Product Measure and Fubini's Theorem

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This article discusses a technique to define measure from two given measures, similar in principle to defining product topology from given topologies. Once we have defined the product measure of two measure spaces, we can then consider measurable and integrable function over a product measure space. Fubini's Theorem then realizes the integral of a measurable function, defined on a product measure space, as an iterated integral over the given measure spaces.

We need some results concerning the uniqueness of given measure on a σ-algebra to show that the product measure so defined is unique.

Uniqueness of Measure.

We assume the familiarity of the notion of the constituent of a measure space, namely, that of a σ-algebra as introduced in my article, *Introduction To Measure Theory*.

Definition 1.

Suppose X is a non-empty set. A collection, $\mathscr{C}$, of subsets of X is called a *d-system* or a *Dynkin class* on X if it satisfies the following three conditions.

(i) $X \in \mathscr{C}$,

(ii) if $A, B \in \mathscr{C}$ and $A \supseteq B$, then $A - B \in \mathscr{C}$ and

(iii) if $\{A_n\}$ is an increasing sequence of sets in $\mathscr{C}$, i.e., for $n \geq 1$, $A_n \subseteq A_{n+1}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{C}$.

A collection, $\mathscr{S}$ of subsets of X is called a *π-system*, if it is closed under the formation of finite intersection.

If X is a set and $\mathscr{M}$ is a σ-algebra on X, then certainly $\mathscr{M}$ is a d-system and also a *π-system*.

It is easily shown that if μ and ν are finite (positive) measures on $\mathscr{M}$ and that $\mu(X) = \nu(X)$, then the collection, $\mathscr{S}$ of all set A in $\mathscr{M}$ such that $\mu(A) = \nu(A)$, is a d-system.

Our first result paving the way towards the uniqueness of measure is the following theorem.

Theorem 2. Suppose X is a non-empty set. Let $\mathscr{C}$ be a *π-system* on X. Then the σ-algebra generated by $\mathscr{C}$ coincides with the d-system generated by $\mathscr{C}$.

Proof.

It is easily seen that the intersection of non-empty d-systems is a d-system. Therefore, the intersections of all d-systems containing $\mathscr{C}$ is the d-system generated by $\mathscr{C}$. Let $\mathscr{D}$ denote this d-system. Let $\sigma(\mathscr{C})$ be the σ-algebra generated by $\mathscr{C}$. Since $\sigma(\mathscr{C})$ is also a d-system, it must contain $\mathscr{D}$. We claim that $\mathscr{D}$ is a *π-system* on X, i.e., $\mathscr{D}$ is closed under the formation of finite intersection.

Note that $X, \emptyset \in \mathscr{D}$. Now let $\mathscr{D}_1 = \{A \in \mathscr{D} : A \cap C \in \mathscr{D}$ for all C in $\mathscr{C}.\}$. Since $\mathscr{C} \subseteq \mathscr{D}$, $X \in \mathscr{D}_1$. If $A, B \in \mathscr{D}_1$ and $A \supseteq B$, then for any C in $\mathscr{C}$, $(A-B)\cap C = A\cap C - B\cap C \in \mathscr{D}$ because $A \cap C$ and $B \cap C$ are in $\mathscr{D}$ and $\mathscr{D}$ is a d-system. Hence, $A - B \in \mathscr{D}_1$. If $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence in $\mathscr{D}_1$, then for any C in $\mathscr{C}$, $\left(\bigcup_{n=1}^{\infty} A_n\right)\cap C = \bigcup_{n=1}^{\infty}(A_n \cap C) \in \mathscr{D}$ because $\mathscr{D}$, is a d-system, each $A_n \cap C$ is in $\mathscr{D}$ and $\{A_n \cap C\}_{n=1}^{\infty}$ is an increasing sequence. Thus, $\mathscr{D}_1$ is closed under the formation of proper difference and the union of increasing sequence of sets in $\mathscr{D}_1$ and so $\mathscr{D}_1$ is a d-system. Since $\mathscr{C}$ is closed under the formation of finite intersection and $\mathscr{C} \subseteq \mathscr{D}$, $\mathscr{C} \subseteq \mathscr{D}_1$. Note that by definition, $\mathscr{D}_1 \subseteq \mathscr{D}$. It follows that $\mathscr{D}_1 = \mathscr{D}$. Let $\mathscr{D}_2 = \{B \in \mathscr{D} : B \cap A \in \mathscr{D}$ for all A in $\mathscr{D}\}$. As $\mathscr{D} = \mathscr{D}_1$, for any A in $\mathscr{D}$ and for all C in $\mathscr{C}$, $C \cap A \in \mathscr{D}$. Hence, $\mathscr{C} \subseteq \mathscr{D}_2$. It is easy to show that $\mathscr{D}_2$ is also a d-system. Moreover, by definition $\mathscr{D}_2 \subseteq \mathscr{D}$. As $\mathscr{D}$ is the smallest d-system containing $\mathscr{C}$, $\mathscr{D} = \mathscr{D}_2$. Hence, $\mathscr{D}$ is closed under the formation of finite intersection and so $\mathscr{D}$ is a *π-system* on X. Now $X \in \mathscr{D}$ so that $\mathscr{D}$ is closed under the formation of complement. Hence, $\mathscr{D}$ is an algebra and since it is closed under the formation of union of increasing sequence of sets in $\mathscr{D}$, $\mathscr{D}$ is a σ-algebra containing $\mathscr{C}$. Thus, $\mathscr{D} \supseteq \sigma(\mathscr{C})$. As $\sigma(\mathscr{C})$ is also a d-system containing $\mathscr{C}$, $\mathscr{D} = \sigma(\mathscr{C})$.

Corollary 3. Suppose $(X, \mathscr{M})$ is a measure space and $\mathscr{C}$ is a *π-system* on X such that $\mathscr{M} = \sigma(\mathscr{C})$. If μ and ν are finite positive measures on $\mathscr{M}$ satisfying $\mu(X) = \nu(X)$ and that $\mu(C) = \nu(C)$ for all $C \in \mathscr{C}$, then $\mu = \nu$.

Proof. Let $\mathscr{D} = \{A \in \mathscr{M}: \mu(A) = \nu(A)\}$. Then plainly, $\mathscr{D} \subseteq \mathscr{M}$, $X \in \mathscr{D}$ and $\mathscr{C} \subseteq \mathscr{D}$. For $A, B \in \mathscr{D}$, $B \subseteq A$ implies that $\mu(A-B) = \mu(A) - \mu(B) = \nu(A) - \nu(B) = \nu(A-B)$ and so $A - B \in \mathscr{D}$. Thus, $\mathscr{D}$ is closed under the formation of proper difference. If $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence in $\mathscr{D}$, then by the continuity from below property of positive measure,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Hence, $\bigcup_{n=1}^{\infty} A_n \in \mathscr{D}$. Thus, $\mathscr{D}$ is closed under the formation of the union of increasing sequence of sets in $\mathscr{D}$. Hence, $\mathscr{D}$ is a *d*-system containing $\mathscr{C}$. Since $\mathscr{M} = \sigma(\mathscr{C})$, by Theorem 2, $\mathscr{D} \supseteq \sigma(\mathscr{C})$. It follows that $\mathscr{D} = \mathscr{M}$ and so $\mu(A) = \nu(A)$ for all A in $\mathscr{M}$.

Corollary 4. Suppose $(X, \mathscr{M})$ is a measure space and $\mathscr{C}$ is a *π-system* on X such that $\mathscr{M} = \sigma(\mathscr{C})$. If μ and ν are positive measures on $\mathscr{M}$ that agree on $\mathscr{C}$ and if there exists an increasing sequence of sets $\{C_n\}$ in $\mathscr{C}$ such that $X = \bigcup_{n=1}^{\infty} C_n$ and $\mu(C_n) = \nu(C_n) < \infty$, for all positive integer $n \geq 1$, then $\mu = \nu$.

Proof.

Let $\{C_n\}$ be an increasing sequence of sets $\{C_n\}$ in $\mathscr{C}$ such that $X = \bigcup_{n=1}^{\infty} C_n$ and $\mu(C_n) = \nu(C_n) < \infty$, for all positive integer $n \geq 1$. For each integer $n \geq 1$, define for A in $\mathscr{M}$, $\mu_n(A) = \mu\left(A \cap C_n\right)$ and $\nu_n(A) = \nu\left(A \cap C_n\right)$. Then for any $C \in \mathscr{C}$, $\mu_n(C) = \nu_n(C)$ for all integer $n \geq 1$. It is easily verified that for integer $n \geq 1$, μ_n and ν_n are positive measures on $\mathscr{M}$ that agree on $\mathscr{C}$. Therefore, by Corollary 3, $\mu_n = \nu_n$ for integer $n \geq 1$.

As $X = \bigcup_{n=1}^{\infty} C_n$, for $A \in \mathscr{M}$,

$$\mu(A) = \mu\left(A \cap X\right) = \mu\left(A \cap \left(\bigcup_{n=1}^{\infty} C_n\right)\right)$$

$$= \lim_{n\to\infty} \mu\left(A \cap C_n\right) = \lim_{n\to\infty} \mu_n\left(A\right),$$

by the continuity from below property of positive measure,

$$= \lim_{n\to\infty} \nu_n\left(A\right) = \lim_{n\to\infty} \nu\left(A \cap C_n\right)$$

$$= \nu\left(\bigcup_{n=1}^{\infty}\left(A \cap C_n\right)\right) = \nu\left(A \cap \left(\bigcup_{n=1}^{\infty} C_n\right)\right)$$

$$= \nu\left(A \cap X\right) = \nu(A).$$

Hence, $\mu = \nu$ on $\mathscr{M}$.

Product Measure

Given two measure spaces, say $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$, we can form the product measure space by specifying an appropriate σ-algebra of subsets of $X \times Y$ and define a positive measure on this σ-algebra. Note that $\mathscr{M}$ is a σ-algebra of subsets of X and $\mathscr{N}$ is a σ-algebra of subsets of Y. If $A \in \mathscr{M}$ and $B \in \mathscr{N}$, we can form the Cartesian product $A \times B$, which is called a *measurable rectangle*.

Let $\mathscr{R} = \{ A \times B : A \in \mathscr{M} \text{ and } B \in \mathscr{N} \}$ be the set of all measurable rectangles. Note that if $A_1 \times B_1$ and $A_2 \times B_2$ are measurable rectangles, then

$$\left(A_1 \times B_1\right) \cap \left(A_2 \times B_2\right) = \left(A_1 \cap A_2\right) \times \left(B_1 \cap B_2\right)$$

is also a measurable rectangle, as $A_1 \cap A_2 \in \mathscr{M}$ and $B_1 \cap B_2 \in \mathscr{N}$. It follows that $\mathscr{R}$ is closed under the formation of finite intersection and so $\mathscr{R}$ is a π-system. Note that $\mathscr{R}$ is not necessary a σ-algebra of subsets of $X \times Y$. Let $\mathscr{B}$ be the σ-algebra generated by the π-system $\mathscr{R}$, i.e., $\mathscr{B} = \sigma(\mathscr{R})$. This is called the product of the σ-algebras $\mathscr{M}$ and $\mathscr{N}$. We also denote it by $\mathscr{M} * \mathscr{N}$. Observe that $(X \times Y, \mathscr{M} * \mathscr{N})$ is a measure space. In due course, we shall define a positive measure on $\mathscr{M} * \mathscr{N}$, which we call the product measure.

In the literature, it is often assumed that both $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are complete measure spaces, i.e., $\mathscr{M}$ is μ complete and $\mathscr{N}$ is ν complete. This is to facilitate the application to Lebesgue measure on $\mathbb{R}^n$ and product measure of $\mathbb{R}^n$ and $\mathbb{R}^m$. We shall take the general approach when measures need not be complete, derive the general Fubini's Theorem and then proceed to rework the corresponding result, with the completion of the product measure, $\mathscr{M} * \mathscr{N}$, with respect to the positive measure defined on $\mathscr{M} * \mathscr{N}$. In this respect, all measure spaces are assumed to be complete and some statements that hold when measures are not required to be complete, need to be modified.

Sections in $X \times Y$

We shall be discussing Fubini's Theorem. It is about when an integral of a function over a product space can be taken as an iterated integral. The components, called *sections* of a subset of $X \times Y$ naturally are the ingredients that we need to use. We shall define this notion and some related ideas.

Definition 5.

Suppose X and Y are non-empty sets. Suppose E is a non-empty subset of $X \times Y$. For each $x \in X$, the *section*, $E_x = \{ y \in Y : (x, y) \in E \}$ is a subset of Y and for each $y \in Y$, the *section*, $E^y = \{ x \in X : (x, y) \in E \}$ is a subset of X.

Suppose $f : X \times Y \to \mathbb{R}, \overline{\mathbb{R}^+}, \overline{\mathbb{R}}$ or $\mathbb{C}$ is a function. Then for each $x \in X$, the *section*, $f_x : Y \to \mathbb{R}, \overline{\mathbb{R}^+}, \overline{\mathbb{R}}$ or $\mathbb{C}$ is given by $f_x(y) = f(x, y)$ for $y \in Y$. For each $y \in Y$, the *section*, $f^y : X \to \mathbb{R}, \overline{\mathbb{R}^+}, \overline{\mathbb{R}}$ or $\mathbb{C}$ is given by $f^y(x) = f(x, y)$ for $x \in X$.

Lemma 6. Suppose $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ are measure spaces.

(a) Suppose $E \subseteq X \times Y$ and $E \in \mathscr{M} * \mathscr{N}$. Then for any $x \in X$, $E_x \in \mathscr{N}$ and for any $y \in Y$, $E^y \in \mathscr{M}$.

(b) Suppose $f : X \times Y \to \mathbb{R}, \overline{\mathbb{R}^+}, \overline{\mathbb{R}}$ or $\mathbb{C}$ is $\mathscr{M} * \mathscr{N}$-measurable on $X \times Y$. Then for any $x \in X$, f_x is $\mathscr{N}$-measurable and for any $y \in Y$, f^y is $\mathscr{M}$-measurable.

Proof.

(a) Take any $x \in X$. Let $\mathscr{F}$ be the collection of all subsets E of $X \times Y$ such that $E_x \in \mathscr{N}$. Then $\mathscr{F}$ contains all the measurable rectangles, i.e., $\mathscr{F} \supseteq \mathscr{R}$. This is because for any $A \in \mathscr{M}$ and any $B \in \mathscr{N}$, $(A \times B)_x$ is either B or the empty set Ø as $(A \times B)_x = B$ if $x \in A$ and $(A \times B)_x = \varnothing$ if $x \notin A$. Moreover, for any $E \subseteq X \times Y$, $\left(E^c\right)_x = \left(E_x\right)^c$. We can deduce this as follows.

$$y \in \left(E^c\right)_x \Leftrightarrow (x, y) \in E^c \Leftrightarrow (x, y) \notin E \Leftrightarrow y \notin E_x \Leftrightarrow y \in \left(E_x\right)^c.$$

It follows that $\left(E^c\right)_x = \left(E_x\right)^c$. For any countable collection, $\{E_n\}$, of subsets of $X \times Y$,

$$y \in \left(\bigcup_{n=1}^{\infty} E_n\right)_x \Leftrightarrow (x, y) \in \bigcup_{n=1}^{\infty} E_n \Leftrightarrow (x, y) \in E_k \text{ for some integer } k \geq 1,$$

$$\Leftrightarrow y \in \left(E_k\right)_x \text{ for some integer } k \geq 1,$$

$$\Leftrightarrow y \in \bigcup_{n=1}^{\infty} \left(E_n\right)_x .$$

Therefore, $\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \bigcup_{n=1}^{\infty} \left(E_n\right)_x$.

Thus, if $E \in \mathscr{F}$, then $E_x \in \mathscr{N}$ so that $\left(E^c\right)_x = \left(E_x\right)^c \in \mathscr{N}$. This means $\mathscr{F}$ is closed under the formation of complement. Moreover, if $\{E_n\}$ is a countable collection of sets in $\mathscr{F}$, then $\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \bigcup_{n=1}^{\infty} \left(E_n\right)_x \in \mathscr{N}$. It follows that $\mathscr{F}$ is closed under the formation of countable union. Hence, $\mathscr{F}$ is a σ-algebra of subsets of $X \times Y$. Note that $\mathscr{M}*\mathscr{N}$ is the smallest σ-algebra generated by $\mathscr{R}$, i.e., $\mathscr{M}*\mathscr{N} = \sigma(\mathscr{R})$. Hence, $\mathscr{F} \supseteq \mathscr{M}*\mathscr{N}$. Therefore, for any $E \in \mathscr{M}*\mathscr{N}$, $E_x \in \mathscr{N}$. We can prove similarly, that for any $y \in Y$ and for any $E \in \mathscr{M}*\mathscr{N}$, $E^y \in \mathscr{M}$.

(b) Suppose $f: X \times Y \to \mathbb{R}, \overline{\mathbb{R}^+}, \overline{\mathbb{R}}$ or $\mathbb{C}$ is a $\mathscr{M}*\mathscr{N}$-measurable function. Then $f_x : Y \to \mathbb{R}, \overline{\mathbb{R}^+}, \overline{\mathbb{R}}$ or $\mathbb{C}$ is given by $f_x(y) = f(x, y)$ for $y \in Y$. For any open set D in $\mathbb{R}, \overline{\mathbb{R}^+}, \overline{\mathbb{R}}$ or $\mathbb{C}$, $f^{-1}(D)$ is $\mathscr{M}*\mathscr{N}$-measurable, i.e., $f^{-1}(D) \in \mathscr{M}*\mathscr{N}$. Now $f^{-1}(D) = \{(x, y) \in X \times Y : f(x, y) \in D\}$ and so for a fixed $y \in Y$,

$$\left(f^{-1}(D)\right)^y = \{x \in X : f(x, y) \in D\} = \{x \in X : f^y(x) \in D\} = \left(f^y\right)^{-1}(D).$$

Thus, since $f^{-1}(D) \in \mathscr{M}*\mathscr{N}$, $\left(f^{-1}(D)\right)^y$ is $\mathscr{M}$-measurable and so $\left(f^y\right)^{-1}(D)$ is $\mathscr{M}$-measurable. It follows that f^y is $\mathscr{M}$-measurable for each y in Y. Similarly, using the fact that $\left(f^{-1}(D)\right)_x = \{y \in Y : f(x, y) \in D\} = \{y \in Y : f_x(y) \in D\} = \left(f_x\right)^{-1}(D)$, we can show that for any $x \in X$, f_x is $\mathscr{N}$-measurable.

Proposition 7. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathscr{M}*\mathscr{N}$, then the function $x \mapsto \nu(E_x)$ is $\mathscr{M}$-measurable and the function $y \mapsto \mu(E^y)$ is $\mathscr{N}$-measurable.

Proof.

We shall prove that the function $x \mapsto \nu(E_x)$ is $\mathscr{M}$-measurable.

We shall consider the case when $\nu(Y) < \infty$ and then extend the proof to the case when Y is σ-finite.

We now assume that $\nu(Y) < \infty$.

Let $\mathscr{F}$ be the collection of all subsets E of $X \times Y$ in $\mathscr{M}*\mathscr{N}$, for which the function $x \mapsto \nu(E_x)$ is $\mathscr{M}$-measurable. By Lemma 6 part (a), $E_x \in \mathscr{N}$ and so $\nu(E_x)$ is defined. Thus, $x \mapsto \nu(E_x)$ is a non-negative function.

If $A \in \mathscr{M}$ and $B \in \mathscr{N}$,

$$\nu\left((A \times B)_x\right) = \nu(B)\chi_A(x),$$

as $(A\times B)_x = B$ if $x\in A$ and $(A\times B)_x = \varnothing$ if $x\notin A$. Since $\chi_A(x)$ is $\mathcal{M}$-measurable, the function $x\mapsto \nu\left(\left(A\times B\right)_x\right) = \nu(B)\chi_A(x)$ is $\mathcal{M}$-measurable. It follows that $A\times B\in\mathcal{F}$. Hence, $X\times Y\in\mathcal{F}$ and $\mathcal{R}\subseteq\mathcal{F}$. Suppose the subsets E and F are measurable rectangles in $\mathcal{R}$ such that $E\subseteq F$. Then

$$(F-E)_x = \left\{y:(x,y)\in F-E\right\} = \left\{y:(x,y)\in F\cap E^c\right\}$$

$$= \left\{y:(x,y)\in F\right\}\cap\left\{y:(x,y)\in E^c\right\} = F_x\cap\left(E^c\right)_x = F_x\cap\left(E_x\right)^c = F_x - E_x .$$

Note that $E_x\subseteq F_x$ and so $\nu\left((F-E)_x\right) = \nu\left(F_x\right)-\nu\left(E_x\right)$. As $x\mapsto\nu(E_x)$ is $\mathcal{M}$-measurable and $x\mapsto\nu(F_x)$ is also $\mathcal{M}$-measurable, it follows that $x\mapsto\nu\left((F-E)_x\right) = \nu\left(F_x\right)-\nu\left(E_x\right)$ is $\mathcal{M}$-measurable. Thus, $F-E\in\mathcal{F}$.

If $\left\{E_n\right\}$ is an increasing sequence of sets in $\mathcal{F}$, then we claim that $\bigcup\limits_{n=1}^{\infty} E_n \in\mathcal{F}$.

We note that $\left(\bigcup\limits_{n=1}^{\infty} E_n\right)_x = \bigcup\limits_{n=1}^{\infty}\left(E_n\right)_x \in\mathcal{N}$. By the continuity from below property of measure,

$$\nu\left(\left(\bigcup_{n=1}^{\infty} E_n\right)_x\right) = \nu\left(\bigcup_{n=1}^{\infty}\left(E_n\right)_x\right) = \lim_{n\to\infty}\nu\left(\left(E_n\right)_x\right).$$

Since each $x\mapsto\nu\left(\left(E_n\right)_x\right)$ is $\mathcal{M}$-measurable and the limit of a sequence of measurable functions is also measurable, $x\mapsto\nu\left(\left(\bigcup\limits_{n=1}^{\infty} E_n\right)_x\right)$ is $\mathcal{M}$-measurable. (See Corollary 14 *of Introduction To Measure Theory*.) Thus, $\bigcup\limits_{n=1}^{\infty} E_n \in\mathcal{F}$. Hence, $\mathcal{F}$ is closed under the formation of proper differences and the formation of unions of increasing sequences of sets. It follows that $\mathcal{F}$ is a *d*-system. Therefore, by Theorem 2, $\mathcal{F}\supseteq\sigma(\mathcal{R}) = \mathcal{M}*\mathcal{N}$. But by definition, $\mathcal{F}\subseteq\mathcal{M}*\mathcal{N}$ and so $\mathcal{F}=\mathcal{M}*\mathcal{N}$. It follows that $x\mapsto\nu(E_x)$ is $\mathcal{M}$-measurable for all E in $\mathcal{M}*\mathcal{N}$.

Now suppose ν is σ-finite. Therefore, there exists a sequence of disjoint sets in $\mathcal{N}$, $\left\{D_n\right\}$, such that $\bigcup\limits_{n=1}^{\infty} D_n = Y$ and that $\nu(D_n)<\infty$ for each integer $n\geq 1$. For each integer $n\geq 1$, define a finite measure, ν_n on $\mathcal{N}$, by $\nu_n(B)=\nu(B\cup D_n)$ for B in $\mathcal{N}$. Note that each ν_n is a positive measure on $\mathcal{N}$. By what we have just proved, for each integer $n\geq 1$, the function $x\mapsto\nu_n(E_x)$ is $\mathcal{M}$-measurable. Since $\nu(E_x)=\sum\limits_{n=1}^{\infty}\nu_n(E_x)$, it follows that $x\mapsto\nu(E_x)$, being the limit of $\mathcal{M}$-measurable functions, is $\mathcal{M}$-measurable.

We can prove in a similar fashion that $y \mapsto \mu(E^y)$ is $\mathscr{N}$-measurable.

Theorem 8. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite measure spaces. Then there is a unique positive measure on the σ-algebra $\mathscr{M} * \mathscr{N}$, denoted by $\mu \times \nu$, such that

$$\mu \times \nu\left(A \times B\right) = \mu(A)\nu(B),$$

for any $A \in \mathscr{M}$ and $B \in \mathscr{N}$. Furthermore, the measure under $\mu \times \nu$ of an arbitrary set E in $\mathscr{M}$ $* \mathscr{N}$ is given by

$$\mu \times \nu(E) = \int_X \nu\left(E_x\right) d\mu(x) = \int_Y \mu(E^y) d\nu(y).$$

The measure $\mu \times \nu$ is called the *product measure* of μ and ν.

Proof.

The measurability of the functions, $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ is given by Proposition 7. Thus, we can now define two functions using these two functions as follows.

We define $\left(\mu \times \nu\right)_1$ and $\left(\mu \times \nu\right)_2$ on $\mathscr{M} * \mathscr{N}$ by

$$\left(\mu \times \nu\right)_1 (E) = \int_Y \mu\left(E^y\right) d\nu(y)$$

and $$\left(\mu \times \nu\right)_2 (E) = \int_X \mu\left(E_x\right) d\mu(x)$$

for $E \in \mathscr{M} * \mathscr{N}$. Plainly, $\left(\mu \times \nu\right)_1 (\varnothing) = \left(\mu \times \nu\right)_2 (\varnothing) = 0$. We shall show next theses two functions are σ-additive on $\mathscr{M} * \mathscr{N}$.

Suppose $\left\{E_n\right\}$ is a sequence of pairwise disjoint sets in $\mathscr{M} * \mathscr{N}$. Take a fixed y in Y. Then $\left\{\left(E_n\right)^y\right\}$ is also a sequence of pairwise disjoint sets in $\mathscr{M}$, as each $\left(E_n\right)^y$ is in $\mathscr{M}$ and members of $\left\{\left(E_n\right)^y\right\}$ are pairwise disjoint. We note that for $i \neq j$, $E_i \cap E_j = \varnothing$ and so as $\left(E_i\right)^y = \left\{x : (x, y) \in E_i\right\}$ and $\left(E_j\right)^y = \left\{x : (x, y) \in E_j\right\}$, we deduce that $\left(E_i\right)^y \cap \left(E_j\right)^y = \varnothing$.

Moreover, $\left(\bigcup_{n=1}^{\infty} E_n\right)^y = \bigcup_{n=1}^{\infty} \left(E_n\right)^y$. Since μ is a positive measure and so is σ-additive on $\mathscr{M}$,

$\mu\left(\left(\bigcup_{n=1}^{\infty} E_n\right)^y\right) = \mu\left(\bigcup_{n=1}^{\infty} (E_n)^y\right) = \sum_{n=1}^{\infty} \mu\left((E_n)^y\right)$. Hence, if we let $E = \bigcup_{n=1}^{\infty} E_n$, then $(\mu\times\nu)_1(E) = \int_Y \mu\left(E^y\right) d\nu(y)$. As each $y \mapsto \mu\left((E_n)^y\right)$ is a non-negative $\mathscr{N}$-measurable function, by Bepo Levi's Theorem or a use of the Monotone Convergence Theorem,

$$(\mu\times\nu)_1(E) = \sum_{n=1}^{\infty} \int_Y \mu\left((E_n)^y\right) d\nu(y) = \sum_{n=1}^{\infty} (\mu\times\nu)_1(E_n).$$

Therefore, $(\mu\times\nu)_1$ is a σ-additive function on $\mathscr{M}*\mathscr{N}$ and so is a positive measure.

Observe that for $A \in \mathscr{M}$ and $B \in \mathscr{N}$, $(A\times B)^y = \begin{cases} A, & y \in B, \\ \varnothing, & y \notin B \end{cases}$. Hence,

$$\mu\left((A\times B)^y\right) = \mu(A)\chi_B(y) .$$

It follows that $(\mu\times\nu)_1(A\times B) = \int_Y \mu\left((A\times B)^y\right) d\nu(y) = \int_Y \mu(A)\chi_B(y) d\nu(y) = \mu(A)\nu(B)$.

We can prove similarly that $(\mu\times\nu)_2$ is σ-additive on $\mathscr{M}*\mathscr{N}$ and so is a positive measure that satisfies the conclusion of the theorem.

Note that the set of measurable rectangles $\mathscr{R}$ in $\mathscr{M}*\mathscr{N}$ is a π-system and $\mathscr{M}*\mathscr{N} = \sigma(\mathscr{R})$. Since $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite measure spaces, we can write $X\times Y$ as a countable union of measurable rectangles on which $(\mu\times\nu)_1$ and $(\mu\times\nu)_2$ agree. Therefore, by Corollary 4, $(\mu\times\nu)_1 = (\mu\times\nu)_2$ on $\mathscr{M}*\mathscr{N}$.

Non-negative Measurable Function

Theorem 9. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite measure spaces. Suppose $f : X\times Y \to [0,\infty] = \overline{\mathbb{R}^+}$ is a $\mathscr{M}*\mathscr{N}$-measurable function. Then

(a) the function $x \mapsto \int_Y f_x(y) d\nu(y)$ is $\mathscr{M}$-measurable and the function $y \mapsto \int_X f^y(x) d\mu(x)$ is $\mathscr{N}$-measurable and f satisfies

(b) $\int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \left(\int_X f^y(x) d\mu(x)\right) d\nu(y) = \int_X \left(\int_Y f_x(y) d\nu(y)\right) d\mu(x)$.

Proof.

By Lemma 6, part (b), f_x is $\mathscr{N}$-measurable and f^y is $\mathscr{M}$-measurable. Hence, $\int_Y f_x(y)d\nu(y)$ is defined for each x in X and $\int_X f^y(x)d\mu(x)$ is defined for each y in Y. Note that the integral may take the value $+\infty$.

Suppose $E \subseteq X \times Y$ is in $\mathscr{M} * \mathscr{N}$ and f is the characteristic function of E, i.e., $f = \chi_E$.

Then $f_x(y) = (\chi_E)_x(y) = \chi_E(x,y) = \chi_{E_x}(y)$ for all y in Y and this shows that $f_x = \chi_{E_x}$. Similarly, we can show that $f^y = \chi_{E^y}$. It follows that $\int_Y f_x(y)d\nu(y) = \int_Y \chi_{E_x}(y)d\nu(y) = \nu(E_x)$. Therefore,

$x \mapsto \int_Y f_x(y)d\nu(y) = \nu(E_x)$ is $\mathscr{M}$-measurable by Proposition 7. Similarly, we can show that $y \mapsto \int_X f^y(x)d\mu(x)$ is $\mathscr{N}$-measurable. Moreover,

$$\int_{X\times Y} f\,d(\mu\times\nu) = \int_{X\times Y} \chi_E\,d(\mu\times\nu) = (\mu\times\nu)(E) = \int_X \nu(E_x)d\mu(x) \text{ by Theorem 8}$$

$$= \int_X \left(\int_Y f_x(y)d\nu(y)\right)d\mu(x).$$

Also, we have, $\int_{X\times Y} f\,d(\mu\times\nu) = (\mu\times\nu)(E) = \int_Y \mu(E^y)d\nu(y) = \int_Y \left(\int_X f^y(x)d\mu(x)\right)d\nu(y)$.

This proves Theorem 9 when f is a characteristic function.

Suppose f is a simple function, i.e., $f = \sum_{i=1}^{n} a_i \chi_{E_i}$, where $E_i \in \mathscr{M} * \mathscr{N}$ and $a_i > 0$ for $1 \le i \le n$.

Then $f_x = \sum_{i=1}^{n} a_i \left(\chi_{E_i}\right)_x$ and $f^y = \sum_{i=1}^{n} a_i \left(\chi_{E_i}\right)^y$. Thus,

$$\int_Y f_x(y)d\nu(y) = \sum_{i=1}^{n} a_i \int_Y \left(\chi_{E_i}\right)_x(y)d\nu(y)$$

and since $x \mapsto \int_Y \left(\chi_{E_i}\right)_x(y)d\nu(y)$ is $\mathscr{M}$-measurable for $1 \le i \le n$, $x \mapsto \int_Y f_x(y)d\nu(y)$ is $\mathscr{M}$-measurable. Moreover,

$$\int_{X\times Y} f\,d(\mu\times\nu) = \sum_{i=1}^{n} \int_{X\times Y} a_i \chi_{E_i} d(\mu\times\nu)$$

$$= \sum_{i=1}^{n} \int_{X\times Y} a_i \chi_{E_i} d(\mu\times\nu) = \sum_{i=1}^{n} \int_X \left(\int_Y a_i \left(\chi_{E_i}\right)_x(y)d\nu(y)\right)d\mu(x)$$

$$= \int_X \left(\sum_{i=1}^{n} \int_Y a_i \left(\chi_{E_i} \right)_x (y) d\nu(y) \right) d\mu(x) = \int_X \left(\int_Y \sum_{i=1}^{n} a_i \left(\chi_{E_i} \right)_x (y) d\nu(y) \right) d\mu(x)$$

$$= \int_X \left(\int_Y f_x(y) d\nu(y) \right) d\mu(x) .$$

In a similar fashion, we can show that $y \mapsto \int_X f^y(x) d\mu(x)$ is $\mathscr{N}$-measurable and that

$$\int_{X\times Y} f \, d\left(\mu \times \nu\right) = \int_Y \left(\int_X f^y(x) d\mu(x) \right) d\nu(y) .$$

Suppose now that $f : X \times Y \to [0, \infty] = \overline{\mathbb{R}^+}$ is a $\mathscr{M} * \mathscr{N}$-measurable function. Then by Theorem 16 of *Introduction To Measure Theory*, there exists an increasing sequence of non-negative measurable simple functions, $\{f_n\}$ converging pointwise to *f*. Note that if f_n are simple functions, then so are $\left(f_n\right)_x$ and $\left(f_n\right)^y$. As $f_n \le f_{n+1}$ for integer $n \ge 1$, $\left(f_n\right)_x \le \left(f_{n+1}\right)_x$ and $\left(f_n\right)^y \le \left(f_{n+1}\right)^y$ for integer $n \ge 1$. Thus, $\left\{\left(f_n\right)_x\right\}$ and $\left\{\left(f_n\right)^y\right\}$ are increasing sequence of measurable functions. We have thus, $f_n \nearrow f$, $\left(f_n\right)_x \nearrow f_x$ and $\left(f_n\right)^y \nearrow f^y$. Therefore, by the Lebesgue Monotone Convergence Theorem, for each *x* in *X*, $\int_Y \left(f_n\right)_x (y) d\nu(y) \nearrow \int_Y f_x(y) d\nu(y)$ and for each *y* in *Y*, $\int_X \left(f_n\right)^y (x) d\mu(x) \nearrow \int_X f^y(x) d\mu(x)$. Now, by what have just proved, each function $x \mapsto \int_Y \left(f_n\right)_x (y) d\nu(y)$ is $\mathscr{M}$-measurable and $y \mapsto \int_X \left(f_n\right)^y (x) d\mu(x)$ is $\mathscr{N}$-measurable. Thus, $g_n : x \mapsto \int_Y \left(f_n\right)_x (y) d\nu(y)$ is an increasing sequence of $\mathscr{M}$-measurable functions converging to $x \mapsto \int_Y f_x(y) d\nu(y)$. Therefore, by the Lebesgue Monotone Convergence Theorem,

$$\int_X g_n(x) d\mu(x) = \int_X \left(\int_Y \left(f_n\right)_x (y) d\nu(y) \right) d\mu(x) \nearrow \int_X \left(\int_Y f_x(y) d\nu(y) \right) d\mu(x) .$$

Similarly, $h_n : y \mapsto \int_X \left(f_n\right)^y (x) d\mu(x)$ is an increasing sequence of $\mathscr{N}$-measurable functions converging to $y \mapsto \int_X f^y(x) d\mu(x)$ and

$$\int_Y h_n(y) d\nu(y) = \int_Y \left(\int_X \left(f_n\right)^y (x) d\mu(x) \right) d\nu(y) \nearrow \int_Y \left(\int_X f^y(x) d\mu(x) \right) d\nu(y) .$$

Since Part (b) is true for simple functions,

$$\int_X \left(\int_Y \left(f_n\right)_x (y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X \left(f_n\right)^y (x) d\mu(x) \right) d\nu(y) = \int_{X\times Y} f_n d\left(\mu \times \nu\right) \nearrow \int_{X\times Y} f \, d\left(\mu \times \nu\right)$$

and so we have that

$$\int_X \left(\int_Y f_x(y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f^y(x) d\mu(x) \right) d\nu(y) = \int_{X\times Y} f \, d\left(\mu \times \nu\right).$$

This completes the proof of Theorem 9.

Fubini's Theorem

Theorem 10. Fubini's Theorem.

Suppose (X, $\mathscr{M}$, μ) and (Y, $\mathscr{N}$, ν) are σ-finite measure spaces. Suppose $f : X \times Y \to [-\infty, \infty] = \overline{\mathbb{R}}$ is a $\mathscr{M}*\mathscr{N}$-measurable function and $\mu \times \nu$-integrable. Then

(a) for μ-almost every x in X, the section f_x is ν-integrable and for ν-almost every y in Y, the section f^y is μ-integrable,

(b) the functions, I_f and J_f defined by,

$$I_f(x) = \begin{cases} \int_Y f_x(y)d\nu(y) \text{ , if } f_x \text{ is } \nu\text{-integrable,} \\ 0, \text{ otherwise} \end{cases}$$

and
$$J_f(y) = \begin{cases} \int_X f^y(x)d\mu(x) \text{ , if } f^y \text{ is } \mu\text{-integrable,} \\ 0, \text{ otherwise} \end{cases}$$

belong to L^1 (X, $\mathscr{M}$, μ, $\mathbb{R}$) and L^1 (Y, $\mathscr{N}$, ν, $\mathbb{R}$) respectively and

(b) $\int_{X\times Y} f \, d(\mu\times\nu) = \int_X I_f d\mu = \int_Y J_f d\nu$.

Proof.

Suppose $f : X \times Y \to [-\infty, \infty] = \overline{\mathbb{R}}$ is a $\mu\times\nu$ integrable function. Let f^+ and f^- be the positive and negative parts of f. Then $f = f^+ - f^-$ and f^+ and f^- are $\mu\times\nu$ integrable and of course $\mathscr{M}*\mathscr{N}$-measurable. By Lemma 6 (b), f_x , $(f^+)_x$ and $(f^-)_x$ are $\mathscr{N}$-measurable and f^y , $(f^+)^y$ and $(f^-)^y$ are $\mathscr{M}$-measurable. Note that $(f_x)^+ = (f^+)_x$, $(f_x)^- = (f^-)_x$, $(f^y)^+ = (f^+)^y$ and $(f^y)^- = (f^-)^y$ for x in X and y in Y. By Theorem 9 (a), the functions, $x \mapsto \int_Y (f^+)_x (y)d\nu(y)$ and $x \mapsto \int_Y (f^-)_x (y)d\nu(y)$ are $\mathscr{M}$-measurable and $y \mapsto \int_X (f^+)^y (x)d\mu(x)$ and $y \mapsto \int_X (f^-)^y (x)d\mu(x)$ are $\mathscr{N}$-measurable. Moreover, by Theorem 9 (b),

$$\int_{X\times Y} f^+ \, d(\mu\times\nu) = \int_Y \left(\int_X (f^+)^y (x)d\mu(x)\right)d\nu(y) = \int_X \left(\int_Y (f^+)_x (y)d\nu(y)\right)d\mu(x) < \infty \text{ and}$$

$$\int_{X\times Y} f^- \, d(\mu\times\nu) = \int_Y \left(\int_X \left(f^-\right)^y (x) d\mu(x)\right) d\nu(y) = \int_X \left(\int_Y \left(f^-\right)_x (y) d\nu(y)\right) d\mu(x) < \infty .$$

It follows that the functions $x \mapsto \int_Y \left(f^+\right)_x (y) d\nu(y)$ and $x \mapsto \int_Y \left(f^-\right)_x (y) d\nu(y)$ are μ-integrable and $y \mapsto \int_X \left(f^+\right)^y (x) d\mu(x)$ and $y \mapsto \int_X \left(f^-\right)^y (x) d\mu(x)$ are ν-integrable.

Hence, $\int_Y \left(f^+\right)_x (y) d\nu(y)$ and $\int_Y \left(f^-\right)_x (y) d\nu(y)$ are finite for μ almost everywhere x,

and $\int_X \left(f^+\right)^y (x) d\mu(x)$ and $\int_X \left(f^-\right)^y (x) d\mu(x)$ are finite for ν almost everywhere y. It follows that $f_x = \left(f_x\right)^+ - \left(f_x\right)^- = \left(f^+\right)_x - \left(f^-\right)_x$ is ν-integrable for μ almost everywhere x and that $f^y = \left(f^y\right)^+ - \left(f^y\right)^- = \left(f^+\right)^y - \left(f^-\right)^y$ is μ-integrable for ν almost everywhere y.

Let N be the subset of X such that $\int_Y \left(f^+\right)_x (y) d\nu(y) = \infty$ or $\int_Y \left(f^-\right)_x (y) d\nu(y) = \infty$. Then N is $\mathscr{M}$-measurable, $\mu(N) = 0$ and for $x \notin N$.

$$\int_Y f_x(y) d\nu(y) = \int_Y \left(f^+\right)_x (y) d\nu(y) - \int_Y \left(f^-\right)_x (y) d\nu(y) < \infty .$$

Define

$$I_f(x) = \begin{cases} \int_Y f_x(y) d\nu(y) \text{ , if } x \in X - N \\ 0, \text{ if } x \in N \end{cases} = \begin{cases} \int_Y f_x(y) d\nu(y) \text{ , if } f_x \text{ is } \nu\text{-integrable,} \\ 0, \text{ otherwise} \end{cases} .$$

Then for all x in X, $I_f(x) \in \mathbb{R}$ and $I_f(x)$ is $\mathscr{M}$-integrable and

$$\int_X I_f(x) d\mu = \int_{X-N} I_f(x) d\mu = \int_{X-N} \left(\int_Y \left(f^+\right)_x (y) d\nu(y)\right) d\mu(x) - \int_{X-N} \left(\int_Y \left(f^-\right)_x (y) d\nu(y)\right) d\mu(x)$$

$$= \int_X \left(\int_Y \left(f^+\right)_x (y) d\nu(y)\right) d\mu(x) - \int_X \left(\int_Y \left(f^-\right)_x (y) d\nu(y)\right) d\mu(x)$$

$$= \int_{X\times Y} f^+ \, d(\mu\times\nu) - \int_{X\times Y} f^- \, d(\mu\times\nu) \text{, by Theorem 9 part (b),}$$

$$= \int_{X\times Y} \left(f^+ - f^-\right) d(\mu\times\nu) = \int_{X\times Y} f \, d(\mu\times\nu) < \infty .$$

Therefore, $I_f \in L^1(X, \mathscr{M}, \mu, \mathbb{R})$.

Similarly, we can show that $J_f \in L^1(Y, \mathscr{N}, \nu, \mathbb{R})$. We elaborate this below.

Let M be the subset of Y such that $\int_X \left(f^+\right)^y (x) d\mu(x) = \infty$ or $\int_X \left(f^-\right)^y (x) d\mu(x) = \infty$. Then M is $\mathscr{N}$-measurable, $\nu(M) = 0$ and for $y \notin M$,

$$\int_X f^y(x)d\mu(x) = \int_X \left(f^+\right)^y(x)d\mu(x) - \int_X \left(f^-\right)^y(x)d\mu(x) < \infty .$$

Define

$$J_f(y) = \begin{cases} \int_X f^y(x)d\mu(x)\ , \text{ if } y \in Y-M \\ 0, \text{ if } y \in M \end{cases} = \begin{cases} \int_X f^y(x)d\mu(x)\ , \text{ if } f^y \text{ is } \mu\text{-integrable,} \\ 0, \text{ otherwise} \end{cases} .$$

Then for all y in Y, $J_f(y) \in \mathbb{R}$ and $J_f(y)$ is $\mathcal{N}$-integrable and

$$\int_Y J_f(y)d\nu = \int_{Y-M} J_f(y)d\nu = \int_{Y-M}\left(\int_X \left(f^+\right)^y(x)d\mu(x)\right)d\nu(y) - \int_{Y-M}\left(\int_X \left(f^-\right)^y(x)d\mu(x)\right)d\nu(y)$$

$$= \int_Y\left(\int_X \left(f^+\right)^y(x)d\mu(x)\right)d\nu(y) - \int_Y\left(\int_X \left(f^-\right)^y(x)d\mu(x)\right)d\nu(y)$$

$$= \int_{X\times Y} f^+\, d\left(\mu\times\nu\right) - \int_{X\times Y} f^-\, d\left(\mu\times\nu\right), \text{ by Theorem 9 part (b),}$$

$$= \int_{X\times Y}\left(f^+ - f^-\right)d\left(\mu\times\nu\right) = \int_{X\times Y} f\, d\left(\mu\times\nu\right) < \infty .$$

Therefore, $J_f \in L^1(Y, \mathcal{N}, \nu, \mathbb{R})$.

Completion of Product Measure

Now we examine the case when all measures involved are to be complete. We shall assume that $(X, \mathcal{M}, \mu)$ is complete with respect to μ and $(Y, \mathcal{N}, \nu)$ is complete with respect to ν. We assume that they are σ-finite measure spaces. Then $(X\times Y, \mathcal{M}*\mathcal{N}, \mu\times\nu)$ is a measure space. Now take $\mathcal{M}\#\mathcal{N}$ to be the completion of $\mathcal{M}*\mathcal{N}$ with respect to the product measure $\mu\times\nu$. We extend the measure $\mu\times\nu$ to $\mathcal{M}\#\mathcal{N}$ in the usual way and denote the extended positive measure on the completion $\mathcal{M}\#\mathcal{N}$ by the same symbol, $\mu\times\nu$.

Lemma 11. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite complete measure spaces. Let $E \subseteq X\times Y$ be a subset in $\mathcal{M}\#\mathcal{N}$, the $\mu\times\nu$-completion of $\mathcal{M}*\mathcal{N}$. Then for μ-almost all x, $E_x \in \mathcal{N}$ and for ν-almost all y, $E^y \in \mathcal{M}$.

Proof.

By Lemma 6 (a), if $E \subseteq X\times Y$ and $E \in \mathcal{M}*\mathcal{N}$. Then for any $x \in X$, $E_x \in \mathcal{N}$ and for any $y \in Y$, $E^y \in \mathcal{M}$.

Suppose now $E \in \mathcal{M}\#\mathcal{N}$. Then there exist $A, B \in \mathcal{M}*\mathcal{N}$ such that $A \subseteq E \subseteq B$ and $\left(\mu\times\nu\right)(B-A) = 0$.

Observe that if $N \in \mathcal{M}\#\mathcal{N}$ and $\mu\times\nu(N)=0$, then there exists $C \in \mathcal{M}*\mathcal{N}$ such that $N \subseteq C$ and $\mu\times\nu(C)=0$. Then by Theorem 8, $\mu\times\nu(C)=\int_X \nu\left(C_x\right)d\mu(x)=\int_Y \mu(C^y)d\nu(y)=0$. Hence, for μ-almost all x, $\nu(C_x)=0$ and for ν-almost all y, $\mu(C^y)=0$.

Therefore, by the above deduction, for μ-almost all x, $\nu((B-A)_x)=\nu(B_x-A_x)=0$ and for ν-almost all y, $\mu((B-A)^y)=\mu(B^y-A^y)=0$. Note that $A_x \subseteq E_x \subseteq B_x$ and $A^y \subseteq E^y \subseteq B^y$. By Lemma 6 (a), A_x and B_x are $\mathcal{N}$-measurable and so for μ-almost all x, $E_x \in \mathcal{N}$ because $\mathcal{N}$ is ν-complete. Similarly, A^y and B^y are $\mathcal{M}$-measurable and so for ν-almost all y, $E^y \in \mathcal{M}$ because $\mathcal{M}$ is μ-complete.

Thus, if $E \in \mathcal{M}\#\mathcal{N}$ and $A, B \in \mathcal{M}*\mathcal{N}$ are such that $A \subseteq E \subseteq B$ and $(\mu\times\nu)(B-A)=0$, then there exists $C \in \mathcal{M}$ such that $\mu(C)=0$ and $E_x \in \mathcal{N}$ for all $x \in X-C$ and there exists $D \in \mathcal{N}$ such that $\nu(D)=0$ and $E^y \in \mathcal{M}$ for all $y \in Y-D$. Moreover, $\mu\times\nu(A)=\mu\times\nu(E)=\mu\times\nu(B)$, for all $x \in X-C$, $\nu(A_x)=\nu(E_x)=\nu(B_x)$ and for all $y \in Y-D$, $\mu(A^y)=\mu(E^y)=\mu(B^y)$.

Note that

$$\begin{aligned}\mu\times\nu(E)&=\mu\times\nu(B)\\ &=\int_X \nu(B_x)d\mu(x)=\int_Y \mu(B^y)d\nu(y)\text{ , by Theorem 8,}\\ &=\int_{X-C} \nu(B_x)d\mu(x)=\int_{Y-D} \mu(B^y)d\nu(y)\\ &=\int_{X-C} \nu(E_x)d\mu(x)=\int_{Y-D} \mu(E^y)d\nu(y).\end{aligned}$$

By Proposition 7, $x \mapsto \nu(B_x)$ is a $\mathcal{M}$-measurable function. As $\nu(E_x)=\nu(B_x)$ for all $x \in X-C$ and $\mu(C)=0$, $x \mapsto \nu^*(E_x)$ defines a $\mathcal{M}$-measurable function on X if we let $\nu^*(E_x)$ to be the outer measure of E_x or $\nu^*(E_x)=\begin{cases}\nu(E_x)=\nu(B_x), x\in X-C\\ 0, x\in C\end{cases}$.

Similarly, we deduce that $y \mapsto \mu^*(E^y)$ defines a $\mathcal{N}$-measurable function on Y if we let $\mu^*(E^y)$ to be the outer measure of E^y or $\mu^*(E^y)=\begin{cases}\mu(E^y)=\mu(B^y), y\in Y-D\\ 0, y\in D\end{cases}$.

If $\mu\times\nu(E)<\infty$, then
$\int_X \nu^*(E_x)d\mu(x)=\int_{X-C} \nu^*(E_x)d\mu(x)=\int_{X-C} \nu^*(B_x)d\mu(x)=\int_X \nu^*(B_x)d\mu(x)=\mu\times\nu(E)<\infty$.

Therefore, $x \mapsto \nu^*(E_x)$ defines a μ-integrable function on *X*. In a similar manner we can show that, $y \mapsto \mu^*(E^y)$ defines a ν-integrable function on *Y*.

We have thus proved the following.

Lemma 12. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite complete measure spaces. Let $E \subseteq X \times Y$ be a subset in $\mathscr{M}\#\mathscr{N}$, the $\mu \times \nu$-completion of $\mathscr{M}*\mathscr{N}$. Suppose $\mu \times \nu(E) < \infty$. Then $x \mapsto \nu^*(E_x)$ defines a μ-integrable function on *X* and $y \mapsto \mu^*(E^y)$ defines a ν-integrable function on *Y*. Furthermore,

$$\mu \times \nu(E) = \int_X \nu^*(E_x) d\mu(x) = \int_Y \mu^*(E^y) d\nu(y) < \infty.$$

Hence, for μ-almost all *x*, $\nu^*(E_x) < \infty$ and for ν-almost all *y*, $\mu^*(E^y) < \infty$.

Lemma 13. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite complete measure spaces. Let $E \subseteq X \times Y$ be a subset in $\mathscr{M}\#\mathscr{N}$. Let χ_E be the characteristic function of *E*. Let

$$\chi_E^*(x,y) = \begin{cases} \chi_E(x,y), x \in X - C \\ 0, x \in C \end{cases} \quad \text{and} \quad \chi_E^*(x,y) = \begin{cases} \chi_E(x,y), y \in Y - D \\ 0, y \in D \end{cases}$$

, where *C* and *D* are as given in the proof of Lemma 11, with $\mu(C) = 0$ and $\nu(D) = 0$. Then

$$\mu \times \nu(E) = \int_{X \times Y} \chi_E d(\mu \times \nu) = \int_X \left(\int_Y \chi_E^*(x,y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X \chi_E^{**}(x,y) d\mu(x) \right) d\nu(y).$$

Proof.

Recall that $\mu \times \nu(E) = \int_X \nu^*(E_x) d\mu(x) = \int_Y \mu^*(E^y) d\nu(y)$. For μ-almost all *x* in *X*, i.e., for *x* in $X - C$, $\nu^*(E_x) = \nu(E_x) = \int_Y \chi_E(x,y) d\nu(y)$. According to Lemma 12, $x \mapsto \nu^*(E_x)$ defines a μ-integrable function on *X* and so $x \mapsto \int_Y \chi_E^*(x,y) d\nu(y)$ is a μ-integrable function on *X*. Likewise, we can show that $y \mapsto \int_X \chi_E^{**}(x,y) d\mu(x)$ is a ν-integrable function on *Y*.

Thus,

$$\mu \times \nu(E) = \int_X \nu^*(E_x) d\mu(x) = \int_{X-C} \nu(E_x) d\mu(x) = \int_{X-C} \left(\int_Y \chi_E(x,y) d\nu(y) \right) d\mu(x)$$

$$= \int_X \left(\int_Y \chi_E^*(x,y) d\nu(y) \right) d\mu(x) \text{ and}$$

$$\mu \times \nu(E) = \int_Y \mu^*(E^y) d\nu(y) = \int_{Y-D} \mu(E^y) d\nu(y) = \int_{Y-D} \left(\int_X \chi_E(x,y) d\mu(x) \right) d\nu(y)$$

$$= \int_Y \left(\int_X \chi_E^{**}(x,y) d\mu(x) \right) d\nu(y).$$

Corollary 14. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite complete measure spaces. Let $E \subseteq X\times Y$ be a subset in $\mathscr{M}\#\mathscr{N}$ and $\mu\times\nu(E) < \infty$. Let χ_E be the characteristic function of E. Then

$$\mu\times\nu(E) = \int_{X\times Y} \chi_E d\left(\mu\times\nu\right) = \int_X \left(\int_Y \chi_E{}^*(x,y)d\nu(y)\right)d\mu(x) = \int_Y \left(\int_X \chi_E{}^{**}(x,y)d\mu(x)\right)d\nu(y),$$

$$\chi_E{}^*(x,y) = \begin{cases} \chi_E(x,y), x\in X-C \\ 0, x\in C \end{cases} \text{ and } \chi_E{}^*(x,y) = \begin{cases} \chi_E(x,y), y\in Y-D \\ 0, y\in D \end{cases}, \text{ where } C \text{ and } D$$

satisfy $\nu(E_x)$ is defined and finite for all $x \in X-C$, $\mu(E^y)$ is defined and finite for all y in $Y-D$, $\mu(C)=0$ and $\nu(D)=0$.

Proof. By Lemma 13, $\mu\times\nu(E) = \int_X \nu^*(E_x)d\mu(x) < \infty$. Therefore, for μ-almost all x in X, $\nu^*(E_x) < \infty$ and so for μ-almost all x in X, $\nu(E_x) < \infty$. Likewise, we deduce that for ν-almost all y in Y, $\mu(E^y) < \infty$. Corollary 14 then follows from Lemma 13.

Remark.

In view of the fact that E_x is $\mathscr{N}$-measurable for μ almost all x and E^y is $\mathscr{M}$-measurable for ν almost all y, we may replace the two functions, $\chi_E{}^*(x,y)$ and $\chi_E{}^{**}(x,y)$ by the following simpler looking definitions, $\chi_E{}^*(x,y) = \begin{cases} \chi_E(x,y), \nu(E_x) < \infty \\ 0, \text{otherwise} \end{cases}$ and

$$\chi_E{}^{**}(x,y) = \begin{cases} \chi_E(x,y), \mu(E^y) < \infty \\ 0, \text{otherwise} \end{cases}.$$

Lemma 15. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite complete measure spaces. Suppose $f : X\times Y \to [0,\infty] = \overline{\mathbb{R}^+}$ is a non-negative, simple $\mathscr{M}\#\mathscr{N}$-measurable function. Suppose f is $\mu\times\nu$-integrable. Then

the functions, I_f and J_f defined by,

$$I_f(x) = \begin{cases} \int_Y f_x(y)d\nu(y), \text{ if } f_x \text{ is } \nu\text{-integrable,} \\ 0, \text{ otherwise} \end{cases}$$

and $$J_f(x) = \begin{cases} \int_X f^y(x)d\mu(x), \text{ if } f^y \text{ is } \mu\text{-integrable,} \\ 0, \text{ otherwise} \end{cases}$$

belong to $L^1(X, \mathscr{M}, \mu, \mathbb{R})$ and $L^1(Y, \mathscr{N}, \nu, \mathbb{R})$ respectively and

$$\int_{X\times Y} f\,d\left(\mu\times\nu\right)=\int_X I_f d\mu=\int_Y J_f d\nu .$$

Proof.

Suppose f is a non-negative simple $\mathscr{M}\#\mathscr{N}$-measurable function, i.e., $f=\sum_{i=1}^{n}a_i\chi_{E_i}$, where $E_i \in \mathscr{M}\#\mathscr{N}$, $a_i>0$ and $\mu\times\nu(E_i)<\infty$ for $1\le i\le n$. Then $f_x=\sum_{i=1}^{n}a_i\left(\chi_{E_i}\right)_x$ and $f^y=\sum_{i=1}^{n}a_i\left(\chi_{E_i}\right)^y$.

By Corollary 14, for each i, there exist $C_i\subseteq X$, $D_i\subseteq Y$ such that $\mu(C_i)=0$, $\nu(D_i)=0$,

$$\mu\times\nu(E_i)=\int_{X\times Y}\chi_{E_i}d\left(\mu\times\nu\right)=\int_X\left(\int_Y\chi_{E_i}*(x,y)d\nu(y)\right)d\mu(x)=\int_Y\left(\int_X\chi_{E_i}**(x,y)d\mu(x)\right)d\nu(y),$$

$$\chi_{E_i}*(x,y)=\begin{cases}\chi_{E_i}(x,y), x\in X-C_i\\ 0, x\in C_i\end{cases} \text{ and } \chi_{E_i}**(x,y)=\begin{cases}\chi_{E_i}(x,y), y\in Y-D_i\\ 0, y\in D_i\end{cases}$$, where

$\nu\left(\left(E_i\right)_x\right)$ is defined and finite for all $x\in X-C_i$ and $\mu\left(\left(E_i\right)^y\right)$ is defined and finite for all $y\in Y-D_i$.

Let $C=\bigcup_{i=1}^{n}C_i$ and $D=\bigcup_{i=1}^{n}D_i$. Then, $\mu(C)=0$ and $\nu(D)=0$. Now let

$$\widetilde{\chi_{E_i}}*(x,y)=\begin{cases}\chi_{E_i}*(x,y), x\in X-C\\ 0, x\in C\end{cases} \text{ and } \widetilde{\chi_{E_i}}**(x,y)=\begin{cases}\chi_{E_i}**(x,y), y\in Y-D\\ 0, y\in D\end{cases} .$$

Note that $x\mapsto\int_Y\widetilde{\chi_{E_i}}*(x,y)d\nu(y)$ is a μ-integrable function for $1\le i\le n$ and

$$\int_{X\times Y}\chi_{E_i}d\left(\mu\times\nu\right)=\int_X\left(\int_Y\widetilde{\chi_{E_i}}*(x,y)d\nu(y)\right)d\mu(x)=\mu\times\nu(E_i)<\infty .$$

Thus, $\int_{X\times Y}f\,d\left(\mu\times\nu\right)=\sum_{i=1}^{n}\int_{X\times Y}a_i\chi_{E_i}d\left(\mu\times\nu\right)=\sum_{i=1}^{n}\int_X\left(\int_Y a_i\chi_{E_i}*(x,y)d\nu(y)\right)d\mu(x)$

$$=\int_X\left(\sum_{i=1}^{n}\int_Y a_i\chi_{E_i}*(x,y)d\nu(y)\right)d\mu(x)=\int_X\left(\int_Y\left(\sum_{i=1}^{n}a_i\chi_{E_i}*(x,y)\right)d\nu(y)\right)d\mu(x)$$

$$=\int_X\left(\int_Y\left(\sum_{i=1}^{n}a_i\widetilde{\chi_{E_i}}*(x,y)\right)d\nu(y)\right)d\mu(x).$$

Observe that for $x\in X-C$, $f_x(y)=\sum_{i=1}^{n}a_i\chi_{E_i}*(x,y)=\sum_{i=1}^{n}a_i\widetilde{\chi_{E_i}}*(x,y)$.

Now define for each x in X, $I_f(x) = \int_Y \left(\sum_{i=1}^{n} a_i \widetilde{\chi_{E_i}}^*(x,y) \right) d\nu(y) < \infty$.

Similarly, we can deduce that for $y \in X - D$, $f^y(x) = \sum_{i=1}^{n} a_i \chi_{E_i}{}^{**}(x,y) = \sum_{i=1}^{n} a_i \widetilde{\chi_{E_i}}^{**}(x,y)$ and that $\int_{X\times Y} f \, d(\mu\times\nu) = \sum_{i=1}^{n} \int_{X\times Y} a_i \chi_{E_i} d(\mu\times\nu) = \sum_{i=1}^{n} \int_Y \left(\int_X a_i \chi_{E_i}{}^{**}(x,y) d\mu(x) \right) d\nu(y)$

$$= \int_Y \left(\int_X \left(\sum_{i=1}^{n} a_i \widetilde{\chi_{E_i}}^{**}(x,y) \right) d\mu(x) \right) d\nu(y) .$$

Define for each y in Y, $J_f(y) = \int_X \left(\sum_{i=1}^{n} a_i \widetilde{\chi_{E_i}}^{**}(x,y) \right) d\mu(x) < \infty$.

We then have, $\int_X I_f(x) d\mu(x) = \int_Y J_f(y) d\nu(y) = \int_{X\times Y} f \, d(\mu\times\nu) < \infty$.

Observe that

$$I_f(x) = \begin{cases} \int_Y f_x(y) d\nu(y), x \in X - C, \\ 0, x \in C \end{cases} = \begin{cases} \int_Y f_x(y) d\nu(y), \text{when } \int_Y f_x(y) d\nu(y) \text{ is defined and finite} \\ 0, \text{ otherwise} \end{cases}$$

for μ-almost all x and that

$$J_f(y) = \begin{cases} \int_X f^y(x) d\mu(x), y \in Y - D, \\ 0, y \in D \end{cases} = \begin{cases} \int_X f^y(x) d\mu(x), \text{when } \int_X f^y(x) d\mu(x) \text{ is defined and finite} \\ 0, \text{ otherwise} \end{cases}$$

for ν-almost all y.

This completes the proof of Lemma 15.

Remark. We have shown that if $f : X\times Y \to [0,\infty] = \overline{\mathbb{R}^+}$ is a non-negative, simple $\mathscr{M}\#\mathscr{N}$-measurable function, then for μ-almost all x, f_x is $\mathscr{N}$-measurable, for ν-almost all y, f^y is $\mathscr{M}$-measurable and if f is $\mu\times\nu$-integrable, then for μ-almost all x, f_x is ν-integrable and for ν-almost all y, f^y is μ-integrable.

Theorem 16. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite measure spaces. Suppose $f : X\times Y \to [0,\infty] = \overline{\mathbb{R}^+}$ is a $\mathscr{M}\#\mathscr{N}$-measurable and $\mu\times\nu$-integrable function. Then

(a) for μ-almost every x in X, the section f_x is ν-integrable and for ν-almost every y in Y, the section f^y is μ-integrable,

(b) $\int_{X\times Y} f \, d(\mu\times\nu) = \int_X I_f(x) d\mu(x) = \int_Y J_f(y) d\nu(y)$, Where,

$$I_f(x)=\begin{cases}\int_Y f_x(y)d\nu(y), \text{when } \int_Y f_x(y)d\nu(y) \text{ is defined and finite}\\ 0, \text{ otherwise}\end{cases}$$

and $J_f(y)=\begin{cases}\int_X f^y(x)d\mu(x), \text{when } \int_X f^y(x)d\mu(x) \text{ is defined and finite}\\ 0, \text{ otherwise}\end{cases}$.

Proof. Since $f: X\times Y\to[0,\infty]=\overline{\mathbb{R}^+}$ is a $\mu\times\nu$-integrable function, there exists an increasing sequence of $\mathscr{M}\#\mathscr{N}$-measurable and $\mu\times\nu$-integrable simple function $\{f_n\}$ such that f_n converges pointwise to *f*.

By Lemma 15, for each integer *n*, $\int_{X\times Y} f_n\, d(\mu\times\nu)=\int_X I_{f_n}(x)d\mu(x)=\int_Y J_{f_n}(y)d\nu(y)$,

where $I_{f_n}(x)=\begin{cases}\int_Y (f_n)_x(y)d\nu(y)<\infty, x\in X-C_n,\\ 0, x\in C_n\end{cases}$,

$J_{f_n}(y)=\begin{cases}\int_X (f_n)^y(x)d\mu(x)<\infty, y\in Y-D_n,\\ 0, y\in D_n\end{cases}$, $\mu(C_n)=0$ and $\nu(D_n)=0$.

Since $\{f_n\}$ is increasing, the sections, $\{(f_n)_x\}$ and $\{(f_n)^y\}$ are increasing sequences of functions. By Lemma 15, $\int_X I_{f_n}(x)d\mu(x)=\int_Y J_{f_n}(y)d\nu(y)=\int_{X\times Y} f_n\, d(\mu\times\nu)<\infty$.

Let $C=\bigcup_{n=1}^{\infty} C_n$ and $D=\bigcup_{n=1}^{\infty} D_n$.

For a fixed *x* in $X-C$, $(f_n)_x(y)\le(f_{n+1})_x(y)$ for all *y* in *Y* and for each integer $n\ge 1$. Moreover $(f_n)_x(y)$ is a ν-integrable function on *Y* for each integer $n\ge 1$. As $(f_n)_x \nearrow f_x$, it follows by the Monotone Convergence Theorem that for each *x* in $X-C$,

$I_{f_n}(x)\nearrow\int_Y f_x(y)d\nu(y)$. Let $\widetilde{I_{f_n}}(x)=\begin{cases}I_{f_n}(x), x\in X-C,\\ 0, x\in C\end{cases}=\begin{cases}\int_Y (f_n)_x(y)d\nu(y)<\infty, x\in X-C,\\ 0, x\in C\end{cases}$

Then, for μ-almost all *x*, $\widetilde{I_{f_n}}(x)=I_{f_n}(x)$ and $\widetilde{I_{f_n}}(x)\nearrow\begin{cases}\int_Y f_x(y)d\nu(y), x\in X-C,\\ 0, x\in C\end{cases}$.

Moreover, for all *x* in *X* and for all integer $n\ge 1$, $\widetilde{I_{f_n}}(x)\le\widetilde{I_{f_{n+1}}}(x)$. Since $I_{f_n}(x)$ is μ-integrable, $\widetilde{I_{f_n}}(x)$ is also μ-integrable. Therefore, $\{\widetilde{I_{f_n}}(x)\}$ is an increasing sequence of integrable functions. Note that

$$\int_X \widetilde{I_{f_n}}(x)d\mu(x) = \int_X I_{f_n}(x)d\mu(x) = \int_{X\times Y} f_n \, d(\mu\times\nu) \le \int_{X\times Y} f \; d(\mu\times\nu) < \infty .$$

Therefore, $g_n(x) = \widetilde{I_{f_n}}(x)$ converges to a μ-integrable function g and

$$\int_X g_n(x)d\mu(x) = \int_X \widetilde{I_{f_n}}(x)d\mu(x) \nearrow \int_X g(x)d\mu(x) .$$

But $\int_X g_n(x)d\mu(x) = \int_X \widetilde{I_{f_n}}(x)d\mu(x) = \int_{X\times Y} f_n \, d(\mu\times\nu) \nearrow \int_{X\times Y} f \, d(\mu\times\nu) < \infty$ and so

$\int_X g(x)d\mu(x) = \int_{X\times Y} f \, d(\mu\times\nu)$. Observe that $g(x) = \begin{cases} \int_Y f_x(y)d\nu(y), x\in X-C, \\ 0, x\in C \end{cases}$. Thus,

$\int_{X-C}\left(\int_Y f_x(y)d\nu(y)\right)d\mu = \int_{X-C} g(x)d\mu(x) = \int_X g(x)d\mu(x) < \infty$. It follows that for μ-almost all x, $\int_Y f_x(y)d\nu(y) < \infty$. This means that for μ-almost all x, f_x is ν-integrable.

This means that there exists a $\mathscr{M}$-measurable set $A \subseteq X$, with $\mu(A) = 0$ such that f_x is ν-integrable for all x in $X - A$ and $g_n(x) \nearrow \int_Y f_x(y)d\nu(y)$. Thus, we may take

$$g(x) = \begin{cases} \int_Y f_x(y)d\nu(y), \text{when } f_x \text{ is } \nu\text{-integrable} \\ 0, \text{ otherwise} \end{cases}$$ and we have

$$\int_{X\times Y} f \, d(\mu\times\nu) = \int_X g(x)d\mu(x) = \int_X I_f(x)d\mu(x).$$

Similarly, we can show that $J_{f_n}(y) \nearrow \int_X f^y(x)d\mu(x)$ and if we let

$$\widetilde{J_{f_n}}(y) = \begin{cases} J_{f_n}(y), y\in Y-D, \\ 0, y\in D \end{cases} = \begin{cases} \int_X (f_n)^y(x)d\mu(x) < \infty, y\in Y-D, \\ 0, y\in D \end{cases}$$

we see that $\left\{\widetilde{J_{f_n}}(y)\right\}$ is an increasing sequence of integrable functions such that

$\widetilde{J_{f_n}}(y) \nearrow \begin{cases} \int_X f^y(x)d\mu(x), y\in Y-D, \\ 0, y\in D \end{cases}$ and $h_n(y) = \widetilde{J_{f_n}}(y)$ converges to a ν-integrable function h. Furthermore, $\int_Y h_n(y)d\nu(y) = \int_Y \widetilde{J_{f_n}}(y)d\nu(y) \nearrow \int_Y h(y)d\nu(y)$ so that as

$\int_Y h_n(y)d\nu(y) = \int_Y \widetilde{J_{f_n}}(y)d\nu(y) = \int_{X\times Y} f_n \, d(\mu\times\nu) \nearrow \int_{X\times Y} f \, d(\mu\times\nu) < \infty$, we have that

$\int_Y h(y)d\nu(y) = \int_{X\times Y} f \, d(\mu\times\nu) < \infty$. Thus, since $h(y) = \begin{cases} \int_X f^y(x)d\mu(x), y\in Y-D, \\ 0, x\in D \end{cases}$,

$\int_{Y-D}\left(\int_X f^y(x)d\mu(x)\right)d\nu(y) = \int_{Y-D} h(y)d\nu(y) = \int_Y h(y)d\nu(y) < \infty$. It follows that for ν-almost all y, f^y is μ-integrable. Thus, we may take

$$h(y) = \begin{cases} \int_X f^y(x)d\mu(x), \text{when } f^y \text{ is } \mu\text{-integrable} \\ 0, \text{ otherwise} \end{cases}$$

and consequently, $\int_{X\times Y} f\, d(\mu\times\nu) = \int_Y h(y)d\nu(y) = \int_Y J_f(y)d\nu(y)$.

Theorem 17. Suppose $(X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite measure spaces. Suppose $f: X\times Y \to \mathbb{R}$ is a $\mathscr{M}\#\mathscr{N}$-measurable and $\mu\times\nu$-integrable function. Then

(a) for μ-almost every x in X, the section f_x is ν-integrable and for ν-almost every y in Y, the section f^y is μ-integrable,

(b) $\int_{X\times Y} f\, d(\mu\times\nu) = \int_X I_f(x)d\mu(x) = \int_Y J_f(y)d\nu(y)$,

Where,

$$I_f(x) = \begin{cases} \int_Y f_x(y)d\nu(y), \text{when } \int_Y f_x(y)d\nu(y) \text{ is defined and finite} \\ 0, \text{ otherwise} \end{cases}$$

and $J_f(y) = \begin{cases} \int_X f^y(x)d\mu(x), \text{when } \int_X f^y(x)d\mu(x) \text{ is defined and finite} \\ 0, \text{ otherwise} \end{cases}$.

Proof.

$f: X\times Y \to \mathbb{R}$ is $\mu\times\nu$-integrable if and only if both f^+ and f^- are $\mu\times\nu$-integrable. Note that both f^+ and f^- are $\mathscr{M}\#\mathscr{N}$-measurable. Moreover,

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_{X\times Y} f^+\, d(\mu\times\nu) - \int_{X\times Y} f^-\, d(\mu\times\nu).$$

By Theorem 16, for μ almost everywhere x, $(f_x)^+$ and $(f_x)^-$ are ν-integrable and for ν almost everywhere y, $(f^y)^+$ and $(f^y)^-$ are μ-integrable. It follows that for μ almost every x, $f_x = (f_x)^+ - (f_x)^- = (f^+)_x - (f^-)_x$ is ν-integrable and that for ν almost everywhere y, $f^y = (f^y)^+ - (f^y)^- = (f^+)^y - (f^-)^y$ is μ-integrable.

By Theorem 16,

$$\int_{X\times Y} f^+\, d(\mu\times\nu) = \int_X I_{f^+}(x)d\mu(x) = \int_Y J_{f^+}(y)d\nu(y)$$

and $\int_{X\times Y} f^- \, d(\mu\times\nu) = \int_X I_{f^-}(x)d\mu(x) = \int_Y J_{f^-}(y)d\nu(y)$.

Therefore,

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_{X\times Y} f^+ \, d(\mu\times\nu) - \int_{X\times Y} f^- \, d(\mu\times\nu) = \int_X I_{f^+}(x)d\mu(x) - \int_X I_{f^-}(x)d\mu(x)$$

$$= \int_X \left(I_{f^+}(x) - I_{f^-}(x)\right) d\mu(x).$$

Note that there exists subsets of X, A and B in $\mathscr{M}$ with $\mu(A)=\mu(B) = 0$, such that

$$I_{f^+}(x) = \begin{cases} \int_Y \left(f^+\right)_x (y)d\nu(y) < \infty, x \in X - A \\ 0, x \in A \end{cases} \text{ and } I_{f^-}(x) = \begin{cases} \int_Y \left(f^-\right)_x (y)d\nu(y) < \infty, x \in X - B \\ 0, x \in B \end{cases}.$$

Let $C = A \cup B$. Then $\mu(C) = \mu(A \cup B) = 0$. Define

$$I_f(x) = \begin{cases} I_{f^+}(x) - I_{f^-}(x) < \infty, x \in X - C \\ 0, x \in C \end{cases} = \begin{cases} \int_Y \left(f^+\right)_x (y)d\nu(y) - \int_Y \left(f^-\right)_x (y)d\nu(y), x \in X - C \\ 0, x \in C \end{cases}$$

$$= \begin{cases} \int_Y f_x(y)d\nu(y), x \in X - C \\ 0, x \in C \end{cases}.$$

Then $\int_X I_f(x)d\mu(x) = \int_X \left(I_{f^+}(x) - I_{f^-}(x)\right) d\mu(x) = \int_X I_{f^+}(x)d\mu(x) - \int_X I_{f^-}(x)d\mu(x)$

$$= \int_{X\times Y} f^+ \, d(\mu\times\nu) - \int_{X\times Y} f^- \, d(\mu\times\nu) = \int_{X\times Y} f\, d(\mu\times\nu) d\mu(x) < \infty .$$

Since $\mu(C) = 0,\ I_f(x) = \begin{cases} \int_Y f_x(y)d\nu(y), \text{when } \int_Y f_x(y)d\nu(y) \text{ is defined and finite} \\ 0, \text{ otherwise} \end{cases}$ for μ almost everywhere x.

Similarly,

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_{X\times Y} f^+ \, d(\mu\times\nu) - \int_{X\times Y} f^- \, d(\mu\times\nu) = \int_X J_{f^+}(y)d\nu(y) - \int_X J_{f^-}(y)d\nu(y)$$

$$= \int_X \left(J_{f^+}(y) - J_{f^-}(y)\right) d\nu(y).$$

By Theorem 16, there exist subsets of Y, E and F in $\mathscr{N}$ with $\nu(E) = \nu(F) = 0$, such that

$$J_{f^+}(y) = \begin{cases} \int_X \left(f^+\right)^y (x)d\mu(x) < \infty, y \in Y - E \\ 0, y \in E \end{cases} \text{ and } J_{f^-}(y) = \begin{cases} \int_X \left(f^-\right)^y (x)d\mu(x) < \infty, y \in Y - F \\ 0, y \in F \end{cases}.$$

Define

$$J_f(y)=\begin{cases} J_{f^+}(y)-J_{f^-}(y)<\infty, y\in Y-D \\ 0, y\in D \end{cases} =\begin{cases} \int_X \left(f^+\right)^y(x)d\mu(x)-\int_X \left(f^-\right)^y(x)d\mu(x), y\in Y-D \\ 0, y\in D \end{cases}$$

$$=\begin{cases} \int_X f^y(x)d\mu(x), y\in Y-D \\ 0, y\in D \end{cases},$$

where $D=E\cup F$. Observe that $\nu(D)=0$ and

$$J_f(y)=\begin{cases} \int_X f^y(x)d\mu(x), \text{when } \int_X f^y(x)d\mu(x) \text{ is defined and finite} \\ 0, \text{ otherwise} \end{cases}.$$

Then

$$\int_Y J_f(y)d\nu(y)=\int_Y \left(J_{f^+}(y)-J_{f^-}(y)\right)d\nu(y)=\int_Y J_{f^+}(y)d\nu(y)-\int_Y J_{f^-}(y)d\nu(y)$$

$$=\int_{X\times Y} f^+\, d\left(\mu\times\nu\right)-\int_{X\times Y} f^-\, d\left(\mu\times\nu\right)=\int_{X\times Y} f\, d\left(\mu\times\nu\right)d\mu(x)<\infty .$$

This completes the proof of Theorem 17.