Sequences and Series.

Part I Sequences.

Definition 1. Let P be the set of positive integers. A *sequence* is simply a function from P into the set of real numbers **R**.
P is of course the set {1,2, … }. Thus a function a: P $\to$ **R** is a sequence.

The image $a(n)$ is called the n-th term of the sequence and is also written as a_n ,
We also write $(a_1 , a_2, \dots)$ or simply (a_n) for the sequence.
Here we use the round bracket for sequences. One should not confused the sequence $(a_1 , a_2, \dots)$ with a row vector.

We are interested in the behaviour of the values or points of the sequences. We want to know if they are bunched together like a cluster or they become further and further apart or oscillatory. We focus on whether the points are bunched together or not. We have a technical term of this bunching together.

Definition 2. Let (a_n) be a sequence in **R**. We say (a_n) tends to a real number a in **R** if for any $\varepsilon > 0$, there exists a positive integer N_0 such that for all n in P with $n \ge N_0$, $|a_n - a| < \varepsilon$.
That is,

$$n \ge N_0 \Rightarrow |a_n - a| < \varepsilon .$$

Notation:
If (a_n) tends to a, we write

$$a_n \to a \text{ as } n \to \infty$$

or

$$\lim_{n\to\infty} a_n = a$$

or just simply, $a_n \to a$.

Definition 3. We say (a_n) *converges* if there exists a real number a such that $a_n \to a$, otherwise (a_n) diverges or is divergent.

Example 4.

1. $a_n = c$ for all n in P. This is a constant sequence obviously $a_n \to c$.
   Given any $\varepsilon > 0$, take any positive integer N obviously for any $n \ge N$
   $|a_n - c| = |c - c| = 0 < \varepsilon$.

2. $a_n = (-1)^n$. Then (a_n) is divergent. There is a quick way to see this. Observe that the value changes from 1 to -1 and so there is no way it can get close to any value.
If you like the following is a proof of this fact.
For any a in **R**, by the triangle inequality,

$$|1-a| + |(-1)-a| \ge |1-a - ((-1)-a)| = 2$$

Hence, either $|1-a| \ge 1$ or $|(-1)-a| \ge 1$.
Take any positive integer N_0 . If $|1-a| \ge 1$, then take any even $n > N_0$ and we have

$|a_n - a| = |1 - a| \geq 1$ and if $|(-1) - a| \geq 1$, then take any odd $n > N_0$ and we have $|a_n - a| = |(-1) - a| \geq 1$.
Thus (a_n) cannot converge to any a and so is divergent.

3. $a_n = 1/n$. Then $a_n \to 0$.
For any $\varepsilon > 0$, there exists a positive integer N_0 such that $0 < \frac{1}{N_0} < \varepsilon$ (by the archimedean property of **R**). Thus for $n \geq N_0$, $\frac{1}{n} \leq \frac{1}{N_0} < \varepsilon$ and this means $|\frac{1}{n} - 0| < \varepsilon$ and so by definition $a_n \to 0$.

We have already come across the notion of continuity and limit of a function, we shall use this notion to derive the properties of the sequence.

Let P^{-1} denotes the set $\{1/n ; n \in P\}$. That is $P^{-1} = \{1, ½, 1/3, ... \}$.
Then let $K = P^{-1} \cup \{0\} = \{0, 1, ½, 1/3, ...\}$.

Here is an easy result:

Proposition 4. Let (a_n) be a sequence in **R**. Define a function,

$$f : K = P^{-1} \cup \{0\} \to \mathbf{R},$$

by $f(1/n) = a_n$ for $n > 0$ and $f(0) = a$.
Then $a_n \to a$ if and only if f is continuous at 0.

We shall omit the proof. It is sufficient to say that this is just a restatement of the convergence of the sequence to a limit form for function.

Example 5.

1. $\frac{1}{n^k} \to 0$ as $n \to \infty$ for all $k > 0$.
   Consider $f : K = P^{-1} \cup \{0\} \to \mathbf{R}$, then
   $f(1/n) = a_n = \frac{1}{n^k} = \left(\frac{1}{n}\right)^k$ and $f(0) = 0$.
   Thus the function is given by $f(x) = x^k$ for $x \geq 0$..
   This function is continuous at $x = 0$ by showing that its limit at 0 is 0.
   $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^k = \lim_{x \to 0^+} e^{k \ln(x)} = \lim_{x \to 0^+} \frac{1}{e^{-k \ln(x)}} = 0,$$
   since $\lim_{x \to 0^+} e^{-k \ln(x)} = \infty$ as $\lim_{x \to 0^+} -k \ln(x) = \infty$.

2. Let $a_n = \frac{27n^2 + 3n - 1}{15n^2 - 2n - 13}$.
   Then $f : K = P^{-1} \cup \{0\} \to \mathbf{R}$ is given by $f(1/n) = a_n = \frac{27n^2 + 3n - 1}{15n^2 - 2n - 13} = \frac{27 + \frac{3}{n} - \frac{1}{n^2}}{15 - \frac{2}{n} - \frac{13}{n^2}}$.
   Thus, $f(x) = \frac{27 + 3x - x^2}{15 - 2x - 13x^2}$ This function is continuous at 0 and $f(0) = \frac{27}{15} = \frac{9}{5}$.
   Therefore, $a_n \to \frac{9}{5}$.

Below we list the properties for sequences, some of which are easy consequences of continuity via Proposition 4.

Properties 8.

1. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

2. If $a_n \to a$, then $\lambda a_n \to \lambda a$ for any real number λ.

3. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

4. If $a_n \to a$ and $a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$.

5. If $a_n \to a$ and $b_n \to b$ with $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$.

6. Comparison Test
If there exists a sequence (b_n) such that
(1) $b_n \to 0$ and
(2) $|a_n - a| \le |b_n|$,
then $a_n \to a$.

Proof. Given $\varepsilon > 0$, by (1), there exists an integer N such that $n \ge N \Rightarrow |b_n| < \varepsilon$. Therefore, for all $n \ge N$, $|a_n - a| \le |b_n| < \varepsilon$. This means $a_n \to a$.

Example.
If $|a| < 1$, then the sequence (a^n) converges to 0.

Since $|a| < 1$, $1/|a| > 1$. Then we can write $1/|a| = 1 + \beta$ and $\beta > 0$.
Hence $|a^n - 0| = \frac{1}{(1+\beta)^n} < \frac{1}{n\beta}$.
The last inequality is because $(1+\beta)^n \ge 1 + n\beta > n\beta$ for positive integer n.
Since $\frac{1}{n} \to 0$, $\frac{1}{n\beta} \to 0$. Thus by the Comparison Test $a^n \to 0$.

7. If (a_n) converges, then (a_n) is bounded.
Proof.
(a_n) converges means there exists an a such that $a_n \to a$. Thus by the definition of convergence, taking $\varepsilon = 1$, there exists an integer N such that

$$n \ge N \Rightarrow |a_n - a| < \varepsilon = 1.$$

Hence, $n \ge N \Rightarrow |a_n| < |a| + 1$.
Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a|+1\}$. Obviously, $|a_n| \le M$ for all positive integer n. This means (a_n) is bounded.

8. If $a_n \to a$ and $b_n \to b$ and there exists an integer N such that $a_n \le b_n$ for all $n \ge N$, then $a \le b$.

9. Squeeze Theorem.
If $a_n \to a$ and $b_n \to a$ and there exists an integer N such that for all $n \ge N$, $a_n \le c_n \le b_n$, then $c_n \to a$.

Definition 10. A real sequence (a_n) is *increasing* if $n > m \Rightarrow a_n \geq a_m$.
It is *decreasing* if $n > m \Rightarrow a_n \leq a_m$.
It is *strictly increasing* if $n > m \Rightarrow a_n > a_m$.
It is *strictly decreasing* if $n > m \Rightarrow a_n < a_m$.
It is a *monotone sequence* if it is either increasing or decreasing.

Proposition 11. Suppose (a_n) is a real bounded monotone sequence. Then (a_n) is convergent.

Proof is omitted. Actually the statement is equivalent to the completeness of **R**.

Example. $(1 - \frac{1}{n})$ is a bounded increasing sequence and so is convergent

Another equivalent statement involves the notion of a Cauchy sequence. This expresses that when a sequence is somehow "bunched" together then it must be convergent.

Definition 12. (a_n) is a Cauchy sequence, if and only if, given any $\varepsilon > 0$, there exists an integer N such that for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$.

An easy consequence of the definition is

Any Cauchy sequence is bounded.

Theorem 13. Cauchy Principle of Convergence.
A sequence (a_n) is convergent if and only if it is Cauchy.

This is the most important theorem. The property that every Cauchy sequence is convergent is equivalent to (order) completeness of **R**. This gives a charaterization of completeness for **R** and also for $\mathbf{R}^n$.

Definition 14. The notion of (a_n) tending to $+\infty$ means
$\lim_{n\to\infty} a(n) = \infty$ regarding the limit as a limit of a function on P.
Similarly, limit of (a_n) tending to $-\infty$ means $\lim_{n\to\infty} a(n) = -\infty$.

The rules for functions translate to the following:

Useful results for computing limits.

Suppose (a_n), (b_n) are two sequences.
1. If $a_n \to +\infty$ or $a_n \to -\infty$, then $\frac{1}{a_n} \to 0$.
2. If $a_n \to +\infty$ *and* $b_n \to a$, a finite, then $a_n + b_n \to +\infty$
3. If $a_n \to -\infty$ *and* $b_n \to a$, a finite, then $a_n + b_n \to -\infty$
4. If $a_n \to +\infty$ *and* $b_n \to a > 0$ a finite, then $a_n b_n \to +\infty$
5. If $a_n \to +\infty$ *and* $b_n \to a < 0$ a finite, then $a_n b_n \to -\infty$

These rules are particular useful when a_n is a rational function of n.

Example

1. $(n + 1/n)$ tends to $+\infty$.
2. $5 - n + 1/2^n$ tends to $-\infty$.
3. $\dfrac{n+1}{n^2+1} = \dfrac{\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^2}} \to 0$.
4. $\dfrac{n^2+1}{2n^2+n+1} = \dfrac{1+\frac{1}{n^2}}{2+\frac{1}{n}+\frac{1}{n^2}} \to \dfrac{1}{2}$.

Part II Series.

Definition 1. Suppose (a_n) is a sequence.
We can form the series

$$a_1 + a_2 + a_3 + \ \ldots$$

More specifically, an (infinite) series consists of
(1) a sequence (a_n)
(2) the sequence (s_n) of partial sums, where $s_n = \sum_{k=1}^{n} a_k$
a_n is called the n-th term of the series and s_n the n-th partial sum of the series.

If (s_n) converges to a real number S, then we say the series converges to S and we write

$$\sum a_n = S \text{ or } \sum_{n=1}^{\infty} a_n = S \text{ or } a_1 + a_2 + \cdots = S .$$

We usually write $\sum a_n$ or $a_1 + a_2 + a_3 + \ \ldots$ for the series.

Example 2. The series $c + c + c + \ \ldots$ converges, if and only if, $c = 0$.

Example 3. The series $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$.
Here $a_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. Thus the n-th partial sum,

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \ldots + a_n \\ &= (\tfrac{1}{1} - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \cdots + (\tfrac{1}{n} - \tfrac{1}{n+1}) = 1 - \tfrac{1}{n+1} \to 1 - 0 = 1. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$.

Example 4. Geometric Series.

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \cdots$$

converges to $\dfrac{1}{1-a}$ if $|a| < 1$.

We begin by letting $c_n = a^n$ and $s_n = c_0 + c_1 + \ldots + c_n$.
Then $s_n = 1 + a + a^2 + \cdots + a^n = \dfrac{(1+a+a^2+\cdots+a^n)(1-a)}{1-a}$ if $a \neq 1$

$$= \frac{1-a^{n+1}}{1-a} = 1 - \frac{a^{n+1}}{1-a} \to \frac{1}{1-a} - 0 = \frac{1}{1-a} \text{ if } |a| < 1.$$

If $|a| > 1$, then s_n will be unbounded and so is divergent.

if $a = 1$, then s_n will be unbounded and so is divergent.

If $a = -1$, then $s_n = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{even} \end{cases}$ and so (s_n) is divergent.

Properties for sequences can now be translated into properties for series .

Properties 7.

(1) If $\sum a_n$ converges then its sum is unique.

(2) If $\sum a_n = a$ and $\sum b_n = b$, then $\sum (a_n + b_n) = a + b$.

(3) If $\sum a_n = a$, then $\sum \lambda a_n = \lambda a$.

Definition 8. $\sum a_n$ is a Cauchy series if the partial sum (s_n) is a Cauchy sequence.
I.e., if given $\varepsilon > 0$, there exists an integer N such that

$$m > n \geq N \Rightarrow |s_n - s_m| < \varepsilon \Rightarrow \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon$$

This is equivalent to saying that there exists an integer N such that for all $n \geq N$ and for all positive integer p, $\left| \sum_{n+1}^{n+p} a_k \right| < \varepsilon$.

Then we have the principle convergence for series.

Theorem 9. $\sum a_n$ is convergent if and only if $\sum a_n$ is Cauchy.

The next theorem is a quick way of telling if certain series is divergent.

Proposition 10. If $\sum a_n$ converges, then $a_n \to 0$.

Proof. If $\sum a_n$ converges, then $\sum a_n$ is Cauchy. Then definition 8, with p =1, shows that $a_n \to 0$.

Exanple

$\sum a^n$ is divergent if $|a| \geq 1$ since (a^n) does not converge to 0.

Converse of Proposition 10 is false.
Counter Example

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent as the observation about its n-th partial sums will reveal

$s_1 = 1$, $s_2 = 1 + 1/2$, $s_4 = 1 + 1/2 + (1/3+1/4) > 1 + 1/2 + 1/2$

$s_8 = 1 + 1/2 + (1/3+1/4) + (1/2+1/6+1/7+1/8) > 1 + 1/2 + 1/2 + 1/2$

....

and so (s_n) is unbounded and so is divergent.

Now we have some nice result, a consequence of the monotone convergence theorem.

Proposition 11. Suppose $\sum a_n$ is a series of real non-negative terms. Then $\sum a_n$ is convergent, if and only if, (s_n) is bounded.

Proposition 12 (Comparison Test)
Let $\sum a_n$ and $\sum b_n$ be two series of real non-negative terms such that

$$a_n \le b_n.$$

Then (1) $\sum a_n$ converges if $\sum b_n$ is convergent
(2) $\sum b_n$ diverges if $\sum a_n$ is divergent.

Example 13. $\sum \frac{1}{n^2}$ is convergent.
Since $\frac{1}{(n+1)^2} \le \frac{1}{n(n+1)}$ and $\sum \frac{1}{n(n+1)}$ is convergent, so by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent . Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent.

Proposition 14. Suppose $\sum |a_n|$ is convergent. Then $\sum a_n$ converges.

Proof is just simply observing that if $\sum |a_n|$ is Cauchy, then so is $\sum a_n$.
This follows from the following inequality and Definition 8 :

$$\left| \sum_{n+1}^{n+p} a_k \right| \le \sum_{n+1}^{n+p} |a_k| < \varepsilon.$$

The converse is not true.

Definition 15. We say the series $\sum a_n$ converges absolutely if $\sum |a_n|$ is convergent

Example 16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but not absolutely. (It converges by Alternating series test.)

Proposition 17. Suppose (a_n) is a bounded sequence. Then $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges.

Example 18. $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ is absolutely convergent for any x.

Proposition 20 (Alternating Series Test, Leibnitz's Test)
If (a_n) is a monotone decreasing, non-negative sequence and $a_n \to 0$, then $\sum (-1)^{n+1} a_n$ is convergent.

The proof consists in showing that $\sum (-1)^{n+1} a_n$ is Cauchy and is omitted. There is also a proof making use of the fact that $s_{2n} = s_{2n-1} - a_{2n}$, both (s_{2n}) and (s_{2n-1}) are bounded and monotone and so are convergent and that $a_{2n} \to 0$.

Theorem 21 (Ratio Test, D'Alembert's Test)
Suppose the series $\sum a_n$ is such that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists and is equal to α.
Then we have:
(i) $\alpha < 1$ implies that $\sum a_n$ is absolutely convergent (hence convergent).
(ii) $\alpha > 1$ implies that $\sum a_n$ is divergent.
(iii) If $\alpha = 1$, then $\sum a_n$ may converge or diverge. No inference can be made. The convergence may be investigated by other methods.

Proof is omitted.

Example $\sum 1/n$ is divergent and $\sum 1/n^2$ is convergent. Ratio test for both gives α as 1.

Example 22.

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent as $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \to 0 < 1$.

2. $\sum_{n=1}^{\infty} n^2 x^n$ for $x > 0$. Let $a_n = n^2 x^n$. Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2 x^{n+1}}{n^2 x^n} = x$.
   Thu s $\sum_{n=1}^{\infty} n^2 x^n$ is convergent for $0 < x < 1$. It is divergent for $x > 1$. For $x = 1$ it is divergent.
3. $\sum_{n=1}^{\infty} \frac{1}{n^s}$ $(s > 0)$.
   This series converges if $s > 1$, diverges if $s \le 1$.

Example 23. $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is convergent .
This is because for $n > 0$, $e^n > 1 + n + n^2/2 + n^3/6 > n^3/6$ and so
$\frac{n}{e^n} < \frac{6}{n^2}$.
Therefore by the Comparison Test $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is convergent because $\sum_{n=1}^{\infty} \frac{6}{n^2}$ is convergent by (3) above.