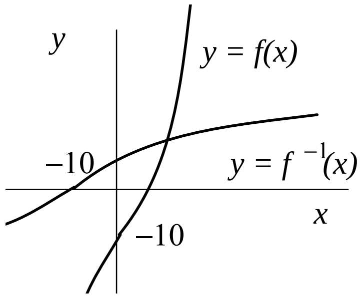
2005/2006 Semester 2 MA1102R Calculus Solution to Tutorial 9

1. a.



b. $f'(x) = 3x^2 + 7$. Thus $f'(1) = 10$. Therefore, the slope of the tangent line at the point (1, -2) is 10.

c. $(f^{-1})'(-2) = \frac{1}{f'(1)} = \frac{1}{10}$. Thus the slope of the tangent to the graph of f^{-1} at the point (-2, 1) is $\frac{1}{10}$.

2. a. Taking logarithm we have $\ln(f(x)) = \tan(x)\ln(x)$. Differentiating $\ln(f(x))$, we have

$$\frac{1}{f(x)} f'(x) = \sec^2(x)\ln(x) + \tan(x)\cdot\frac{1}{x}.$$ Therefore,

$$f'(x) = (x)^{\tan(x)}(\sec^2(x)\ln(x) + \tfrac{1}{x}\tan(x)).$$

b. $g'(x) = \dfrac{1}{\int_0^{\sin^3(x)} \frac{1}{1+t^2+\sin^2(t+t^2)}dt} \cdot \dfrac{3\sin^2(x)\cos(x)}{1+\sin^6(x)+\sin^2(\sin^3(x)+\sin^6(x))}$.

c. $h'(x) = x^{(e^{(x^2+x)})} e^{(x^2+x)}((2x+1)\ln(x) + \frac{1}{x})$.

d. $\ln(u(x)) = \ln(x+1)\ln(\ln(x+1))$. Differentiating $\ln(u(x))$,

$$\frac{u'(x)}{u(x)} = \frac{1}{x+1}\ln(\ln(x+1)) + \ln(x+1)\cdot\frac{1}{\ln(x+1)}\cdot\frac{1}{x+1} = \frac{1}{x+1}(\ln(\ln(x+1)) + 1)$$

Therefore, $u'(x) = (\ln(x+1))^{\ln(x+1)}\frac{1}{x+1}(\ln(\ln(x+1)) + 1)$.

e. $g'(x) = \ln(17)(17^x)^{7^x}7^x(1 + x\ln(7))$.

f. $k(x) = \dfrac{\ln(\ln(x+2)) - \ln(\ln(7))}{\ln(5)}$. Therefore, $k'(x) = \dfrac{1}{\ln(5)(x+2)\ln(x+2)}$.

3. a. $\int x^2 17^{x^3}dx = \int x^2 e^{x^3\ln(17)}dx = \frac{1}{3\ln(17)}\int e^{x^3\ln(17)}3x^2\ln(17)dx = \frac{1}{3\ln(17)}17^{x^3} + C$.

b. $\int xe^{3+2x^2}dx = \frac{1}{4}e^{3+2x^2} + C$.

c. $\int(x+1)e^x 13^{xe^x}dx = \int(x+1)e^x e^{xe^x\ln(13)}dx = \frac{1}{\ln(13)}\int e^{xe^x\ln(13)}\ln(13)(x+1)e^x dx$

$$= \frac{1}{\ln(13)}e^{xe^x\ln(13)} + C = \frac{1}{\ln(13)}13^{xe^x} + C.$$

4. $\tan^{-1}(x+5y) = \ln(y)$. Differentiating implicitly, $\dfrac{1+5\frac{dy}{dx}}{1+(x+5y)^2} = \dfrac{1}{y}\dfrac{dy}{dx}$. Therefore,

$\dfrac{1}{1+(x+5y)^2} + \dfrac{5}{1+(x+5y)^2}\dfrac{dy}{dx} = \dfrac{1}{y}\dfrac{dy}{dx}$. Hence $\dfrac{dy}{dx} = \dfrac{y}{x^2+25y^2+10xy-5y+1}$.

5. a. $\int\dfrac{1}{(x^2+5)(6+x^2)}dx = \int\left(\dfrac{1}{x^2+5} - \dfrac{1}{6+x^2}\right)dx = \frac{1}{\sqrt{5}}\tan^{-1}(\frac{x}{\sqrt{5}}) - \frac{1}{\sqrt{6}}\tan^{-1}(\frac{x}{\sqrt{6}}) + C$.

b. $\int\dfrac{1}{\sqrt{x}\sqrt{8-x}}dx = 2\int\dfrac{1}{\sqrt{8-u^2}}du = 2\sin^{-1}(\dfrac{u}{\sqrt{8}}) + C = 2\sin^{-1}(\sqrt{\dfrac{x}{8}}) + C$, where $u = \sqrt{x}$

. Thus, $\int_4^6\dfrac{1}{\sqrt{x}\sqrt{8-x}}dx = 2(\sin^{-1}(\dfrac{\sqrt{3}}{2}) - \sin^{-1}(\dfrac{1}{\sqrt{2}})) = 2(\dfrac{\pi}{3} - \dfrac{\pi}{4}) = \dfrac{\pi}{6}$.

c. $\int_0^1\dfrac{x}{1+x^4}dx = \dfrac{1}{2}\int_0^1\dfrac{1}{1+u^2}du = \dfrac{1}{2}[\tan^{-1}(u)]_0^1$, where $u = x^2$ is used

$$= \frac{1}{2}(\tan^{-1}(1) - \tan^{-1}(0)) \; = \frac{1}{2}(\frac{\pi}{4} - 0) = \frac{\pi}{8}.$$

6. a. $\int \sec^{-1}(x)dx = x\sec^{-1}(x) - \int x\frac{1}{x\sqrt{x^2-1}}dx = x\sec^{-1}(x) - \int \frac{1}{\sqrt{x^2-1}}dx$

$= x\sec^{-1}(x) - \ln\left|x + \sqrt{x^2-1}\,\right| + C$ using trigonometric substitution $x = \sec(\theta)$.

b. $\int x^2\cos(2x)dx = \frac{1}{2}x^2\sin(2x) - \frac{1}{2}\int \sin(2x)2xdx$

$= \frac{1}{2}x^2\sin(2x) - \int \sin(2x)xdx$

$= \frac{1}{2}x^2\sin(2x) - (-x\frac{1}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)dx)$

$= \frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{2}\int \cos(2x)xdx$

$= \frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{2}(\frac{1}{2}\sin(2x)) + C$

$= \frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C.$

c. $\int \sin(\ln(x^2))dx = x\sin(\ln(x^2)) - \int x\cos(\ln(x^2))\frac{2}{x}dx = x\sin(\ln(x^2)) - 2\int \cos(\ln(x^2))dx.$

$\int \cos(\ln(x^2))dx = x\cos(\ln(x^2)) + \int x(\sin(\ln(x^2))\frac{2}{x})dx = x\cos(\ln(x^2)) + 2\int \sin(\ln(x^2))dx.$

Hence $\int \sin(\ln(x^2))dx = x(\sin(\ln(x^2)) - 2x\cos(\ln(x^2))) - 4\int \sin(\ln(x^2))dx.$

Therefore $\int \sin(\ln(x^2))dx = \frac{1}{5}x(\sin(\ln(x^2)) - 2\cos(\ln(x^2))) + C.$

7. (a) $\int \sqrt{x}\,\tan^{-1}(\sqrt{x})dx = \frac{2}{3}x^{\frac{3}{2}}\tan^{-1}(\sqrt{x}) - \int \frac{2}{3}x^{\frac{3}{2}}\frac{1}{1+x}\frac{1}{2\sqrt{x}}dx$

$= \frac{2}{3}x^{\frac{3}{2}}\tan^{-1}(\sqrt{x}) - \frac{1}{3}\int(1 - \frac{1}{1+x})dx = \frac{2}{3}x^{\frac{3}{2}}\tan^{-1}(\sqrt{x}) - \frac{1}{3}(x - \ln(1+x)) + C.$

$\therefore \int_{\frac{1}{3}}^{3} \sqrt{x}\,\tan^{-1}(\sqrt{x})dx = \frac{1}{3}\left[2x^{\frac{3}{2}}\tan^{-1}(\sqrt{x}) - x + \ln(1+x))\right]_{\frac{1}{3}}^{3}$

$= \frac{1}{3}(2\cdot 3\sqrt{3}\,\tan^{-1}(\sqrt{3}) - 3 + \ln(4) - \frac{2}{3\sqrt{3}}\tan^{-1}(\sqrt{\frac{1}{3}}) + \frac{1}{3} - \ln(\frac{4}{3}))$

$= \frac{1}{3}(6\sqrt{3}\,\frac{\pi}{3} - \frac{2}{3\sqrt{3}}\frac{\pi}{6} - 2\frac{2}{3} + \ln(3)) = \frac{1}{3}(2\sqrt{3}\,\pi - \frac{\sqrt{3}}{27}\pi - 2\frac{2}{3} + \ln(3)).$

(b) $\int \ln(x^2+1)dx = x\ln(x^2+1) - \int x\frac{2x}{1+x^2}dx = x\ln(x^2+1) - 2\int(1 - \frac{1}{1+x^2})dx$

$= x\ln(x^2+1) - 2x + 2\tan^{-1}(x) + C.$

$\therefore \int_0^4 \ln(x^2+1)dx = [x\ln(x^2+1) - 2x + 2\tan^{-1}(x)]_0^4 = 4\ln(17) - 8 + 2\tan^{-1}(4).$

8. (a) $\int \sin^6(3x)\cos(3x)dx = \frac{1}{3}\int \sin^6(3x)3\cos(3x)dx = \frac{1}{3}\int \sin^6(3x)\frac{du}{dx}dx = \frac{1}{3}\int u^6du$

where $u = \sin(3x)$

$= \frac{1}{21}u^7 + C = \frac{1}{21}\sin^7(3x) + C.$

(b) $\int \sin^6(x)\cos^4(x)dx = \frac{1}{16}\int \sin^4(2x)\sin^2(x)dx = \frac{1}{32}\int \sin^4(2x)(1 - \cos(2x))dx$

$= \frac{1}{32}\int \sin^4(2x)dx - \frac{1}{32}\int \sin^4(2x)\cos(2x))dx = \frac{1}{32}\int(\frac{1}{2}(1 - \cos(4x)))^2dx - \frac{1}{320}\sin^5(2x)$

$= \frac{1}{128}\int(1 - 2\cos(4x) + \cos^2(4x))dx - \frac{1}{320}\sin^5(2x)$

$= \frac{1}{128}x - \frac{1}{256}\sin(4x) + \frac{1}{256}\int(1 + \cos(8x))dx - \frac{1}{320}\sin^5(2x)$

$= \frac{1}{128}x - \frac{1}{256}\sin(4x) + \frac{1}{256}x + \frac{1}{2048}\sin(8x) - \frac{1}{320}\sin^5(2x) + C$

$= \frac{3}{256}x - \frac{1}{256}\sin(4x) + \frac{1}{2048}\sin(8x) - \frac{1}{320}\sin^5(2x) + C.$