When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous? The answer and application to generalized change of variable for Lebesgue integral.

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This is an intriguing question. Besides checking if the variation of a function is actually bounded above and the condition of absolute continuity is satisfied, we may use a useful but little known criterion for deciding when a function is of bounded variation and absolutely continuous, given by Saks in his monograph "*Theory of The Integral*". We state the result as Theorem 1 below.

Theorem 1. Suppose $f: [a, b] \to \mathbf{R}$ is a continuous function. Suppose E is a measurable subset of $[a, b]$ such that at each point x outside of E, f is differentiable, i.e., $f'(x)$ exists finitely and that the Lebesgue measure of $f(E)$, $m(f(E))$, is zero. Suppose further there exists a Lebesgue integrable function $g : [a, b] \to \mathbf{R}$ such that

$$f'(x) \leq g(x)$$

for $x \in [a, b] - E$. Then f is of bounded variation and absolutely continuous.

Remark. If f is absolutely continuous, then f is of bounded variation. The condition given in Theorem 1 is sufficient to prove both bounded variation and absolute continuity. On the other hand, if f is absolutely continuous, then the condition in Theorem 1 is fulfilled with g taken to be f' and E the complement of the set on which f is differentiable.

An immediate consequence is the following.

Corollary 2. Suppose $f: [a, b] \to \mathbf{R}$ is a continuous function, differentiable everywhere except perhaps on a subset E of $[a, b]$, which is at most denumerable. If f' is Lebesgue integrable or summable, then f is absolutely continuous.

Proof. Note that trivially $m(f(E)) = 0$. Let g be f'. Then by Theorem 1, f is absolutely continuous.

Remark. Corollary 2 can be deduced from Theorem 7 of "*Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*".

Next we present a simple technical Lemma, a weaker form of a similar result in Saks monograph (Theorem 6.6 Chapter 9) restated by F. S. Cater replacing the Banach's condition (T_2) by a stronger condition, which implies that the function is an N function or a function satisfying Lusin's condition. A function is a N function, if it maps sets of measure zero to sets of measure zero. Banach has proved that any continuous N function necessarily satisfies Banach condition (T_2). A function f is said to have Banach's condition (T_2) if each value of the image of f, *except possibly for a set of measure zero,* is assumed by at most a denumerable number of points in the domain. In this weaker form, it is much easier to prove than the stronger version.

Lemma 3. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Suppose P is a subset of $[a, b]$ such that f is differentiable at each point of P and that . $m(f([a, b] - P)) = 0$. Let $P_+ = \{ x \in P : f'(x) \geq 0\}$ and $P_- = \{ x \in P : f'(x) \leq 0\}$. Then $\max(0, f(b) - f(a)) \leq m^*(f((a, b) \cap P_+))$ and $\max(0, f(a) - f(b)) \leq m^*(f((a, b) \cap P_-))$. Consequently,

$$-m^*(f((a, b) \cap P_-)) \leq f(b) - f(a) \leq m^*(f((a, b) \cap P_+)).$$

Proof. Suppose $f(a) < f(b)$. By hypothesis $m(f((a, b) - P)) = 0$. Therefore,

$$m^*((f(a), f(b)) - f((a, b) - P)) = m^*((f(a), f(b))) = f(b) - f(a).$$

Next we show that

$$(f(a), f(b)) - f((a, b) - P) \subseteq f((a, b) \cap P_+).$$

Take y in $(f(a), f(b)) - f((a, b) - P)$. Then $f(a) < y < f(b)$ and $y \notin f((a, b) - P)$. Then $f^{-1}(y)$ is a subset of P and so f is differentiable at each point of $f^{-1}(y)$. Now since f is continuous, $f^{-1}(y)$ is compact and so is closed and bounded. Let $e = \max\{x: x \in f^{-1}(y)\}$. Then $f'(e) \geq 0$. This is because if $f'(e) < 0$, then by definition of the derivative there exists $x' > e$ such that $b > x'$ and $f(b) > y = f(e) > f(x')$. Thus, by the Intermediate Value Theorem, there exists a point d such that $b > d > x'$ and $f(d) = y$. Hence, $d \in f^{-1}(y)$ and $d > e$. This contradicts that $e = \max\{x: x \in f^{-1}(y)\}$. Hence, $y = f(e) \in f((a, b) \cap P_+)$. It follows that

$$(f(a), f(b)) - f((a, b) - P) \subseteq f((a, b) \cap P_+).$$

Therefore,

$$m^*((f(a), f(b)) - f((a, b) - P)) \leq m^*(f((a, b) \cap P_+))$$

and consequently,

$$f(b) - f(a) \leq m^*(f((a, b) \cap P_+)).$$

Suppose $f(a) > f(b)$. Using a similar argument, we show that

$$f(a) - f(b) \leq m^*(f((a, b) \cap P_-)).$$

It follows that $\max(0, f(b) - f(a)) \leq m^*(f((a, b) \cap P_+))$ and $\max(0, f(a) - f(b)) \leq m^*(f((a, b) \cap P_-))$.

Before we embark on the proof of Theorem 1, we show that under the hypothesis of Lemma 3, f is a N function.

Lemma 4. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Suppose P is a subset of $[a, b]$ such that f is differentiable at each point of P and that $m(f([a, b] - P)) = 0$. Then f is an N function.

Proof. Let E be a subset of $[a, b]$ of measure 0. Write $E = (E \cap P) \cup (E \cap ([a, b] - P))$. Then $m(E \cap P) = 0$ and $m(E \cap ([a, b] - P)) = 0$. By hypothesis,

$$m(f(E \cap ([a, b] - P)) = 0.$$

Since $m(E \cap P) = 0$, $E \cap P$ is measurable. Then by Theorem 2 of "*Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*",

$$m^*(f(E \cap P) \leq \int_{E \cap P} |f'| = 0.$$

It follows that $m(f(E \cap P)) = 0$. Since

$$m^*(f(E)) \leq m^*(f(E \cap P)) + m^*(f(E \cap ([a, b] - P)) = 0,$$

$m(f(E)) = m^*(f(E)) = 0$. Hence f is a N function.

Proof of Theorem 1.

The key is to show that either f has bounded positive variation or bounded negative variation. Then since f is continuous and bounded, it follows that f is of bounded variation. We can deduce this as follows.

Suppose f has bounded positive variation. Take any partition Q: $a = x_0 < x_1 \ldots < x_n = b$. Let $p(Q)$ be the positive variation with respect to Q and $n(Q)$ be the negative variation with respect to Q. Then

$$f(b) - f(a) = p(Q) - n(Q).$$

It follows that $n(Q) = p(Q) + f(a) - f(b) \le f(a) - f(b) + p(f)$, where $p(f)$ is the positive variation of f, for any partition Q. This shows that f has bounded negative variation and so f has bounded total variation. Conversely, we can show similarly, that if f has bounded negative variation, then f has bounded positive variation and so is of bounded variation.

Let $P = [a, b] - E$. Then f is differentiable at each point of P and $m(f([a, b] - P)) = m(f(E)) = 0$.

Let $[a_i, b_i]$ be any closed subinterval in $[a, b]$. Then by Lemma 3,

$$\max(0, f(b_i) - f(a_i)) \le m^*(f((a_i, b_i) \cap P_+)).$$

Note that P is measurable and so P_+ is also measurable. By Theorem 2 of "*Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*",

$$m^*(f((a_i, b_i) \cap P_+)) \le \int_{(a_i,b_i)\cap P_+} |f'| = \int_{(a_i,b_i)\cap P_+} f'.$$

But by hypothesis, $f'(x) \le |g(x)|$ for x in P, and so

$$\int_{(a_i,b_i)\cap P_+} f' \le \int_{(a_i,b_i)\cap P_+} |g| \le \int_{(a_i,b_i)} |g| = \int_{a_i}^{b_i} |g|.$$

Therefore, it follows by Lemma 3 that

$$\max(0, f(b_i) - f(a_i)) \le \int_{a_i}^{b_i} |g|. \quad \text{-------------------------} \quad (1)$$

Note that since g is Lebesgue integrable, | g | is also Lebesgue integrable.

So take any any partition Q: $a = x_0 < x_1 \ldots < x_n = b$. Then by (1),

$$p(Q) = \sum_{i=1}^{n} \max(f(x_i) - f(x_{i-1}), 0) \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |g| = \int_a^b |g|.$$

Hence, f is of bounded positive variation. Consequently f is of bounded variation. Therefore, f is differentiable almost everywhere and f' is Lebesgue integrable. Then using the same set P as above and replacing g by f' for any closed interval $[u, v]$ in $[a, b]$, by (1) we get, if $f(u) < f(v)$,

$$f(v) - f(u) \le \int_u^v |f'|.$$

Also if $f(u) > f(v)$, by Lemma 3,

$$f(u) - f(v) \le m^*(f((u, v) \cap P_-)) \le \int_u^v |f'|.$$

Hence,

$$|f(v) - f(u)| \le \int_u^v |f'|. \quad \text{--------------------------} \quad (2)$$

Note that the function $F(x)$ defined by $F(x) = \int_a^x |f'|$ is absolutely continuous because it is the indefinite integral of a Lebesgue integrable function.

Thus given $\varepsilon > 0$, there exists $\delta > 0$ such that for any non-overlapping sequence of closed intervals $\{ [a_i, b_i] ; i = 1, \ldots, n\}$ with $\sum_{i=1}^{n} |b_i - a_i| < \delta$, we have that

$$\sum_{i=1}^{n} \int_{a_i}^{b_i} |f'| < \varepsilon.$$

Therefore, for any non-overlapping sequence { [a_i , b_i] } with $\sum_i |b_i - a_i| < \delta$,

$$\sum_{i=1}^{n} | f(b_i) - f(a_i)| \le \sum_{i=1}^{n} \int_{a_i}^{b_i} |f'| < \varepsilon \ .$$

This shows that f is absolutely continuous. This completes the proof.

Remark. 1. The hypothesis of Theorem 1 implies that $m(f(\{x \in [a, b] : f'(x) = \pm\infty\})) = 0$. Furthermore, it also implies that f is of bounded variation. Thus, it follows by Theorem 13 of "*Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*", that f is absolutely continuous. Note that $m(f(\{x \in [a, b] : f'(x) = \pm\infty\}))$ = 0 is a necessary condition for absolute continuity.

2. By Lemma 4, any function f satisfying the hypothesis of Theorem 1 is a N function. So Theorem 1 resembles the Banach Zarecki Theorem since the hypothesis implies that f is of bounded variation. Recall that Banach Zarecki Theorem states that any continuous function of bounded variation, which is also a N function, is absolutely continuous. Theorem 1 is a little more convenient since one need not verify that the function f is of bounded variation.

If it is known that the function $f: [a, b] \to \mathbf{R}$ is a continuous N function, which is differentiable almost everywhere on $[a, b]$. Then f maps its set of non differentiability (finite or infinite) into a set of measure zero. Consequently by Theorem 1, for f to be absolutely continuous it is sufficient and necessary that f' be dominated from above by a Lebesgue integrable function. We state this result below.

Theorem 5. Suppose $f: [a, b] \to \mathbf{R}$ is a continuous N function. Furthermore, suppose that f is differentiable almost everywhere on $[a, b]$. Then f is absolutely continuous, if and only if, there exists a Lebesgue integrable function g such that $f' \le g$ almost everywhere on $[a, b]$.

The following is a consequence of Theorem 5.

Corollary 6. Suppose $f: [a, b] \to \mathbf{R}$ is a continuous function. Furthermore, suppose that f is differentiable everywhere on $[a, b]$ except perhaps on a subset which is at most denumerable. Then f is absolutely continuous if there exists a Lebesgue integrable function g such that $f' \le$ g almost everywhere on $[a, b]$.

Proof. Note that if $f: [a, b] \to \mathbf{R}$ is continuous and differentiable everywhere on $[a, b]$ except perhaps on a subset which is at most denumerable, then f is a N function. The result then follows immediately from Theorem 5.

Remark. 1. Compare Theorem 5 with Theorem 7 of "*Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem*", which states that, given the hypothesis of Theorem 5, f is absolutely continuous, if and only if, f' is Lebesgue integrable. Theorem 5 is easier to use as we only need to look for a Lebesgue integrable function dominating the derivative of f, where the derivative of f exists (finitely).

We now apply Theorem 1 to prove Goodman's version of a change of variable formula for the Lebesgue integral. We state the Theorem as follows.

Theorem 7. Suppose g: $[a, b] \to \mathbf{R}$ is continuous and $f: [c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let $F: [c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$. Suppose g maps its set of non differentiability into a set of measure zero. Define the function g* :$[a, b] \to \mathbf{R}$ by

$$g^*(x) = \begin{cases} g'(x), & \text{when } g'(x) \text{ exists (finitely)} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases}$$

Then $\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx$ if the integral on the right exists.

Proof. Note that F and g are both continuous. It is sufficient to show that if $\int_a^b f(g(x))g^*(x)dx$ exists, then the function $F \circ g$ is absolutely continuous and that

$$(F \circ g)'(x) = f(g(x))g^*(x)$$

almost everywhere on $[a, b]$.

Observe that if g is differentiable at x and F is differentiable at g(x), then $F \circ g$ is differentiable at x. Now F is absolutely continuous and so F is differentiable almost everywhere on $[c, d]$. Thus there exists a subset E such that $m(E) = 0$, F is differentiable on $[c, d] - E$ and $F' = f$ on $[c, d] - E$. By hypothesis, g is differentiable except on a set A, where m (g(A)) = 0. By lemma 4, g is a N function. Let $B = A \cup g^{-1}(E)$. Since the Lebesgue measure on $[a, b]$ is regular, there exists a measurable subset $C \supseteq B$ such that $m(C - B) = 0$. Then for $x \notin C$, g is differentiable at x and F is differentiable at g(x) and so $F \circ g$ is differentiable at *every* x in $[a, b] - C$ and

$$(F \circ g)'(x) = f(g(x))g'(x).$$

Now $g(B) \subseteq E \cup g(A)$. Since $m(E) = 0$ and m (g(A)) = 0 , m (g(B)) = 0. Since F is absolutely continuous, F is a N function and so $m(F \circ g(B)) = 0$. As $m(C - B) = 0$, we have too that $m(F \circ g(C - B)) = 0$. It follows that $m(F \circ g(C)) = 0$. Moreover, for every x in $[a, b] - C$,

$$(F \circ g)'(x) = f(g(x))g'(x) \le f(g(x))g^*(x)$$

and by hypothesis $f(g(x))g^*(x)$ is Lebesgue integrable. Therefore, by Theorem 1, $F \circ g$ is absolutely continuous on $[a, b]$. Consequently, $F \circ g$ is differentiable almost everywhere on $[a, b]$ and so $F \circ g$ is differentiable almost everywhere on B. As $m(F \circ g(B)) = 0$, by Theorem 2 of "*Change of Variables Theorem*", $(F \circ g)' = 0$ almost everywhere on $B = A \cup g^{-1}(E)$. Note that g is differentiable on $g^{-1}(E) - A$ and $m(g(g^{-1}(E) - A)) = 0$. Then by Theorem 2 of "*Change of Variables Theorem*", $g' = 0$ almost everywhere on $g^{-1}(E) - A$ and so $(F \circ g)'(x) = f(g(x))g'(x) = 0$ almost everywhere on $g^{-1}(E) - A$.

Hence,

$$(F \circ g)'(x) = \begin{cases} f(g(x))g'(x), & \text{when } g'(x) \text{ exists (finitely)} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases}$$

almost everywhere on $[a, b]$

$= f(g(x))g^*(x)$ almost everywhere on $[a, b]$. -------------- (1)

By the absolute continuity of F,

$$\int_{g(a)}^{g(b)} f(x)dx = F(g(b)) - F(g(a)). \text{ --------------------- } (2)$$

By the absolute continuity of $F \circ g$

$$F(g(b)) - F(g(a)) = \int_a^b (F \circ g)'(x)dx$$

$$= \int_a^b f(g(x))g^*(x)dx \text{ ------------------------ (3)}$$

by (1).
It then follows from (2) and (3) that

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx.$$

Goodman stated a more generalized change of variable theorem, requiring only that g satisfies Lusin's condition, i.e., g is a continuous *N* function.

Theorem 8. Suppose g: [*a*, *b*] → **R** is a continuous *N* function and *f* : [*c*, *d*] → **R** is a Lebesgue integrable function such that the range of g is contained in [*c*, *d*]. Let *F* : [*c*, *d*] → **R** be defined by $F(x) = \int_c^x f(t)dt$. Define the function g* :[*a*, *b*] → **R** by

$$g^*(x) = \begin{cases} g'(x), & \text{when } g'(x) \text{ exists (finitely)} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases}$$

Then $\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx$ if the integral on the right exists or more precisely if the integral $\int_D f(g(x))g^*(x)dx$ exists, where *D* is the set on which g is differentiable finitely.

As in the proof of Theorem 1 in "*Change of Variables Theorems*", the main step is to show that $F \circ g$ is absolutely continuous and that the generalized chain rule for the composite function $F \circ g$ holds almost everywhere on [*a*, *b*].

First we state the following chain rule for $F \circ g$.

Theorem 9. Suppose g: [*a*, *b*] → **R** is a continuous *N* function and *f* : [*c*, *d*] → **R** is a Lebesgue integrable function such that the range of g is contained in [*c*, *d*]. Let *F* : [*c*, *d*] → **R** be defined by $F(x) = \int_c^x f(t)dt$.
Then there is (i) a subset *L* such that $m((F \circ g)(L) = 0$ and for any *x* in *L*, if $(F \circ g)'(x)$ exists, $(F \circ g)'(x) = 0$ except for such *x* in a set of measure zero and if $g'(x)$ exists, either $g'(x) = 0$ except for such *x* in a subset of *L* of measure zero or *f* (g(*x*))= 0, (ii) a subset *K* in the complement of *L,* where both $(F \circ g)'(x)$ and $g'(x)$ do not exist for every *x* in *K* and (iii)

$$(F \circ g)'(x) = f(g(x))g'(x)$$

for *x* in the complement of $L \cup K$.

Proof. Since *F* is an indefinite integral of an integrable function, *F* is absolutely continuous. Therefore, *F* is differentiable almost everywhere on [*c*, *d*]. Thus, there exists a subset *E* of [*c*, *d*] such that *m*(*E*) = 0 and *F* is differentiable (finitely) on [*c*, *d*] – *E and F'*(*x*) = *f* (*x*) for *x* in [*c*, *d*] – *E* . Note that *F* is also a *N* function since it is absolutely continuous and so *m*(*F*(*E*)) = 0. Let $E_0 = \{ x \in [c, d]\text{: } F \text{ is differentiable at } x \text{ and } F'(x) = 0\}$. Then by Theorem 3 of "*Functions having Finite Derivatives,*

Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallee Poussin's Theorem", $m(F(E_0)) = 0$.
Let $B = g^{-1}(E \cup E_0)$.
Suppose g'(x) does not exist finitely at every point of a subset A and differentiable (finitely) at every point outside of A. Let A_∞ be the subset of A where the derivative is $\pm\infty$. Then by the Theorem of Denjoy Saks and Young (Theorem 12), $m(A_\infty) = 0$.
Let $C = A \cup B$. Then for any x in $[a, b] - C$, F is differentiable at g(x) and g is differentiable at x, consequently, $F \circ g$ is differentiable at x and

$$(F \circ g)'(x) = f(g(x))g'(x).$$

Note that since both F and g are N functions, the composite, $F \circ g$, is also a N function.
Let $C_1 = A_\infty \cup B$. Then $m(F \circ g(C_1)) = 0$ because $m(F \circ g(A_\infty)) = 0$ and because $g(B) \subseteq E \cup E_0$ so that $m(F \circ g(B)) \leq m(F(E \cup E_0)) = 0$.
Now consider $K = A - C_1$. Then F is differentiable at g(x) for every x in K and $F'(g(x)) \neq 0$. Thus for x in K, we can write,

$$\frac{F \circ g(x+h) - F \circ g(x)}{h} = H(g(x+h) - g(x)) \cdot \frac{g(x+h) - g(x)}{h}$$

where,

$$H(k) = \begin{cases} \dfrac{F(g(x)+k) - F(g(x))}{k}, & \text{when } k \neq 0 \\ F'(g(x)), & \text{when } k = 0 \end{cases} , \quad \text{------------------- (1)}$$

and H is continuous.
It follows from (1) that both $F \circ g$ and g are not differentiable finitely or infinitely at every point in K.
Let $L = C_1$. Then $L \cup K = C$.
Now since $m(F \circ g(C_1)) = 0$, on the subset of C_1, where $F \circ g$ is differentiable finitely or infinitely, $(F \circ g)'(x) = 0$ almost everywhere by Theorem 2 of "*Change of Variables Theorems*". If x is in $C_1 - A$, then g'(x) exists finitely. Now since $m(g(g^{-1}(E) \cap (C_1 - A))) = 0$, by Theorem 2 of "*Change of Variables Theorems*", g'(x) = 0 almost everywhere on $g^{-1}(E) \cap (C_1 - A)$ and since $g((C_1 - A) - (g^{-1}(E) \cap (C_1 - A))) \subseteq E_0 - E$, $f(g(x)) = 0$ for every x in $(C_1 - A) - (g^{-1}(E) \cap (C_1 - A))$.
Consequently,

$$(F \circ g)'(x) = f(g(x))g^*(x) = 0$$

whenever $(F \circ g)'(x)$ exists in L. This completes the proof.

If we can show that $m((F \circ g)(K) = 0$, then we can invoke Theorem 1 together with the hypothesis of Theorem 8 to show that , $F \circ g$ is absolutely continuous.

We shall need the following theorem due to Banach (see Saks monograph chapter 9, Theorem 7.3).

Theorem 10 (Banach). Suppose $f: [a, b] \to \mathbf{R}$ is a continuous N function. Then f satisfies Banach condition (T_2) on $[a, b]$.

Theorem 11. Suppose $f: [a, b] \to \mathbf{R}$ is a continuous N function. Let K be the subset of $[a, b]$ such that $f'(x)$ does not exist finitely or infinitely for any x in K. Then $m(f(K)) = 0$.

Proof. Since f is a continuous N function, f satisfies Banach condition (T_2). Consequently, barring a set of measure zero, every value in $f(K)$ is assumed at most a denumerable number of times on $[a, b]$, hence on K. So without lost of generality we may assume that for every y in $f(K)$, $f^{-1}(y)$ is at most a denumerable set. If $f^{-1}(y)$ is a single point x_y, then we associate this value of y with x_y.
Now since f is continuous, the subset $D = f^{-1}(y)$ is closed in $[a, b]$.
If D is finite and contains more than one point, then take any point x_y in D. Plainly x_y is an isolated point.
If D is denumerable, its derived set D' is at most denumerable because $D' \subseteq D$ as D is closed. Hence D cannot be a perfect set, i.e., it must have an isolated point because a perfect set is uncountable. Suppose e is an isolated point of D. Then let $x_y = e$.
In this way we have associated with each y in $f(K)$ a point x_y in K such that x_y is an isolated point of $f^{-1}(y)$. Thus by examining the sign of $f(x) - f(x_y)$ for x in a small neighbourhood of x_y and not equal to x_y, $f(x) - f(x_y)$ either changes sign as x passes through x_y or the sign remains the same.
If $f(x) - f(x_y)$ does not change sign as x passes through x_y, then $f(x_y)$ is either a strict local maximum or a strict local minimum. It follows that such a point x_y belongs to a set, which is at most denumerable and hence of measure zero. (See Theorem 13 below.)
If the sign of $f(x) - f(x_y)$ changes as x passes through x_y in a small punctured neighbourhood of x_y, then the four Dini derivatives have the same sign. As $f'(x_y) \neq \pm\infty$, we have that either $0 \leq \min(D_+f(x_y), D_- f(x_y)) < \infty$ or $0 \geq \max(D^+f(x_y), D^- f(x_y)) > -\infty$. Since f is not differentiable at x_y, by the Denjoy-Saks-Young Theorem (Theorem 12), x_y belongs to a set of measure zero. Hence the collection $\{x_y : y \in f(K)\}$ is a set of measure zero. It follows that $f(K)$ is a set of measure zero since f is a N function.

Proof of Theorem 8.
By the proof of Theorem 9, there is a subset C (given in the proof of Theorem 9) such that outside C both $F \circ g$ and ,g are differentiable. Moreover, $m((F \circ g)(C) = 0$. This is because , $m((F \circ g)(C - K) = 0$ and $m((F \circ g)(K) = 0$ by Theorem 11. We can replace C by a measurable set C' such that $C \subseteq C'$ and $m(C' - C) = 0$. Then $m((F \circ g)(C')) = 0$. Moreover, by the hypothesis of Theorem 8, as in the proof of Theorem 9, for every x in $[a, b] - C'$,

$$(F \circ g)'(x) = f(g(x))g^*(x)$$

and $\int_a^b f(g(x))g^*(x)dx$ is Lebesgue integrable. Therefore, by Theorem 1, $F \circ g$ is absolutely continuous. Consequently, $F \circ g$ is differentiable almost everywhere,

$$F(g(b)) - F(g(a)) = \int_a^b (F \circ g)'(x)dx$$

and $(F \circ g)'(x) = 0$ almost everywhere on C *by* Theorem 2 of *"Change of Variables Theorems"* since $m((F \circ g)(C) = 0$. It follows that

$$(F \circ g)'(x) = f(g(x))g^*(x)$$

almost everywhere on $[a, b]$. Therefore,

$$\int_{g(a)}^{g(b)} f(x)dx = F(g(b)) - F(g(a)) = \int_a^b f(g(x))g^*(x)dx.$$

This completes the proof.

We state next the theorem of Denjoy, Saks and Young for reference.

Theorem 12 (Denjoy-Saks-Young Theorem).
Suppose $f : [a, b] \to \mathbf{R}$ is a real valued function defined on the closed and bounded interval $[a, b]$. Then at every point of $[a, b]$ except for a set of measure zero, either
(1) there is a finite derivative; or
(2) $D^+ f$ and $D_- f$ are finite and equal, $D^- f = +\infty$, and $D_+ f = -\infty$; or
(3) $D_+ f$ and $D^- f$ are finite and equal, $D_- f = -\infty$, and $D^+ f = +\infty$; or
(4) $D^+ f = D^- f = +\infty$ and $D_+ f = D_- f = -\infty$,
where the Dini derivatives are defined as follows:
$$D^+ f = \limsup_{h \to 0^+} \frac{f(x+h)-f(x)}{h}, \quad D_+ f = \liminf_{h \to 0^+} \frac{f(x+h)-f(x)}{h},$$
$$D^- f = \limsup_{h \to 0^-} \frac{f(x+h)-f(x)}{h} \text{ and } D_- f = \liminf_{h \to 0^-} \frac{f(x+h)-f(x)}{h}.$$

Theorem 13. Suppose $f : [a, b] \to \mathbf{R}$ is a real valued function defined on the closed and bounded interval $[a, b]$. Then the set of points at which f assumes a strict maximum or minimum is at most denumerable.

Proof. Let E be the set of points where f assumes a strict maximum**.** For any such local maximizer x *in E,* there exists an integer n such that for all $y \neq x$ in $(x - 1/n,\ x + 1/n)$, $f(y) < f(x)$. Moreover, it is obvious that $(x - 1/n,\ x + 1/n)$ cannot contain more than one maximizer. Consequently, the collection $A_n = \{ x \in E : f(y) < f(x)$ for all y in $(x - 1/n,\ x + 1/n)\}$ is a set of isolated points. Therefore, A_n is at most denumerable. This can be seen as follows. The collection $\mathcal{C} = \{(x - 1/(2n),\ x + 1/(2n)) : x \in A_n \}$ is a collection of disjoint open intervals covering A_n , such that each interval $(x - 1/(2n),\ x + 1/(2n))$ contains exactly one point in A_n . Since the set of real numbers is of the second countable, the collection $\mathcal{C}$ is at most denumerable. Now $E = \bigcup_{n=1}^{\infty} A_n$ and so it follows that E is at most denumerable. In a similar way we can show that the set of strict local minimizers is at most denumerable. Therefore, the union of these two sets is at most denumerable. This means that the set of points at which f assumes a strict maximum or minimum is at most denumerable.

Variation of Theorem 1.

Theorem 14. Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function. Suppose E is a measurable subset of $[a, b]$ such that at each point x outside of E, f is differentiable, i.e., $f'(x)$ exists finitely and that the Lebesgue measure of $f(E)$, $m(f(E))$, is zero. Let $P_+ = \{ x \in [a, b] - E : f'(x) \geq 0\}$ and suppose further there exists a function $g : [a, b] \to \mathbf{R}$ such that
$$f'(x) \leq g(x)$$
for $x \in P_+$ and g is integrable or summable on P_+. Then f is absolutely continuous.

Proof. Exactly as in Theorem 1, we apply Lemma 3 to conclude that for any partition Q: $a = x_0 < x_1 \ldots < x_n = b,$ of $[a, b]$, the positive variation $p(Q)$ satisfies,
$$p(Q) = \sum_{i=1}^{n} \max(f(x_i) - f(x_{i-1}), 0) \leq \sum_{i=1}^{n} m^*(f((x_{i-1}, x_i) \cap P_+)$$

$$\leq \sum_{i=1}^{n} \int_{(x_{i-1}, x_i)\cap P_+} |f'| = \sum_{i=1}^{n} \int_{(x_{i-1}, x_i)\cap P_+} f'$$
$$\leq \sum_{i=1}^{n} \int_{(x_{i-1}, x_i)\cap P_+} |g| = \int_{(a, b)\cap P_+} | g| .$$

Hence f is of bounded positive variation. Since f is continuous, f is of bounded variation. Therefore, f is differentiable almost everywhere on $[a, b]$ and f' is Lebesgue integrable. From here we deduce in exactly the same manner as in the proof of Theorem 1, that f is absolutely continuous.

Theorem 15 (Banach).
Suppose $f: [a, b] \to \mathbf{R}$ is a continuous function. Then f is absolutely continuous if and only if f is a N function and that f' is Lebesgue integrable on $P_+ = \{ x \in [a, b] : f'(x) \geq 0\}$, i.e.,

$$\int_{P_+} f' < \infty.$$

Proof. If f is absolutely continuous, then of course, f is of bounded variation and a N function (see Proposition 9 of " *Change of Variable or substitution in Riemann and Lebesgue integration*"). Since f is of bounded variation, f is differentiable almosrt everywhere and f' is Lebesgue integrable on $[a, b]$ and so on P_+.

Conversely, suppose f is a N function and that f' is Lebesgue integrable on P_+ . Let E be the subset of $[a, b]$, where $f'(x)$ does not exist finitely or infinitely for each x in E and $E_\infty = \{x \in [a, b] : f'(x) = \pm\infty\}$. By the Denjoy-Saks-Young Theorem (Theorem 12), $m(E_\infty) = 0$. Since f is a N function, $m(f(E_\infty)) = 0$. By Theorem 11, $m(f(E)) = 0$. Consequently, $m(f(E_\infty \cup E)) = 0$. Then by Theorem 14, f is absolutely continuous.

Theorem 16.
Suppose $f: [a, b] \to \mathbf{R}$ is a continuous N function. Then f must be differentiable at every point of a set of positive measure.

Proof. Suppose f is not differentiable almost everywhere. Then by Theorem 11, $m(f([a, b]) = 0$. Since f is continuous, $f([a, b])$ is compact and connected. Since the only compact connected subset of measure zero is the singleton set, f must be a constant function. Hence f is differentiable everywhere, contradicting that f is not differentiable (finitely or infinitely) almost everywhere on $[a, b]$.

Another proof of Theorem 8.
By Theorem 9, there is a subset C such that both g and $F \circ g$ are differentiable at every point outside C, $m((F \circ g)(C)) = 0$ and

$$(F \circ g)'(x) = f(g(x))g'(x)$$

for x not in C.
Moreover if $f(g(x))g'(x)$ is integrable on $D = \{x \in [a, b]: g'(x) \text{ exists finitely}\}$, then $F \circ g$ is integrable on $[a, b] - C$.
Now $C = L \cup K$. $F \circ g$ and g are not differentiable finitely or infinitely at every point in K. Note that L and K are disjoint. Since $m((F \circ g)(L)) = 0$, on the subset M of L, where $F \circ g$ is differentiable, $(F \circ g)'(x) = 0$ almost everywhere on M. On $L - M$, $F \circ g$ is not differentiable. We may replace L by M. We may remove the subset of measure zero, where $(F \circ g)'(x) \neq 0$ from M . Thus, we may assume that

$(F\circ g)'(x)=0$ on M. If x is in $M-A$, then, $g'(x)$ exists finitely and either $F'(g(x))$ does not exist or $F'(g(x)) = 0$. Then on the subset of $M-A$, where $F'(g(x))$ does not exist, $g'(x)=0$ almost everywhere. Removing the appropriate subset from M, we may assume that when $F'(g(x))$ does not exist, $g'(x)=0$. If x is in $M\cap A$, then $g'(x)$ does nor exist or $g'(x)=\pm\infty$.
Hence, $(F\circ g)'(x)=f(g(x))g^*(x)'$. By hypothesis, the integral $\int_D f(g(x))g^*(x)dx$ exists, where D is the set on which g is differentiable finitely. This implies that

$$\int_{P_+}(F\circ g)'<\infty,$$

where $P_+ = \{x\in[a,b]:\ (F\circ g)'\geq 0\}$.
Then by Theorem 15, $F\circ g$ is absolutely continuous. Therefore,

$$F(g(b))-F(g(a)) = \int_a^b (F\circ g)'(x)dx = \int_D (F\circ g)'(x)dx = \int_D f(g(x))g'(x)dx.$$

Consequently,

$$\int_{g(a)}^{g(b)} f(x)dx = \int_D f(g(x))g'(x)dx\ .$$

In "*Change of Variables Theorems*", I made a remark after Theorem 1, querying if there are integrable function f and finite function g, not having finite derivatives almost everywhere on $[a, b]$, such that $F\circ$ g is absolutely continuous on $[a, b]$ but $(F\circ$ g $)'(x)\neq f$ (g(x)) g$'(x)$ almost every where on $[a, b]$. Goodman in "*N-Functions and Integration By Substitution, Milan Journal of Mathematics vol 47 (1977) 123-134,*" gave such an example due to Ruziewicz, where g is a function not differentiable on a set of positive measure but its square g^2 is absolutely continuous on [0,1]. However we can make some interesting observation as follows.

Theorem 17. Suppose g: $[a, b]\to \mathbf{R}$ is a finite function and $f: [c, d]\to\mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let $F:[c, d]\to\mathbf{R}$ be defined by $F(x)=\int_c^x f(t)dt$. Suppose $F\circ g$ is absolutely continuous on $[a, b]$.
Then $(F\circ g)'(x)=f(g(x))g^*(x)$ almost everywhere on $[a, b]$, where

$$g^*(x)=\begin{cases} g'(x)\,, & \text{when } g'(x) \text{ exists (finitely)} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite}\end{cases},$$

$f(g(x))g^*(x)$ is Lebesgue integrable on $[a, b]$ and

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx = \int_D f(g(x))g^*(x)dx,$$

where $D=\{x\in[a, b]: g'(x)$ exists finitely.$\}$.

Proof.

Since F is an indefinite integral of an integrable function, F is absolutely continuous. Therefore, F is a N function, differentiable almost everywhere on $[c, d]$ and $F'(x) = f(x)$ almost everywhere on $[c, d]$.
Let $E=\{x\in[c, d]:\ F'(x)$ does not exist finitely or $F'(x)=\pm\infty$ or $F'(x)\neq f(x)\}$.
Then $m(E)=0$ since $m\{x:\ F'(x)=\pm\infty\}=0$ by the Denjoy-Saks-Young Theorem (Theorem 12). It follows that for x in $[c, d]-E$, $F'(x)$ exists finitely and $F'(x)= f(x)$. Let $E_0=\{x\in[c, d]:\ F'(x)=0\}$. By Theorem 3 of "*Functions having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem*

and de La Vallee Poussin's Theorem", $m(F(E_0)) = 0$. Since F is a N function, $m(F(E)) = 0$. Consequently, $m(F(E_0 \cup E)) = 0$.
Let $B = g^{-1}(E_0 \cup E)$. Suppose $A = \{ x \in [a, b]$: g'(x) does not exist finitely or infinitely$\}$. If $x \in [a, b] - A$, then either g'(x) is finite or g'(x) = $\pm\infty$.
Observe that $(F \circ g)'(x) = f(g(x))g'(x)$ for x in $[a, b] - (A \cup B)$.
We now examine the derivative of $F \circ g$ on $A \cup B$. By hypothesis, $F \circ g$ is absolutely continuous and so $F \circ g$ is differentiable almost everywhere on $[a, b]$.
Now for x in $A - B$, g(x) $\notin E_0 \cup E$ and so we have F is differentiable at g(x) and $F'(g(x)) = f(g(x)) \neq 0$. But for x in $A - B$, g'(x) does not exist finitely or infinitely. It follows that for x in $A - B$, $(F \circ g)'(x)$ does not exist finitely or infinitely. Since $F \circ g$ is differentiable almost everywhere on $[a, b]$, $A - B$ must be of measure zero. Thus, we may assume without loss of generality, that $A \subseteq B$. (We may simply remove the set $A - B$ of measure zero from A.)
Next we examine the set B. Since $g(B) \subseteq E_0 \cup E$ and $m(F(E_0 \cup E)) = 0$, it follows that $m((F \circ g)(B)) = 0$. Since $F \circ g$ is differentiable almost everywhere on B, it follows then by Theorem 2 of "*Change of Variables Theorems*", that

$$(F \circ g)'(x) = 0$$

almost everywhere on B. Hence $(F \circ g)'(x) = 0$ almost everywhere on A.

Consider $g^{-1}(E) - A$. Since $m(g(g^{-1}(E) - A)) = 0$ and g is differentiable finitely on $g^{-1}(E) - A$, by Theorem 2 of "*Change of Variables Theorems*", g'(x) = 0 almost everywhere on $g^{-1}(E) - A$.
Note that for x in $g^{-1}(E_0) - g^{-1}(E) \cup A$, g'($x$) exists finitely and $f(g(x)) = 0$.
Consequently, $(F \circ g)'(x) = f(g(x))g'(x) = 0$ *almost* everywhere on $B - A$.
It follows that $(F \circ g)'(x) = f(g(x))g^*(x)$ almost everywhere on B. Hence $(F \circ g)'(x) = f(g(x))g^*(x)$ almost everywhere on $[a, b]$. Therefore, $f(g(x))g^*(x)$ is Lebesgue integrable on $[a, b]$ and

$$\int_{g(a)}^{g(b)} f(x)dx = F(g(b)) - F(g(a)) = \int_a^b (F \circ g)'(x)dx$$
$$= \int_D (F \circ g)'(x)dx = \int_D f(g(x))g^*(x)dx,$$

where $D = \{x \in [a, b]$: $g'(x)$ exists finitely$\}$.
This completes the proof.

There is a partial converse to Theorem 17. We impose the requirement that g be a continuous N function.

Theorem 18. Suppose g: $[a, b] \to \mathbf{R}$ is a continuous N function and f: $[c, d] \to \mathbf{R}$ is a Lebesgue integrable function such that the range of g is contained in $[c, d]$. Let F : $[c, d] \to \mathbf{R}$ be defined by $F(x) = \int_c^x f(t)dt$. Suppose $f(g(x))g^*(x)$ is Lebesgue integrable on $D = \{x \in [a, b]$: $g'(x)$ exists finitely.$\}$, where

$$g^*(x) = \begin{cases} g'(x), & \text{when } g'(x) \text{ exists (finitely)} \\ 0, & \text{when } g'(x) \text{ does not exist or is infinite} \end{cases}.$$

Then $F \circ g$ is absolutely continuous on $[a, b]$ and

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g^*(x)dx = \int_D f(g(x))g^*(x)dx.$$

Proof. This is just Theorem 8. We deduce as in the proof of Theorem 8 that under the hypothesis of Theorem 18, $F \circ g$ is absolutely continuous on [a, b]. The remaining conclusion then follows from Theorem 17.

Remark. Note that the integrability of $f(g(x))g^*(x)$ is not sufficient to ensure that $F \circ g$ is absolutely continuous on [a, b]. Take for example, $f(x) = 2x$ and g to be the Cantor ternary function. Then g is increasing and continuous, $g' = 0$ almost everywhere on [0, 1] but g is not absolutely continuous on [0, 1] and therefore not a N function. With the terminology of Theorem 18, $F \circ g = g^2$ and g^2 is not absolutely continuous on [0, 1]. We can observe this by noting that $f(g(x))g^*(x) = 0$ almost everywhere on [0,1] but g^2 is not constant.

Revised Jan 2017

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