See General Advice for Tutorial 4 in the course website:
http://www.math.nus.edu.sg/~matngtb/Calculus/tutorials.html
For a deeper appreciation check out
http://www.math.nus.edu.sg/~matngtb/Calculus/Extreme%20Value/bound.htm
Students are expected to **master the skill of evaluation of limits.** For some parts that are straightforward, only answers are given for the purpose of checking. For questions that most students have difficulty (such as 1(b), 2(c) etc) or when there are some specific points to take note (such as 1(c), 1(d) , 3 etc), solutions with explanation are provided. Do try to compare with your work and make sure you really understand the working and reasoning. When a solution is not given, it is similar to examples in lecture or textbook. You should find this out. You should know the **definition of continuity at a point**, and be able to determine whether a function is continuous at a point. **Properties of continuity** such as addition or product or composite of continuous functions give a continuous function. You should understand the statement of The **Intermediate Value Theorem** and know how to apply it. As before, the **solution given here serves as a guide**. For those of you who have difficulty in presenting solution, use this solution as a guide in learning how to present your work. For those who have difficulty in working out the problem, use it to clarify your understanding of the subject. **Lastly, use the solution wisely, and not blindly**.

1 a. $\lim_{x\to\infty} \sqrt[3]{\frac{8x^2+7}{27x^2-1}} = \frac{2}{3}$.

b. $\lim_{x\to-\infty} \frac{\sqrt{x^2+9}}{1+4x} = \lim_{x\to-\infty} -\frac{\sqrt{x^2+9}\,/\sqrt{x^2}}{(1+4x)/\sqrt{x^2}} = \lim_{x\to-\infty} -\frac{\sqrt{x^2+9}\,/\sqrt{x^2}}{(1+4x)/(-x)} = \lim_{x\to-\infty} \frac{\sqrt{1+\frac{9}{x^2}}}{1-\frac{4}{x}} = -\frac{1}{4}$.

(Note that for $x < 0$, $\sqrt{x^2} = |x| = -x$.)

Alternatively, for $x < -1/4$, note that $1+4x < 0$ so that $1+4x = -\sqrt{(1+4x)^2}$

$\lim_{x\to-\infty} \frac{\sqrt{x^2+9}}{1+4x} = \lim_{x\to-\infty} -\sqrt{\frac{x^2+9}{(1+4x)^2}} = -\sqrt{\lim_{x\to-\infty} \frac{x^2+9}{(1+4x)^2}} = ... = -\frac{1}{4}$.

c. $\lim_{x\to(-6)^-} \frac{3x}{36-x^2} = \lim_{x\to(-6)^-} \frac{3x}{6-x}\cdot\frac{1}{6+x} = +\infty$ since $\lim_{x\to(-6)^-} \frac{1}{6+x} = -\infty$ and $\lim_{x\to(-6)^-} \frac{3x}{6-x} = \frac{-3}{2} < 0$, use Theorem 4 part 2 of this tutorial set.

d. $\lim_{x\to4^+} \frac{x-4}{\sqrt{8x-x^2}-4} = \lim_{x\to4^+} \frac{(x-4)(\sqrt{8x-x^2}+4)}{(\sqrt{8x-x^2}-4)(\sqrt{8x-x^2}+4)} = \lim_{x\to4^+} \frac{(x-4)(\sqrt{8x-x^2}+4)}{8x-x^2-16} = \lim_{x\to4^+} \frac{\sqrt{8x-x^2}+4}{4-x} = -\infty$

since $\lim_{x\to4^+} \frac{1}{4-x} = -\infty$ and $\lim_{x\to4^+} (\sqrt{8x-x^2}+4) = 8 > 0$., by Theorem 4 (part 1), Tut set 4 .

2. a. $\lim_{x\to\infty} \frac{23x^2-5x^3+7}{13x^3+3} = -\frac{5}{13}$.

b. $\lim_{x\to-\infty} \left(\frac{x^3}{4x^2-2} - \frac{x^2}{4x+3}\right) = \lim_{x\to-\infty} \left(\frac{3x^3+2x^2}{(4x^2-2)(4x+3)}\right) = ... = \frac{3}{16}$

c. $\lim_{x\to-\infty} \frac{\sqrt{9x^2-2}}{3-x} = \lim_{x\to-\infty} \sqrt{\frac{9x^2-2}{(3-x)^2}} = \lim_{x\to-\infty} \sqrt{\frac{9-\frac{2}{x^2}}{\frac{9}{x^2}-\frac{6}{x}+1}} = 3$. (Here, for $x < 0$, we have $3-x>0$ so that $\sqrt{(3-x)^2} = (3-x)$.)

d. $\lim_{x\to\infty} \left(\sqrt{x^2+2500} - x\right) = \lim_{x\to\infty} \frac{(\sqrt{x^2+2500}-x)(\sqrt{x^2+2500}+x)}{\sqrt{x^2+2500}+x} = \lim_{x\to\infty} \frac{2500}{\sqrt{x^2+2500}+x} = 0$

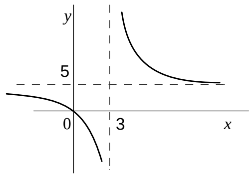
(since $\lim_{x\to\infty} \sqrt{x^2+2500} + x = \infty$, and use Useful results (1) on page 2 of Tutorial set 4.)

3. a. We can write $f(x) = \frac{5x}{x-3} = 5 + \frac{15}{x-3}$.

To find horizontal asymptote, find the following limits at infinity: $\lim\limits_{x\to\pm\infty} f(x) = 5$.
Thus $y = 5$ is the only horizontal asymptote.
To find vertical asymptotes, note that

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \left(5 + \frac{15}{x-3}\right) = +\infty \text{ and } \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \left(5 + \frac{15}{x-3}\right) = -\infty.$$

Therefore, a vertical asymptote for the graph of f is given by the equation $x = 3$ and a horizontal asymptote of the graph of f is given by the equation $y = 5$.



b. Notice that $f(x) = \frac{1}{2x^2-x-17}$. Then $\lim\limits_{x\to+\infty} f(x) = 0$,

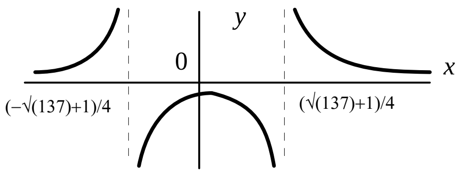
$$\lim_{x\to \frac{\sqrt{137}+1}{4}^+} f(x) = \lim_{x\to \frac{\sqrt{137}+1}{4}^+} \left(\frac{1}{2(x+\frac{\sqrt{137}-1}{4})} \cdot \frac{1}{(x-\frac{\sqrt{137}+1}{4})}\right) = +\infty,$$

$$\lim_{x\to \frac{\sqrt{137}+1}{4}^-} f(x) = \lim_{x\to \frac{\sqrt{137}+1}{4}^-} \left(\frac{1}{2(x+\frac{\sqrt{137}-1}{4})} \cdot \frac{1}{(x-\frac{\sqrt{137}+1}{4})}\right) = -\infty,$$

$$\lim_{x\to \frac{-\sqrt{137}+1}{4}^+} f(x) = \lim_{x\to \frac{-\sqrt{137}+1}{4}^+} \left(\frac{1}{2(x+\frac{\sqrt{137}-1}{4})} \cdot \frac{1}{(x-\frac{\sqrt{137}+1}{4})}\right) = -\infty \text{ and}$$

$$\lim_{x\to \frac{-\sqrt{137}+1}{4}^-} f(x) = \lim_{x\to \frac{-\sqrt{137}+1}{4}^-} \left(\frac{1}{2(x+\frac{\sqrt{137}-1}{4})} \cdot \frac{1}{(x-\frac{\sqrt{137}+1}{4})}\right) = +\infty.$$

Therefore, the vertical and horizontal asymptotes are the lines $x = \frac{\sqrt{137}+1}{4}$, $x = -\frac{\sqrt{137}-1}{4}$ and $y = 0$ respectively. The graph of f is shown below.



4. To check for continuity at $x = -1$, one must check if $\lim\limits_{x\to(-1)} f(x) = f(-1)$.

5 a. Note that f is continuous on $\mathbf{R} - \{1, 3\}$. (since on each part, the expression is a polynomial.)
Now $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (2x^2+3) = 5$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (5-3x) = 2$ and so these two limits are not the same and consequently f is not continuous at $x = 1$.
Now $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (5-3x) = -4$ and $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (x-7) = -4 = f(3)$ and so $\lim\limits_{x\to 3} f(x) = f(3)$. Thus, f is continuous at $x = 3$. Therefore, f is continuous on $\mathbf{R} - \{1\}$.

b. Observe that $g(x) = \begin{cases} \frac{|x-4|}{x-4}, & x \neq 4 \\ 2, & x = 4 \end{cases} = \begin{cases} 1, & x > 4 \\ 2, & x = 4 \\ -1, & x < 4 \end{cases}$. Thus

$\lim\limits_{x\to 4^-} g(x) = -1 \neq 1 = \lim\limits_{x\to 4^+} g(x)$. Therefore, g is not continuous at $x = 7$.
On $(-\infty, 4)$, $g(x) = -1$, which is a constant function, hence g is continuous on $(-\infty, 4)$.
On $(4, \infty)$, $g(x) = 1$, which is a constant function, hence g is continuous on $(4, \infty)$.
Hence, g is continuous on $\mathbf{R} - \{4\}$.

6. a. $\lim_{x\to 0} \frac{\sin(7x)}{\sin(5x)} = \frac{7}{5}$.

   b. $\lim_{x\to 0} \frac{\tan^4(2x)}{4x^4} = 4$.

   c. $\lim_{x\to 0} \frac{1-\cos(8x)}{\sin(8x)} = \lim_{x\to 0} \left(\frac{1-\cos(8x)}{8x} \frac{8x}{\sin(8x)}\right) = \lim_{x\to 0} \frac{1-\cos(8x)}{8x} \lim_{x\to 0} \frac{8x}{\sin(8x)} = 0 \cdot 1 = 0.$

   d. $\lim_{x\to 0} \frac{\sin(\sin(x^2))}{23x} = 0$

   e. $\lim_{x\to 0} x^3 \cos(\frac{1}{x^7}) = 0$ by the *Squeeze Theorem* since $-|x|^3 \le x^3 \cos(\frac{1}{x^7}) \le |x|^3$ and $\lim_{x\to 0} |x|^3 = 0$.

7. Since $1-4x^2 \le g(x) \le \cos(2x)$ for x in $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have, by substituting $x = 0$,
   $$1-4\cdot 0^2 \le g(0) \le \cos(0) = 1 \text{ and so } g(0) = 1.$$
   Also $1 = \lim_{x\to 0} (1-4x^2) = \lim_{x\to 0} \cos(2x) = 1$ so that by the Squeeze Theorem, $\lim_{x\to 0} g(x) = 1$.
   Hence $\lim_{x\to 0} g(x) = g(0)$ and so g is continuous at $x = 0$.

8. **You should give the reason for obtaining the equation involving k.**
   $\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} (7x+5k^2) = 5k^2$ and assuming $k \ne 0$
   $\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} \frac{\tan(kx)}{x} = \lim_{x\to 0^-} \left(\frac{\sin(kx)}{kx} \frac{k}{\cos(kx)}\right) = 1 \cdot \frac{k}{1} = k.$
   Therefore, **for the limit at x = 0 to exist**, we must have $\lim_{x\to 0^+} h(x) = \lim_{x\to 0^-} h(x)$, *i.e.* $5k^2 = k$,
   that is, $k = \frac{1}{5}$. Thus, with this value of k, we have
   $\lim_{x\to 0} h(x) = \frac{1}{5}$ which is the same as $h(0)$. Hence h is now continuous at $x = 0$.

9. a. **Refer to similar examples in lectures or textbook.**
   Let $g(x) = 2x^3 + x^2 + 2$. Then $g(-2) < 0$ and $g(-1) = 1 > 0$. Note that g is continuous on the closed and bounded interval $[-2,-1]$. Then apply the *Intermediate Value Theorem*.

   b. Let $g(x) = x - 2\sin(x) - 1$. Then g is continuous on $[0, \frac{3\pi}{2}]$. Now $g(0) = -1 < 0$ and $g(\frac{3\pi}{2}) = \frac{3\pi}{2} - 2\sin(\frac{3\pi}{2}) - 1 = 1 + \frac{3\pi}{2} > 0$. Therefore, by the *Intermediate Value Theorem*, there is a c in $(0, \frac{3\pi}{2})$ such that $g(c) = 0$. Hence $c - 2\sin(c) = 1$.

10. Observe that for x in $[0, 2]$ we have $0 \le f(x) \le 2$. Thus, we have that $0 \le f(0)$ and $f(2) \le 2$. Define a function $g : [0,2] \to \mathbf{R}$ by $g(x) = f(x) - x$.
    Then g is continuous since f is continuous on $[0, 2]$. Now $g(0) = f(0) - 0 \ge 0$ and $g(2) = f(2) - 2 \le 0$ and so by the *Intermediate Value Theorem* there exists a c in $[0, 2]$ such that $g(c) = 0$, that is, $f(c) = c$.

11. (Optional) False. There are plenty of counter examples. For instance, take $g(x) = x^2(x-1)$. Then $g(-1) < 0$ and $g(2) > 0$. But there are exactly two zeros in $[-1, 2]$.

12. (**Optional.**)Discussion after Activity 8 should give some idea of the equivalent definition of continuity:
    f is continuous at $x = a$ if and only if for any sequence $\{x_n\}$ that converges to a we have that the sequence $\{f(x_n)\}$ converges to $f(a)$. That is to say if $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} f(x_n) = f(a)$. We only need the one way implication. This is how the proof goes.
    Suppose f is continuous at $x = a$. That means given any $\varepsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. We shall make use of this implication. Now suppose $\{x_n\}$ converges to $x = a$. That means taking the same $\delta > 0$ above, there is an integer N_δ such

that for all $n > N_\delta$ we have $|x_n - a| < \delta$. Consequently by the above implication $|f(x_n) - f(a)| < \varepsilon$. This means $\{f(x_n)\}$ converges to $f(a)$.

Now for the opposite implication. This is equivalent to: if f is not continuous at $x = a$, then there exists a sequence $\{x_n\}$ that converges to a but $\{f(x_n)\}$ does not converge to $f(a)$. Now f is not continuous at $x = a$ means we can find a $\varepsilon > 0$ such that for any $\delta > 0$, we can find a x_δ such that $|x_\delta - a| < \delta$ but $|f(x_\delta) - f(a)| \geq \varepsilon$. Thus taking $\delta = 1/n$ for n a natural number, and let $x_n = x_\delta$ we have a sequence $\{x_n\}$ that converges to a. This is seen as follows for any $\eta > 0$, there exists an integer N_η such that $1/N_\eta < \eta$. Thus for all $n > N_\eta$ $|x_n - a| < 1/n < 1/N_\eta < \eta$. Therefore, $\{x_n\}$ converges to a. But we have for any natural number n, $|f(x_n) - f(a)| \geq \varepsilon$. Hence $\{f(x_n)\}$ does not converge to $f(a)$. Now we apply this result to our question.

Suppose $f(x) = 0$ for all rational number x. To prove that the function is zero is to prove that $f(x) = 0$ for irrational number. Let a be an irrational number. Then by the density of the rational numbers, for each natural number n there exists a rational number a_n such that $|a_n - a| < 1/n$ (i.e., $a_n \in (a-1/n, a+1/n)$. The density theorem states that between any two numbers, there exists a rational number. Thus between $a-1/n$ and $a+1/n$ there is a rational number a_n .) Then obviously the sequence $\{a_n\}$ converges to a. Therefore, $\{f(a_n)\}$ converges to $f(a)$. But $f(a_n) = 0$ for all natural number n, and so $\{f(a_n)\}$ converges to 0. Therefore, by the uniqueness of limit, $f(a) = 0$. Hence $f(x) = 0$ for all x.

13. (**Optional.**) Recall $f(x + y) = f(x) + f(y)$ for any x and y. Thus for any natural number $n > 1$, $f(n) = f(1 + 1 + \ldots + 1) = f(1) + f(1) + \ldots + f(1) = n\, f(1)$.

Also we have that $f(1) = f(n \cdot 1/n) = f(1/n + 1/n + \ldots + 1/n)$

$$= f(1/n) + f(1/n) + \ldots + f(1/n) = n\, f(1/n).$$

Thus $f(1/n) = f(1)\cdot 1/n$ --------------------- (1)

Now note that $f(0) = f(0 + 0) = f(0) + f(0) = 2\, f(0)$ and so

$f(0) = 0$ --------------- (2)

We then have $0 = f(0) = f(1/n + (-1/n)) = f(1/n) + f(-1/n)$ implying that

$f(-1/n) = -f(1/n) = -f(1)\cdot 1/n = f(1)\,(-1/n)$ ---------------- (3)

Now for any rational number q, q is of the form m/n with $n > 0$. Supose $m > 0$,

$f(m/n) = f(m \cdot 1/n) = f(1/n + 1/n + \ldots + 1/n) = f(1/n) + f(1/n) + \ldots + f(1/n)$ (m times)

$= m\, f(1/n) = m\, f(1)\cdot 1/n = f(1)\cdot (m/n)$ by (1)

Suppose $m < 0$, $f(m/n) = f(-m \cdot (-1/n)) = f((-1/n) + (-1/n) + \ldots + (-1/n))$

$= f(-1/n) + f(-1/n) + \ldots + f(-1/n)$ ($-m$ times)

$= (-m)\, f(-1/n) = (-m)\, f(1)\cdot(-1/n) = f(1)\cdot(m/n)$ by (3).

Since $f(0) = 0$, we have thus proved that $f(m/n) = f(1)\cdot(m/n) = k \cdot (m/n)$, where $k = f(1)$. This means for any rational number x, $f(x) = k\,x$. This proves part (1)

Part (2). Suppose f is continuous on **R**. Then the function $g(x) = f(x) - k\,x$ is continuous on **R**. By part (1), $g(x) = 0$ for all rational number x. Therefore, by Question 12, $g(x) = 0$ for all x. Hence $f(x) = k\,x$ for all x.