark.**

1. If $g$ is continuous, then Theorem 1 is trivial. This is because $f \cdot g$ is continuous and so it follows by the Fundamental Theorem of Calculus, that $f \cdot g$ has a primitive on $[a, b]$.

2. If $f$ is continuous and $g$ is Riemann integrable, then $f \cdot g$ is Riemann integrable and so has a primitive almost everywhere on $[a, b]$, i.e., there exists a function $H$ such that $H' = f g$ almost everywhere on $[a, b]$.

3. If $f$ is continuous and $g$ is Lebesgue integrable, then $f \cdot g$ is Lebesgue integrable and $f \cdot g$ has a primitive almost everywhere on $[a, b]$.

4. Note that if $f$ is Lebesgue integrable and $g$ is bounded, then $f \cdot g$ has a primitive almost everywhere on $[a, b]$. This is because $g$ has a primitive and so $g$ is measurable by a Theorem of Banach and if $g$ is bounded, then $f \cdot g$ is Lebesgue integrable.

We shall prove Theorem 1 for the special case when $f : [a, b] \to \mathbb{R}$ is a continuous injective function. Now a continuous injective function on $[a, b]$ is strictly monotonic. Thus we shall prove the special case of theorem 1 when $f$ is continuous and strictly monotonic on $[a, b]$.

**Lemma 2.** Suppose $f : [a, b] \to \mathbb{R}$ is a continuous strictly increasing function and $g : [a, b] \to \mathbb{R}$ is a function that has a primitive. Then the product $f \cdot g$ has a primitive on $[a, b]$.

The conclusion is also valid if $f$ is a strictly decreasing function.

**Proof.** Since $f : [a, b] \to \mathbb{R}$ is a strictly increasing and continuous function, the image $f([a, b])$ is a non-trivial interval $[c, d]$ with $c < d$. The inverse function $f^{-1} : [c, d] \to [a, b]$ is also continuous and strictly increasing. Let $G : [a, b] \to \mathbb{R}$ be a primitive of $g$. Then $G' = g$ on $[a, b]$.

$G$ is continuous and so the composite $h = G \circ f^{-1} : [c, d] \to \mathbb{R}$ is a continuous function. Hence $h$ has a primitive $H : [c, d] \to \mathbb{R}$. This means $H' = h = G \circ f^{-1}$.
Define a function \( K: [a, b] \to \mathbb{R} \) by \( K(x) = f(x)G(x) - H \circ f(x) = f(x)G(x) - H(f(x)) \) for \( x \) in \([a, b]\). Then we claim that \( K'(x) = f(x)g(x) \) for all \( x \) in \([a, b]\).

Take \( x \) in \((a, b)\). We shall show that the left and right derivatives of \( K \) at \( x \) is \( f(x)g(x) \).

Let \( y > x \) and \( y \) is in \([a, b]\). Then

\[
K(y) - K(x) = f(y)G(y) - H(f(y)) - (f(x)G(x) - H(f(x)))
\]

\[
= f(x)(G(y) - G(x)) + (f(y) - f(x))G(y) - (H(f(y)) - H(f(x)))
\]

\[
\text{--------------------------------------} \quad (1)
\]

Now since \( H \) is differentiable and \( f(y) > f(x) \) because \( f \) is strictly increasing, by the Mean Value Theorem, there exists \( \zeta \) such that \( f(y) > \zeta > f(x) \) and

\[
H(f(y)) - H(f(x)) = (f(y) - f(x)) H'(\zeta) = (f(y) - f(x)) G(f^{-1}(\zeta)) \quad \quad \text{------- (2)}
\]

But since \( f \) is continuous and strictly increasing, by the Intermediate Value Theorem, there exists \( \zeta_y \) such that \( y > \zeta_y > x \) and \( f(\zeta_y) \) = \( \zeta \). Hence it follows from (2) that

\[
H(f(y)) - H(f(x)) = (f(y) - f(x)) G(f^{-1}(f(\zeta_y))) = (f(y) - f(x)) G(\zeta_y) \quad \quad \text{----- (3)}
\]

Therefore, it follows from (1) and (3) that

\[
K(y) - K(x) = f(x)(G(y) - G(x)) + (f(y) - f(x))G(y) - (f(y) - f(x)) G(\zeta_y)
\]

\[
= f(x)(G(y) - G(x)) + (f(y) - f(x))(G(y) - G(x)) - (f(y) - f(x)) (G(\zeta_y) - G(x))
\]

\[
= f(y)(G(y) - G(x)) - (f(y) - f(x)) (G(\zeta_y) - G(x))
\]

Therefore,

\[
\frac{K(y) - K(x)}{y - x} = f(x)\frac{G(y) - G(x)}{y - x} - (f(y) - f(x))\frac{G(\zeta_y) - G(x)}{y - x} \quad \text{------- (4)}
\]

Now \( \lim_{y \to x} f(y)\frac{G(y) - G(x)}{y - x} = f(x)g(x) \). \quad \text{------------------------ (5)}

Note that \( \frac{G(\zeta_y) - G(x)}{y - x} = \frac{G(\zeta_y) - G(x)}{\zeta_y - x} \cdot \frac{\zeta_y - x}{y - x} \). Since \( G'(x) \) exists and is equal to \( g(x) \), there exists \( \delta > 0 \) such that for \( x < y < x + \delta \), \( \left| \frac{G(y) - G(x)}{y - x} \right| < |g(x)| + 1 \). It follows that for \( 0 < y < x + \delta \), \( \left| \frac{G(\zeta_y) - G(x)}{\zeta_y - x} \cdot \frac{\zeta_y - x}{y - x} \right| < |g(x)| + 1 \). That is, \( \frac{G(\zeta_y) - G(x)}{y - x} \) is bounded on \((x, x + \delta)\). Therefore,
\[
\lim_{y \to x'} (f(y) - f(x)) \frac{G(\zeta, y) - G(x)}{y - x} = 0 \quad \text{---------------------- (6)}
\]

because \( \lim_{y \to x'} (f(y) - f(x)) = 0 \) by continuity at \( x \).

Thus it follows from (4), (5) and (6) that,

\[
\lim_{y \to x'} \frac{K(y) - K(x)}{y - x} = f(x)g(x).
\]

This proves that the right derivative of \( K \) at \( x \) is \( f(x)g(x) \).

If \( x = a \), the above argument shows that \( K'(a) = f(a)g(a) \). Similarly we can show that the left derivative of \( K \) at \( x \) is \( f(x)g(x) \). The same argument shows that \( K'(b) = f(b)g(b) \).

Hence for \( x \) in \((a, b)\), \( K'(x) = f(x)g(x) \) and so \( K' = f \cdot g \). That is, \( K \) is a primitive of \( f \cdot g \).

If \( f \) is strictly decreasing and continuous, then \( -f \) is strictly increasing and continuous. Therefore, by what we have just proved, \( -f \cdot g \) has a primitive say \( K \). Then \( -K \) is a primitive for \( f \cdot g \).

Next we show that the conclusion of Lemma 2 is valid for an increasing and continuous function.

**Corollary 3.** Suppose \( f : [a, b] \to \mathbb{R} \) is a continuous increasing function and \( g : [a, b] \to \mathbb{R} \) is a function that has a primitive. Then the product \( f \cdot g \) has a primitive on \([a, b]\).

**Proof.** If \( f \) is increasing and continuous, then \( h(x) = f(x) + x \) is strictly increasing and continuous on \([a, b]\). Therefore, by Lemma 2, \( h(x)g(x) \) has a primitive, say \( K(x) \) on \([a, b]\).

Also since \( x \) is strictly increasing and continuous, \( xg(x) \) has a primitive \( H(x) \) on \([a, b]\). Then \( K - H \) is a primitive for \( f \cdot g \) as \((K - H)'(x) = K'(x) - H'(x) = h(x)g(x) - xg(x) = f(x)g(x)\).

**Proof of Theorem 1.**

If \( f : [a, b] \to \mathbb{R} \) is continuous and of bounded variation, then \( f \) is the difference of two continuous increasing functions. Hence \( f = f_1 - f_2 \), where \( f_1 \) and \( f_2 \) are increasing continuous functions. If \( g \) has a primitive, then by Corollary 3, \( f_1g \) and \( f_2g \) both have primitives. It follows that \( f \cdot g = f_1g - f_2g \) has a primitive.
We next present a special characterization of a function of bounded variation satisfying the conclusion of Theorem 1.

First a technical lemma.

**Lemma 4.** Suppose \( f : [a, b] \to \mathbb{R} \) is of bounded variation and has the Darboux property, i.e., \( f \) has the intermediate value property. Then \( f \) is continuous.

**Proof.** If \( f \) is of bounded variation, then it can have only jump discontinuities. But since \( f \) has the intermediate value property, \( f \) cannot have any jump discontinuity and so \( f \) is continuous.

**Lemma 5.** Suppose \( f : [a, b] \to \mathbb{R} \) can be represented as the quotient of two functions having primitives. That is, \( f = g / h \), \( g \) and \( h \) have primitives and \( h \neq 0 \). Then \( f \) is a Darboux function, i.e., \( f \) has the intermediate value property.

**Proof.** Suppose \( f(c) < f(d) \) for some \( c, d \) in \([a, b]\), and \( c \neq d \). For any \( k \) such that \( f(c) < k < f(d) \). That is,

\[
g(c) / h(c) < k < g(d) / h(d). \tag{1}
\]

Define \( H(x) = g(x) - k \cdot h(x) \) for \( x \) in \([a, b]\). Then since \( g \) and \( h \) have primitives, \( H \) too has a primitive. Therefore, by Darboux Theorem, \( H \) has the intermediate value property.

Since \( h \neq 0 \) and \( h \) has the intermediate value property, \( h > 0 \) or \( h < 0 \). Therefore, it follows from (1) that

\[
g(c) < k \cdot h(c) \text{ and } k \cdot h(d) < g(d), \text{ if } h > 0 \tag{2}
\]

or

\[
g(c) > k \cdot h(c) \text{ and } k \cdot h(d) > g(d), \text{ if } h < 0. \tag{3}
\]

Hence, it follows from (2) and (3) that

\[
H(c) < 0 < H(d) \text{ or } H(c) > 0 > H(d).
\]

Therefore, since \( H \) has the intermediate value property, there exists \( x \) between \( c \) and \( d \) such that \( H(x) = 0 \). That is, \( g(x) = k \cdot h(x) \) and so \( f(x) = g(x)/h(x) = k \). This shows that \( f \) has the intermediate value property.

**Corollary 6.** Suppose \( f : [a, b] \to \mathbb{R} \) can be represented as the quotient of two functions having primitives. That is, \( f = g / h \), \( g \) and \( h \) have primitives and \( h \neq 0 \). Furthermore if \( f \) is of bounded variation, then \( f \) is continuous.

**Proof.** By Lemma 5, \( f \) has the intermediate value property. Since \( f \) is also of bounded variation, by Lemma 4, \( f \) is continuous.
Now we state the characterization theorem.

**Theorem 7.** Suppose \( f : [a, b] \to \mathbb{R} \) is of bounded variation. Suppose there exists a non-zero function \( h : [a, b] \to \mathbb{R} \) possessing primitives such that the product \( f \cdot h \) possesses primitives. Then \( f \cdot g \) possesses primitives for any \( g \) possessing primitives.

**Proof.** Let \( K = f \cdot h \). Then \( K \) has primitives by hypothesis. Then \( f = K / h \) since \( h \neq 0 \). And so \( f \) is a quotient of two functions possessing primitives and is also of bounded variation and so by Corollary 6, \( f \) is continuous. This means that \( f \) is a continuous function of bounded variation. Therefore, by Theorem 1, \( f \cdot g \) possesses primitives for any \( g \) possessing primitives.

**Remark.** By Theorem 7, a function of bounded variation having the property that there exists a non-zero function \( h \) possessing primitives such that the product \( f \cdot h \) has primitives is necessarily continuous. Thus if \( f \) is a discontinuous function of bounded variation, then for any non-zero function \( h \) possessing primitives, \( f \cdot h \) has no primitives.

Corollary 6 is a criterion of deciding when a function of bounded variation is continuous. By isolating the use of the intermediate value property we can prove the following weaker result in exactly the same way.

**Theorem 8.** Suppose \( f : [a, b] \to \mathbb{R} \) is a function of bounded variation that can be represented as the quotient of two Darboux functions, i.e., \( f = g / h \), where \( h \) is a non-zero Darboux function possessing primitives and \( g \) is a Darboux function satisfying that \( g + k \) is a Darboux function for any non-zero Darboux function \( k \) possessing primitives. Then \( f \) is continuous.

Following A. Bruckner, we can use the product formula for derivatives to deduce the following result.

**Theorem 9.** Suppose \( g : [a, b] \to \mathbb{R} \) is a function possessing primitives and is Lebesgue integrable. Then for any differentiable function \( F : [a, b] \to \mathbb{R} \), \( F \cdot g \) possesses primitives, which are all absolutely continuous. In particular, if \( g \) is the derivative of a differentiable function of bounded variation or equivalently a differentiable absolutely continuous function, then for any differentiable \( F \), \( F \cdot g \) possesses primitives, which are all absolutely continuous.

**Proof.** Suppose \( G : [a, b] \to \mathbb{R} \) is a primitive of \( g \). Then since \( g \) is Lebesgue integrable, \( G \) is absolutely continuous and so \( G \) is continuous of bounded variation. Since \( F \) is differentiable, by Theorem 1, \( G \cdot F' \) possesses primitives. Now by the product rule for derivatives,

\[
(G \cdot F)' = G' \cdot F + G \cdot F' = F \cdot g + G \cdot F'.
\]

Thus, if \( H \) is a primitive of \( G \cdot F' \), then \( G \cdot F - H \) is a primitive of \( F \cdot g \) since
\[(G \cdot F - H)' = (G \cdot F)' - H' = F \cdot g + G \cdot F' - G \cdot F = F \cdot g.\]

Since \(F\) is differentiable and so is continuous, \(F\) is a bounded Lebesgue integrable function. Because \(g\) is Lebesgue integrable, \(F \cdot g\) is Lebesgue integrable. It follows by Theorem 6 of “Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallé Poussin’s Theorem” that \(G \cdot F - H\) is absolutely continuous. Therefore, \(F \cdot g\) possesses an absolutely continuous primitive for any differentiable \(F\). Hence all its primitives are absolutely continuous.

In particular if \(G\) is differentiable and of bounded variation, then \(G' = g\) is Lebesgue integrable and so \(G\) is absolutely continuous. Note that a differentiable function on \([a, b]\) is absolutely continuous if and only if it is of bounded variation. It follows as above that \(F \cdot g\) possesses absolutely continuous primitives for any differentiable \(F\).

**Remark.** Note that any differentiable function \(G : [a, b] \rightarrow \mathbb{R}\) is necessarily a continuous \(N\) function. (See lemma 4 of “ When is a function on a closed and bounded interval be of bounded variation, absolutely continuous?”.) Therefore, by Theorem 6 of “Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallé Poussin’s Theorem”, such a differentiable function \(G\) is absolutely continuous if and only if \(G'\) is Lebesgue integrable.

In view of the remark above we may state Theorem 9 as follows.

**Theorem 10.** Suppose \(g : [a, b] \rightarrow \mathbb{R}\) is the derivative of a differentiable absolutely continuous function. Then for any differentiable function \(F : [a, b] \rightarrow \mathbb{R}\), \(F \cdot g\) possesses primitives, which are all absolutely continuous.

If \(g\) is positive and \(g\) possesses primitives, then any primitive of \(g\) is strictly increasing. For such function we have the following result.

**Lemma 11.** Suppose \(g : [a, b] \rightarrow \mathbb{R}\) is a function possessing primitives and \(g > 0\) or \(g(x) > 0\) for \(x \neq a, b\). Then for any continuous function \(f : [a, b] \rightarrow \mathbb{R}\), \(f \cdot g\) possesses primitives, which are all absolutely continuous.

**Proof.** Suppose \(G : [a, b] \rightarrow \mathbb{R}\) is a primitive of \(g\). Then \(G'(x) = g(x) > 0\) for \(x \neq a, b\). It follows that \(G\) is strictly increasing and continuous. Hence \(G([a, b])\) is compact and so a closed and bounded interval \([c, d]\) and \(G^{-1} : [c, d] \rightarrow [a, b]\) is also strictly increasing and continuous. Thus, for any continuous function \(f : [a, b] \rightarrow \mathbb{R}\), \(f \circ G^{-1} : [c, d] \rightarrow \mathbb{R}\) is continuous and so has a primitive \(H : [c, d] \rightarrow \mathbb{R}\) such that \(H' = f \circ G^{-1}\). Then \(H \circ G : [a, b] \rightarrow \mathbb{R}\) is differentiable and by the Chain Rule,

\[(H \circ G)'(x) = H'(G(x)) \cdot G'(x) = f \circ G^{-1}(G(x)) \cdot g(x) = f(x) \cdot g(x).\]

It follows that \(H \circ G\) is a primitive of \(f \cdot g\). Note that \(g\) being the derivative of a monotone function is Lebesgue integrable. Since \(f\) is continuous and so is integrable and bounded, \(f \cdot g\)
is Lebesgue integrable. It follows that $H \circ G$ is absolutely continuous. Hence all primitives of $f \cdot g$ are absolutely continuous. We can also deduce that $H \circ G$ is absolutely continuous by observing that $H$ satisfies a Lipschitz condition and $G$ is absolutely continuous so that their composite $H \circ G$ is absolutely continuous.

**Theorem 12.** Suppose $g : [a, b] \to \mathbb{R}$ is a function possessing primitives and $g \geq 0$ Then for any continuous function $f : [a, b] \to \mathbb{R}$, $f \cdot g$ possesses primitives, which are all absolutely continuous.

**Proof.** Let $h = g + 1$. Then $h > 0$. Therefore, by Lemma 11, $f \cdot h$ possesses an absolutely continuous primitive, say $H$. Since $f$ is continuous, $f$ has an absolutely continuous primitive, say $F$. Then $H - F$ is absolutely continuous, differentiable and 

$$(H - F)' = H' - F' = f \cdot h - f = f \cdot g + f - f = f \cdot g.$$ 

Thus, $f \cdot g$ possesses an absolutely continuous primitive and so all its primitives are absolutely continuous.

The proof of Theorem 12 suggests the following slight generalization of Theorem 12.

**Theorem 13.** Suppose $g : [a, b] \to \mathbb{R}$ is a function possessing primitives. Suppose there exists a function $h : [a, b] \to \mathbb{R}$ possessing primitives such that $h \leq 0$ and $g \geq h$. Then for any continuous function $f : [a, b] \to \mathbb{R}$, $f \cdot g$ possesses primitives, which are all absolutely continuous.

**Proof.** Let $k = g - h \geq 0$. Then for any continuous function $f : [a, b] \to \mathbb{R}$, by Theorem 12, $f \cdot k$ has an absolutely continuous primitive $K$ and $-f \cdot h$ has an absolutely continuous primitive $H$. Hence $K - H$ is absolutely continuous and $(K - H)' = f \cdot g$ and so $f \cdot g$ has an absolutely continuous primitive. It follows that all its primitives are absolutely continuous.

Note that in the proof of Theorem 13, both $K$ and $H$ are differentiable increasing (therefore absolutely continuous) functions. We can prove the following in exactly the same way as Theorem 13.

**Theorem 14.** Suppose $g : [a, b] \to \mathbb{R}$ is a function possessing a primitive expressible as the difference of two differentiable increasing functions. Then for any continuous function $f : [a, b] \to \mathbb{R}$, $f \cdot g$ possesses primitives, which are all absolutely continuous.