1. (a) $\int \frac{x^2}{\sqrt{25-x^2}}dx$ using the substitution $x = 5\sin(\theta)$

$= \int 25\sin^2(\theta)d\theta = \frac{25}{2}\int(1-\cos(2\theta))d\theta = \frac{25}{2}(\theta - \frac{1}{2}\sin(2\theta)) + C = \frac{25}{2}(\theta - \sin(\theta)\cos(\theta)) + C$

$= \frac{25}{2}\left(\sin^{-1}(\frac{x}{5}) - \frac{x}{5}\sqrt{1-\frac{x^2}{25}}\right) + C = \frac{25}{2}\sin^{-1}(\frac{x}{5}) - \frac{x}{2}\sqrt{25-x^2} + C.$

(b) $\int \frac{x^2}{\sqrt{9+x^2}}dx = \int 9\tan^2(\theta)\sec(\theta)d\theta$ using the substitution $x = 3\tan(\theta)$. Now

$\int \tan^2(\theta)\sec(\theta)d\theta = \int \sec(\theta)\tan(\theta)\tan(\theta)d\theta = \sec(\theta)\tan(\theta) - \int \sec(\theta)\sec^2(\theta)d\theta$

$= \sec(\theta)\tan(\theta) - \int\sec(\theta)(1+\tan^2(\theta))d\theta = \sec(\theta)\tan(\theta) - \int\sec(\theta)d\theta - \int\tan^2(\theta)\sec(\theta)d\theta$

$= \sec(\theta)\tan(\theta) - \ln(|\sec(\theta)+\tan(\theta)|) - \int\tan^2(\theta)\sec(\theta)d\theta$.

$\therefore \int\tan^2(\theta)\sec(\theta)d\theta = \frac{1}{2}(\sec(\theta)\tan(\theta) - \ln(|\sec(\theta)+\tan(\theta)|)) + C.$

$\therefore \int\frac{x^2}{\sqrt{9+x^2}}dx = \frac{9}{2}(\sec(\theta)\tan(\theta) - \ln(|\sec(\theta)+\tan(\theta)|)) + C$

$= \frac{9}{2}(\frac{x}{3}\sqrt{1+\frac{x^2}{9}} - \ln\left(\frac{x}{3} + \sqrt{1+\frac{x^2}{9}}\right)) + C = \frac{x}{2}\sqrt{9+x^2} - \frac{9}{2}\ln\left(\frac{x+\sqrt{9+x^2}}{3}\right) + C.$

(c) $\int\frac{\sqrt{x^2-25}}{x}dx$ using the substitution $x = 5\sec(\theta)$

$= 5\int\tan^2(\theta)d\theta = 5\int(\sec^2(\theta)-1)d\theta = 5\tan(\theta) - 5\theta + C$

$= 5\sqrt{\frac{x^2}{25}-1} - 5\sec^{-1}(\frac{x}{5}) + C = \sqrt{x^2-25} - 5\sec^{-1}(\frac{x}{5}) + C.$

(d) $\frac{x^5+x^4+x^3+x^2+x+1}{x^3-x} = x^2 + x + 2 + \frac{3}{x-1} - \frac{1}{x}.$

$\therefore \int\frac{x^5+x^4+x^3+x^2+x+1}{x^3-x}dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 3\ln(|x-1|) - \ln(|x|) + C.$

(e) $\int\frac{x^2+3}{x(x^2+x+1)}dx = \int(\frac{3}{x} - \frac{2x+3}{x^2+x+1})dx = 3\ln(|x|) - \int\frac{(2x+1)+2}{x^2+x+1}dx$

$= 3\ln(|x|) - \ln(|x^2+x+1|) - \int\frac{2}{(x+\frac{1}{2})^2+\frac{3}{4}}dx = \ln\left(\left|\frac{x^3}{x^2+x+1}\right|\right) - \frac{4}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + C$.

(f) $\int\frac{x^2+3x+5}{(x^2+2x+5)^2}dx = \int(\frac{1}{x^2+2x+5} + \frac{x}{(x^2+2x+5)^2})dx$

$= \int\frac{1}{(x+1)^2+4}dx + \int\frac{\frac{1}{2}(2x+2)-1}{(x^2+2x+5)^2})dx = \frac{1}{2}\tan^{-1}(\frac{x+1}{2}) - \frac{1}{2}\frac{1}{x^2+2x+5} - \int\frac{1}{(x^2+2x+5)^2}dx.$

Now $\int\frac{1}{(x^2+2x+5)^2}dx = \int\frac{1}{((x+1)^2+4)^2}dx = \frac{1}{8}\int\frac{\sec^2(\theta)}{(\tan^2(\theta)+1)^2}d\theta$, where $x+1 = 2\tan(\theta)$

$= \frac{1}{8}\int\cos^2(\theta)d\theta = \frac{1}{16}\int(1+\cos(2\theta))d\theta = \frac{1}{16}\theta + \frac{1}{32}\sin(2\theta) + C$

$= \frac{1}{16}\theta + \frac{1}{16}\sin(\theta)\cos(\theta) + C = \frac{1}{16}\theta + \frac{1}{16}\frac{\tan(\theta)}{\sec^2(\theta)} + C = \frac{1}{16}\tan^{-1}(\frac{x+1}{2}) + \frac{1}{16}\frac{x+1}{2}\frac{1}{1+(\frac{x+1}{2})^2} + C$

$= \frac{1}{16}\tan^{-1}(\frac{x+1}{2}) + \frac{x+1}{8}\frac{1}{x^2+2x+5} + C.$

Therefore, $\int\frac{x^2+3x+5}{(x^2+2x+5)^2}dx = \frac{7}{16}\tan^{-1}(\frac{x+1}{2}) - \frac{x+5}{8}\frac{1}{x^2+2x+5} + C'.$

(g) $\int\sqrt{9-\sqrt{x}}\,dx$. We shall use the substitution $u = \sqrt{9-\sqrt{x}}$. Then $u^2 = 9-\sqrt{x}$.

Differentiating $2udu = -\frac{1}{2\sqrt{x}}dx = -\frac{1}{2(9-u^2)}dx$. Thus

$\int\sqrt{9-\sqrt{x}}\,dx = -\int 4u^2(9-u^2)du = -12u^3 + \frac{4}{5}u^5 + C = -12(9-\sqrt{x})^{\frac{3}{2}} + \frac{4}{5}(9-\sqrt{x})^{\frac{5}{2}} + C.$

2 (i) The function for the curve is $f(x) = x^{3/2}$. Therefore, the arc length from (0,0) to (1,1) is given by $\int_0^1\sqrt{1+(f'(x))^2}\,dx = \int_0^1\sqrt{1+\frac{9}{4}x}\,dx = \frac{8}{27}\left[(1+\frac{9}{4}x)^{3/2}\right]_0^1 = \frac{8}{27}(\frac{13^{3/2}}{8} - 1).$

(ii) The function for the curve is $f(x) = \ln(x)$. Therefore, the arc length from $x = \frac{1}{2}$ to $x = 2$ is given by $\int_{1/2}^2\sqrt{1+(f'(x))^2}\,dx = \int_{1/2}^2\sqrt{1+\frac{1}{x^2}}\,dx = \int_{1/2}^2\frac{1}{x}\sqrt{1+x^2}\,dx$

$$= \left[\sqrt{1+x^2} - \ln(\frac{\sqrt{1+x^2}+1}{x})\right]_{1/2}^2 = \frac{\sqrt{5}}{2} - \ln(\frac{3}{2} - \frac{\sqrt{5}}{2})$$

(For the computation of the antiderivative, use substitution $x = \tan(\theta)$. Then

$$\int\frac{1}{x}\sqrt{1+x^2}\,dx = \int\frac{\sec^3(\theta)}{\tan(\theta)}d\theta = \int\frac{\sec(\theta)(1+\tan^2(\theta))}{\tan(\theta)}d\theta$$

$$= \int \sec(\theta)\tan(\theta)d\theta + \int \csc(\theta)d\theta$$
$$= \sec(\theta) + \ln|\csc(\theta) - \cot(\theta) + C$$
$$= \sqrt{1+\tan^2(\theta)} + \ln(\sqrt{1+\cot^2(\theta)} - \cot(\theta) + C$$
$$= \sqrt{1+x^2} + \ln(\sqrt{1+\frac{1}{x^2}} - \frac{1}{x}) + C \ .)$$

(iii) The volume of the solid of revolution is given by

$$\int_0^2 \pi\, e^{-2x}dx = \pi\left[-\tfrac{1}{2}e^{-2x}\right]_0^2 = \tfrac{\pi}{2}(1-e^{-4}) \ .$$

(iv) The volume of the solid of revolution is given by

$$\int_0^b \pi\frac{x^4}{a^2}\, dx = \pi\left[\frac{x^5}{5a^2}\right]_0^b = \frac{\pi b^5}{5a^2}$$

(v) The area of the surface of revolution is given by

$$\int_2^6 2\pi\, f(x)\sqrt{1+(f'(x))^2}\, dx, \qquad \text{where } f(x) = \sqrt{x}$$
$$= \int_2^6 2\pi\ \sqrt{x}\ \sqrt{1+(\tfrac{1}{2\sqrt{x}})^2}\, dx = \int_2^6 2\pi\ \sqrt{x}\ \sqrt{1+\ \tfrac{1}{4x}}\, dx = \pi\int_2^6 \sqrt{1+4x}\, dx$$
$$= \pi[\tfrac{1}{6}(1+4x)^{3/2}]_2^6 = \tfrac{\pi}{6}(125-27) = \frac{49\pi}{3}$$

(vi) The area of the surface of revolution is given by

$$\int_0^{2/3} 2\pi\, f(x)\sqrt{1+(f'(x))^2}\, dx, \qquad \text{where } f(x) = x^3$$
$$= \int_0^{2/3} 2\pi x^3\sqrt{1+9x^4}\, dx = \int_1^{1+16/9} \tfrac{\pi}{18}\sqrt{u}\, du \ \text{ where } u = 1+9x^4$$
$$= \tfrac{\pi}{27}[u^{3/2}]_1^{25/9} = \tfrac{\pi}{27}(\tfrac{125}{27}-1) = \tfrac{98\pi}{729}$$

3. (a) $\lim\limits_{x\to\pi} \frac{\sin^2(2x)}{1+\cos(5x)} = \lim\limits_{x\to\pi} \frac{4\sin(2x)\cos(2x)}{-5\sin(5x)} = \lim\limits_{x\to\pi} \frac{8\cos(4x)}{-25\cos(5x)} = \frac{8}{25}.$

(b) $\lim\limits_{x\to 0} \frac{2x-\ln(2x+1)}{1-\cos(3x)} = \lim\limits_{x\to 0} \frac{2-\frac{2}{1+2x}}{3\sin(3x)} = \lim\limits_{x\to 0} \frac{4x}{3(1+x)\sin(3x)} = \frac{4}{3}\lim\limits_{x\to 0} \frac{1}{\sin(3x)+(1+x)3\cos(3x)} = \frac{4}{9}.$

(c) $\lim\limits_{x\to 1} (\frac{1}{\ln(x)} - \frac{1}{x-1}) = \lim\limits_{x\to 1} \frac{x-1-\ln(x)}{(x-1)\ln(x)} = \lim\limits_{x\to 1} \frac{1-\frac{1}{x}}{\ln(x)+1-\frac{1}{x}} = \lim\limits_{x\to 1} \frac{x-1}{x\ln(x)+x-1} = \lim\limits_{x\to 1} \frac{1}{1+\ln(x)+1} = \frac{1}{2}.$

(d) $\lim\limits_{x\to 2} (\frac{1}{x-2} - \frac{1}{e^{x-2}-1}) = \lim\limits_{x\to 2} \frac{e^{x-2}-x+1}{(x-2)(e^{x-2}-1)} = \lim\limits_{x\to 2} \frac{e^{x-2}-1}{(x-2)e^{x-2}+e^{x-2}-1} = \lim\limits_{x\to 2} \frac{e^{x-2}}{2e^{x-2}+(x-2)e^{x-2}} = \frac{1}{2}.$

(e) $\lim\limits_{x\to 0} \frac{\int_0^x \sin(t^2+t)dt}{\tan(x^2)} = \lim\limits_{x\to 0} \frac{\sin(x^2+x)}{\sec^2(x^2)2x} = \lim\limits_{x\to 0} \frac{\sin(x^2+x)}{x^2+x} \cdot \frac{\cos^2(x^2)}{2} \cdot (1+x) = \frac{1}{2}.$

(f) $\lim\limits_{x\to 0} \frac{\tan^2(x^2)}{x^3} = \lim\limits_{x\to 0} \frac{2\tan(x^2)\sec^2(x^2)2x}{3x^2} = \lim\limits_{x\to 0} \frac{\tan(x^2)}{x^2} \cdot \sec^2(x^2)\frac{4}{3}x = 1\cdot 1\cdot 0 = 0.$

(g) Let $y = x^{\sin(x^3)}$ for $x > 0$.Then $\ln(y) = \sin(x^3)\ln(x)$.

$\lim\limits_{x\to 0^+} \ln(y) = \lim\limits_{x\to 0^+} \frac{\ln(x)}{\csc(x^3)} = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\csc(x^3)\cot(x^3)3x^2} = -\lim\limits_{x\to 0^+} \frac{\sin^2(x^3)}{3x^3\cos(x^3)} = -\lim\limits_{x\to 0^+} \frac{\sin^2(x^3)}{x^6}\frac{x^3}{3\cos(x^3)} = 1\cdot 0 = 0.$

$\therefore\ \lim\limits_{x\to o^+} y = e^{\lim_{x\to o^+}\ln(y)} = e^0 = 1.$

(h) Let $y = (e^x+7x)^{\frac{1}{x}}$. Then $\ln(y) = \frac{1}{x}\ln(e^x+7x)$.

Thus $\lim\limits_{x\to 0} \ln(y) = \lim\limits_{x\to 0} \frac{\ln(e^x+7x)}{x} = \lim\limits_{x\to 0} \frac{e^x+7}{e^x+7x} = 8.\ \therefore\ \lim\limits_{x\to 0} y = e^{\lim_{x\to 0}\ln(y)} = e^8.$

4. (a) $\lim\limits_{x\to\infty} \frac{x^7}{e^x} = \lim\limits_{x\to\infty} \frac{7x^6}{e^x} = \lim\limits_{x\to\infty} \frac{42x^5}{e^x} = \lim\limits_{x\to\infty} \frac{210x^4}{e^x} = \lim\limits_{x\to\infty} \frac{840x^3}{e^x} = \lim\limits_{x\to\infty} \frac{2520x^2}{e^x} = \lim\limits_{x\to\infty} \frac{5040x}{e^x} = \lim\limits_{x\to\infty} \frac{5040}{e^x} = 0.$

(b) $\lim\limits_{x\to\infty} \frac{(\ln(x))^7}{x} = \lim\limits_{x\to\infty} \frac{7(\ln(x))^6}{x} = \lim\limits_{x\to\infty} \frac{42(\ln(x))^5}{x} = \lim\limits_{x\to\infty} \frac{210(\ln(x))^4}{x} = \lim\limits_{x\to\infty} \frac{840(\ln(x))^3}{x} = \lim\limits_{x\to\infty} \frac{2520(\ln(x))^2}{x}$

$= \lim\limits_{x\to\infty} \frac{5040(\ln(x))}{x} = \lim\limits_{x\to\infty} \frac{5040}{x} = 0$

(c) $\lim\limits_{x\to 0^+} \tan(x^3)\ln(x) = \lim\limits_{x\to 0^+} \frac{\ln(x)}{\cot(x^3)} = -\lim\limits_{x\to 0^+} \frac{1.}{x\csc^2(x^3)3x^2} = -\lim\limits_{x\to 0^+} \frac{\sin^2(x^3)}{3x^3} = \lim\limits_{x\to 0^+} \frac{\sin^2(x^3)}{x^6} \cdot \frac{1}{3}x^3 = 1\cdot 0 = 0.$

(d) Let $y = (\sin(x))^{x^3}$ for $x > 0$.Then $\ln(y) = x^3\ln(\sin(x))$.

$\lim\limits_{x\to 0^+} x^3\ln(\sin(x)) = \lim\limits_{x\to 0^+} \frac{\ln(\sin(x))}{\frac{1}{x^3}} = -\lim\limits_{x\to 0^+} \frac{\cos(x)}{-3\frac{1}{x^4}\sin(x)} = -\lim\limits_{x\to 0^+} \frac{x}{\sin(x)}\frac{x^3\cos(x)}{3} = 1\cdot 0 = 0.$

Therefore, $\lim\limits_{x\to 0^+} (\sin(x))^{x^3} = e^{\lim_{x\to 0^+}\ln((\sin(x))^{x^3})} = e^0 = 1.$

5. Let $f(x) = \ln(1+x)$ for $x > -1$. Then $\ln(1.2) = f(0.2)$. The derivatives of f are given by $f'(x) = \frac{1}{1+x}, f''(x) = -\frac{1}{(1+x)^2}, f^{(3)}(x) = 2\frac{1}{(1+x)^3}, \cdots, f^{(n)}(x) = (-1)^{n-1}(n-1)!\frac{1}{(1+x)^n}$.

The modulus of the remainder term,

$$|R_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\right| = \left|\frac{n!}{(n+1)!(1+\xi)^{n+1}}x^{n+1}\right| = \left|\frac{1}{(n+1)(1+\xi)^{n+1}}\right||x|^{n+1}.$$

Therefore $|R_n(0.2)| \le \frac{1}{n+1}(0.2)^{n+1}$ and so in order for $|R_n(0.2)| < 0.0001$ it suffices to take $n = 4$. This is because $|R_3(0.2)| \le 0.0004$ and $|R_4(0.2)| \le 0.000064$.

Now $f(0) = 1$, $f'(0) = 1$, $f''(0) = -1$. And so the Taylor polynomial of degree 4 is $p_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$. $p_4(0.2) = 0.1822666$. Therefore the value of $\ln(1.2)$ is given by $f(0.2)$ and is approximately 0.182 accurate to 3 decimal places.

6. (i) $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$

(ii) $\lim_{n\to\infty} \frac{n+1}{n^3+4} = \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{1}{n^3}}{1+\frac{4}{n^3}} = 0$

7 (i) $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$ since $0 \le \left|\frac{\sin(n)}{n}\right| \le \frac{1}{n}$ for ann positive n and $\lim_{n\to\infty} \frac{1}{n} = 0$.

(ii) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ by the Squeeze Theorem because

$0 \le \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdot \cdots n}{n\cdot n\cdot n\cdot \cdots n} \le \frac{1}{n}$ for any positive integer n

and that $\lim_{n\to\infty} \frac{1}{n} = 0$

8 (i) $\sum_1^\infty \frac{1}{1+n^2}$ converges by the comparison test.

Note that it is a series of nonnegative terms and

$$\frac{1}{1+n^2} \le \frac{1}{n^2}$$

Since we know $\sum_1^\infty \frac{1}{n^2}$ converges by teh Comparison Test, $\sum_1^\infty \frac{1}{1+n^2}$ converges.

(ii) $\sum_1^\infty \frac{n+1}{n(n+2)}$ is divergent

since $\frac{n+1}{n(n+2)} \ge \frac{1}{n+2}$ and , $\sum_1^\infty \frac{1}{n+2}$ is divergent by the Comparison Test

(iii) $\sum_1^\infty \frac{\ln(n)}{\sqrt{n+1}}$ is divergent

since $\frac{\ln(n)}{\sqrt{n+1}} \ge \frac{1}{\sqrt{n+1}}$ for $n \ge 3$ and $\sum_1^\infty \frac{1}{\sqrt{n+1}}$ is divergent by the Comparison Test

9. (i) $\sum_1^\infty \frac{(n!)^2}{(2n)!}$ is convergent by the Ratio Test as follows.

$$\frac{((n+1)!)^2}{(2(n+1))!} \Big/ \frac{(n!)^2}{(2n)!} = \frac{(n+1)^2}{(2n+1)(2n+1)} \to \frac{1}{4} < 1$$

and so by the Ratio Test the series is convergent.

(ii) $\sum_1^\infty \frac{n!}{n^n}$

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{(n+1)}} \Big/ \frac{n!}{n^n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{\lim_{n\to\infty}(1+\frac{1}{n})^n} = \frac{1}{e} < 1$$

Therefore, by the Ratio Test the series is convergent.

(iii) $\sum_1^\infty \frac{n^2}{2^n}$

Now $\frac{(n+1)^2}{2^{(n+1)}} \Big/ \frac{n^2}{2^n} = \frac{1}{2}(\frac{n+1}{n})^2 \to \frac{1}{2} < 1$.

Thus, by the Ratio Test the series is also convergent.