National University of Singapore

Department of Mathematics

Level 1000 Semester 2 (2005/06) MA1102R Calculus Tutorial set 2

Dirichlet: y is a function of x when to each value of x in a given interval there corresponds a unique value of y. It does not matter whether throughout this interval y depends upon x according to one law or more or whether the dependence of y on x can be expressed by mathematical operations.

Definition 1. A *function* $f : A \to B$ *from A to B* is a rule which assigns to each element $a \in A$ *one and only one* element b of B. We write $f(a)$ for b. The set A is called the *domain* of f and the set B is called the *codomain* of f.

Discuss: In your opinion, Dirichlet's definition and the definition above are the same. Are they precise enough? If not, where do they fail to be precise?

The set $\{b \in B : \text{there exists } a \in A \text{ such that } f(a) = b\} = \{ f(a) : a \in A\}$ is called the *range* of f. We say $b = f(a)$ is the *image* of a under f. We may write for a subset $U \subseteq A$, $f(U) = \{ f(x) : x \in U\}$ and call this the *image* of U under f. Thus the range of f is equal to $f(A)$ the image of A.

Activity 1. Consider the expression $f(x) = x^2 + 1$. What is the largest subset of $\mathbf{R}$ for which the expression determines a function with it as the domain? What would you take as the codomain for such a function? For what values of y can we solve the equation $f(x) = y$ for x in the domain of f ? Note that this is precisely the range of f . How would you describe the range of f ? Is it sufficient to plot the graph of f and make a deduction for the range of f ? What is wrong in doing this ?

Definition 2. A function $f : A \to B$ is said to be *injective* (or *one-one*) if and only if the following statement is true: If $f(a) = f(b)$, then $a = b$.

Activity 2. 1. Take the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x$. Explain why it is injective.
2. Explain why the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 + 1$ is not injective.

Definition 3. A function $f : A \to B$ is said to be *surjective* (or *onto*) if for each element b in B, there exists an element a in A such that $f(a) = b$, *equivalently* if $f(A) = B$.

Activity 3. Explain why the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 3x$ is surjective. (Take any y in $\mathbf{R}$. Show that we can find an x in $\mathbf{R}$ with $f(x) = y$). Why is it inadequate to plot the graph of f for a finite interval on a piece of paper and deduce from the graph that f is surjective ?

Definition 4. A function $f : A \to B$ is said to be *bijective* if and only if f is both injective and surjective.

Activity 4. Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x$ is bijective.

Definition 5. Let $f : B \to C$ and $g : A \to B$ be functions. Then their *composition* (or *composite*) is a function $f \circ g : A \to C$ defined by $f \circ g(x) = f(g(x))$ for each $x \in A$.

Activity 5. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 3$ and $g(x) = 5x^2$ respectively. Find $f \circ g(x)$ and $g \circ f(x)$.

Definition 6. Suppose $f : A \to B$ is a *bijection*. Then it has an *inverse function* $f^{-1} : B \to A$ defined as follows. Take *any* b in B. Since f is surjective, there is an element a in A such that $f(a) = b$. We define $f^{-1}(b) = a$. f^{-1} assigns only one value to b and f^{-1} is a function with domain B and codomain A.

Activity 6. Take f :**R**→**R** defined by $f(x) = x + 8$ which is bijective. Show that its inverse function f^{-1} :**R**→**R** is given by $f^{-1}(y) = y - 8$.

Definition 7. A function $f : D \to \mathbf{R}$ is said to be *even* if $f(-x) = f(x)$ for all x in D. It is said to be *odd* if $f(-x) = -f(x)$ for all x in D.

Activity 7. Give an example of an odd function and an example of an even function.

Definition 8. A *rational function* is a function h defined by $h(x) = \frac{f(x)}{g(x)}$, where f and g are polynomial functions and the domain of h is given by $Domain(h) = \{x \in \mathbf{R}: g(x) \neq 0\}$.

Activity 8. What is the domain of the rational function $g(x) = \frac{2x+7}{(x^2+4x+3)}$**?**

Inequalities.

Activity 9. Write the following subsets of **R** in interval notation.

1. $\{x : 1 < x \leq 6\}$ 2. $\{x : 3 < x < 7\}$ 3. $\{x : -4 < x\}$ 4.$\{x : x \leq 9\}$

Properties 9. *The real numbers* **R** *is the only* ***complete totally ordered field*** *(upto isomorphism). It has exactly one linear ordering determined by* $\mathbf{R}_+$ *which contains the positive rational numbers.*

Note that $\mathbf{R}_+$ *is a distinguished* subset of **R** that does not contain 0 and satisfies the following (1) Any real number x is either in $\mathbf{R}_+$ *or* its reflection $-\mathbf{R}_+ = \{-x : x \in \mathbf{R}_+\}$or is equal to 0 *and* (2) a, b *in* $\mathbf{R}_+$ *implies that* $a + b \in \mathbf{R}_+$ and $ab \in \mathbf{R}_+$. We define $a > b$ *if and only if* $a - b \in \mathbf{R}_+$. *Using this definition of " > " and the properties of* $\mathbf{R}_+$, *convince yourself the truth of the following statements.*

1. Given $a, b \in \mathbf{R}$ exactly one of the following is true: $a > b$, $a = b$, $a < b$.
2. $a > b$ and $b > c \Rightarrow a > c$. 3. For any c in **R**, $a > b \Rightarrow a + c > b + c$. 4. $a > b$ and $c > 0 \Rightarrow ac > bc$
5. $a > b$ and $c < 0 \Rightarrow ac < bc$. 6. $a > b$ and $c > 0 \Rightarrow \frac{a}{c} > \frac{b}{c}$. 7. $a > b$ and $c < 0 \Rightarrow \frac{a}{c} < \frac{b}{c}$.
8. $ab > 0 \Leftrightarrow (a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$.
9. $ab < 0 \Leftrightarrow (a > 0$ and $b < 0)$ or$(a < 0$ and $b > 0)$.
10. If $a > b > 0$ and $n \in N$, then $a^n > b^n$. 11. If $a > b > 0$ and $n \in N$, then $a^{\frac{1}{n}} > b^{\frac{1}{n}}$.

Definition 10. We say a is *greater than or equal to* b if $a > b$ *or* $a = b$. We write $a \geq b$ (or $b \leq a$).

Definition 11. Define the modulus function, $|\ \ |$:$\mathbf{R} \to \mathbf{R}_+ \cup \{0\}$by $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$.

Activity 10. Explain $|x| \geq 0$ and $-x,\ x \leq |x|$ for any real number x.

Properties 9 (Continued).

12. Suppose $a \geq 0$, then $|b| \leq a \Leftrightarrow b \leq a$ and $-b \leq a \Leftrightarrow -a \leq b \leq a$.

13. $|b| \geq a \Leftrightarrow b \geq a$ or $-b \geq a \Leftrightarrow b \geq a$ or $b \leq -a$.

Activity 11. 1. Show that $|a| \leq |a - b| + |b|$ and $|b| \leq |a - b| + |a|$ and deduce that $||a| - |b|| \leq |a - b|$
2. Show that $|x - 3| < 2 \Leftrightarrow 1 < x < 5$.

1. Let $f(x)$ be the expression $\frac{5x-19}{3x-6}$.

   a. Find the largest subset $D \subseteq \mathbf{R}$ so that $f(x)$ does determine a function $f: D \to \mathbf{R}$ with domain D and codomain $\mathbf{R}$.

   b. With the domain D as given in part a, find the range of f.

   c. Let E be the range of f found in part b. Let f' be the function obtained from f by replacing the codomain of f with its range E, i.e., $f': D \to E$ is given by $f'(x) = f(x)$ for all x in D. Show that f' is bijective and find an inverse to f'.

2. Let $g: \mathbf{R} - \{\frac{2}{9}\} \to \mathbf{R}$ be the function given by $g(x) = \frac{1}{9x-2}$.

   a. Find the *range* of g.

   b. Show that, when considered as a function $g: \mathbf{R} - \{\frac{2}{9}\} \to$ Range g, g is a bijection and find its inverse.

3. For each of the following expressions, determine the largest domain on which it defines a function with values in the real numbers $\mathbf{R}$ (= its codomain) and its range. Sketch their graphs.

   a. $f(x) = \sqrt{25 - x^2}$.

   b. $g(x) = |x-3| + |x+2|$.

4. For each of the following functions $f: \mathbf{R} \to \mathbf{R}$, determine whether it is injective, surjective or bijective.

   a. $f(x) = 7x + 5$.

   b. $f(x) = x^2 - 6x + 5$.

5. Let $g: A \to B$ and $h: B \to C$ be functions. Show that

   a. if $h \circ g$ is surjective, then h is surjective.

   b. if $h \circ g$ is injective, then g is injective.

6. Find the solution set of each of the following inequalities:

   a. $x^2 - x - 6 > 0$. b. $\frac{4x+1}{x-3} < 1$. c. $\frac{5}{x} - 4 \geq \frac{3}{x} - 10$. d. $|x+2| + |x-5| \geq 6$.

7. If $b, d > 0$ and $\frac{a}{b} < \frac{c}{d}$, then show that $\frac{a}{b} < \frac{a+101c}{b+101d} < \frac{c}{d}$.

8. Solve the following inequalities for x in $\mathbf{R}$.

   a. $|5x^2 + 9| > 4$. b. $\left|\frac{2}{x}\right| \geq 7$.

9. Prove that if a_1, a_2 and a_3 are positive and $a_1 a_2 a_3 = 1$ then $(1+a_1)(1+a_2)(1+a_3) \geq 8$.

10. **(Optional)** Show that if P is a subset of $\mathbf{Q}$ satisfying 1. $\mathbf{Q} = P \cup \{0\} \cup (-P)$, $0 \notin P$, $-P = \{-a\colon a \in P\}$, $P \cap (-P) = \varnothing$ and 2. $a, b \in P$ implies that $a + b$, $ab \in P$, then ***P* is precisely the set of positive rational numbers.** Show then that Properties 1 to 10 listed under the Inequalities heading hold for rational numbers. [Hint for the first part: Start by showing $1 \in P$.]

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