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Department of Mathematics

Level 1000 Semester 2 (2005/2006) **MA1102R Calculus** Tutorial set 1

“ *Given the cardinal or natural number, it is possible to build up the real and complex number systems, functions, and in fact all of analysis (calculus).”*

Definition 1. A *set* is a collection of objects, which we call *elements* or *members*.

If a is an element of the set *A*, we say *a belongs* to *A* and this is written $a \in A$. If *a* is *not* an element of *A*, then we say *a does not belong* to *A* and this is written $a \notin A$.

Activity 1. 1. Give an example of a set with elements chosen from the integers.

2. Make up a statement, from your example, involving "$\in$" or "$\notin$".

Two sets *A* and *B* are *equal* if they have the same elements and we write $A = B$, otherwise $A \neq B$.

3. **N** = the set of natural numbers $= \{x : x \text{ is a natural number}\} = \{1, 2, 3, \cdots\}$.
Explain the meaning of "$\cdots$" here.

Food for Thoughts: *1. What is the set of natural numbers? This question is not a simple one from the point of view of mathematics. It involves a deeper meaning of the concept of numbers. Ref: The Concept of Number , B Artmann, John Wiley 1982. 2. How do we count the elements of a set? What do we mean by a countable set, an infinite set? If we have a notion of “size” , then can the “size” of the set of real numbers* **R** *be the same as* $\mathbf{R}^2$ *? (Hint: Cantor’s Theorem.) The set of real numbers is fundamentally important in analysis. But its initial discovery is somewhat tumultuous. Hippasus was supposedly tossed into the Mediterranean by fellow pythagoreans for discovering irrational numbers. Readings: Daubney, Journey Through Genius: Chapter 11.*

Examples 1.

1. $A = \{1, 4, 9, 16\}$, $B = \{9, 1, 4, 16\}$, $C = \{1, 2, 4, 8, 16\}$ and $D = \{2^n : n \in \{0, 1, 2, 3, 4\}\}$.
   Then $A = B$ and $C = D$. But $A \neq C$ because $9 \in A$ but $9 \notin C$.

2. $E = \{1, 5, 6, 7, 9\}$ and $F = \{1, 5, 7, 6, 9, 5, 6\}$. Then $E = F$.

Definition 2. *A* is a *subset* of *B* if every element of *A* is also an element of *B*. We write $A \subseteq B$ or $B \supseteq A$ (and we say *B* contains *A*). Thus if *A* has an element which is not an element of *B*, then *A* is not a subset of *B* and we write $A \not\subseteq B$ or $B \not\supseteq A$.

Activity 2. Give an example of three sets *A*, *B* and *C* such that $A \subseteq B$ and $A \subseteq C$ but $B \not\subseteq C$.

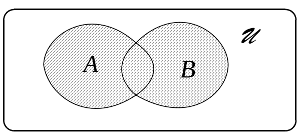
Note : $A \subseteq B$ if and only if *every element of A is also an element of B*, i.e., if and only if the following statement is true:

$$x \in A \Rightarrow x \in B \quad (\Rightarrow \text{ reads "implies"}) \text{ or } \quad \text{if } x \in A, \text{ then } x \in B.$$

Activity 3. Give examples of three sets *A*, *B* and *C* illustrating the following two statements.

1. For any set *A*, $A \subseteq A$, $\phi \subseteq A$.
2. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition 3. The *union* of *A* and *B* is defined by $A \cup B = \{x : x \in A \text{ or } x \in B\}$. This set is indicated by the shaded region in the following Venn diagram.



Activity 4.

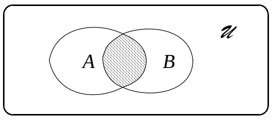
1. Take two finite subsets of the integers and form their union.

2. Take two bounded intervals of the real numbers and form their union. What is the resulting subset? Is it still a bounded interval?

[A bounded interval is a subset of the real numbers of the following form.

$\{x \in \boldsymbol{R} : a < x < b\}$or $\{x \in \boldsymbol{R} : a \le x < b\}$or $\{x \in \boldsymbol{R} : a < x \le b\}$or $\{x \in \boldsymbol{R} : a \le x \le b\}$.]

Definition 4. The *intersection* of A and B is defined by $A \cap B = \{x : x \in A \text{ and } x \in B\}$. It is denoted by the shaded region below.

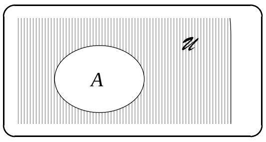


Activity 5.

1. Take two finite subsets of the natural numbers **N** and form their intersection.

2. Take two bounded intervals and form their intersection. What kind of subset of the real numbers can we obtain?

Definition 5. The *complement* of A, $A' = \{x \in \mathcal{U} : x \notin A\}$. It is denoted by the shaded region below.

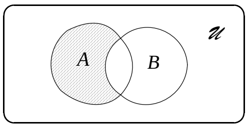


Activity 6.

1. Let $\mathcal{U} = \{1, 2, 3, 4, \cdots, 10\}$ and $A = \{2, 5, 9\}$. What is the complement of A?

2. Let $\mathcal{U} = \mathbf{R}$ and $A = \{x \in \boldsymbol{R} : 3 \le x < 9\}$. Find A'.

Definition 6. The *relative complement* of B in A is $A - B = \{x : x \in A \text{ and } x \notin B\}$. It is denoted by the shaded region below.



Activity 7.

1. Let $A = \{2, 3, 4, 5, 7\}$ let $B = \{1, 5, 7\}$. Find $A - B$.

2. Find $\{x \in \boldsymbol{R} : 1 < x \le 9\} - \{x \in \boldsymbol{R} : 4 \le x < 7\}$.

Activity 8. Give *examples* of sets A, B and C for each of the following statement.

1. If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

2. If $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

Draw a Venn diagram for each statement.

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To know a set we must be able to identify or recognise its elements. Most definitions or statements in set theory are stated in terms of elements. Therefore in order to answer any question which is set theoretic in nature, we must first have a good idea of the elements of the sets involved.

1. Let $A = \{2, 3, 5, 8, 10\}$, $B = \{2, 3, 8\}$, $C = \{2, 5, 10\}$ and $D = \{2, 8\}$.

   a. List the elements of A, B, C and D respectively.

   b. Which elements of B are also elements of A? Is every element of B an element of A? Is B a subset of A?

   c. Which elements of C are also elements of A? Is every element of C an element of A? Is C a subset of A?

   d. Is D a subset of A? Is it also a subset of B and C? Is C a subset of B?

2. For each of the following statements, state whether it is true or false and give reasons.

   a. $\phi = 0$.

   b. $\{1, 9, 2, 7\} \in \{1, 2, 7, 9\}$.

   c. $\{4, 5, 8, 5, 4, 4, 8\} \subseteq \{4, 5, 8\}$.

   d. $\phi \subseteq \{\phi\}$.

   e. $1173 \in \{1173\}$.

   f. $4 = \{4\}$.

   g. $\{x \in \mathbf{R} : 1 \le x < 6\} = \{1, 2, 3, 4, 5\}$

   h. $\{x \in \mathbf{R} : 1 \le x < 1\} = \phi$

   i. $\{37\} \subseteq \{\{37\}\}$.

   Note: Part (a) has surprising answers. They depend on the meaning that you will assign to "0" as mathematical objects and the meaning of "=". Contrast with part (f) and (i).

3. The set $\{x \in \boldsymbol{N} : 3 \le x \le 5\} = \{3, 4, 5\}$. Obtain similar identities for

   a. $\{x \in \boldsymbol{N} : x^2 - 2x - 24 = 0\}$.

   b. $\{x \in \boldsymbol{N} : x^2 - 6x + 10 = 0\}$.

   c. $\{x \in \boldsymbol{N} : x^2 + 7x + 10 = 0\}$.

4. Let the universal set $\mathcal{U} = \{x \in \boldsymbol{R} : 0 < x \le 11\}$ and let

   $A = \{x \in \boldsymbol{R} : 3 \le x \le 6\}$, $B = \{x \in \boldsymbol{R} : 4 \le x \le 9\}$ and $C = \{x \in \boldsymbol{R} : 5 \le x \le 8\}$.

   Find the following sets:

   a. $A \cup B$.

   b. A'.

   c. $A \cap B'$.

   d. $B \cap C'$.

   e. $B - A'$.

   f. $A \cup (B \cap C')$.

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Note that we may use the interval notation here.

$\{x \in \mathbf{R}: a < x < b\}$ is denoted by (a, b); $\{x \in \mathbf{R}: a \le x < b\}$by $[a, b)$

r$\{x \in \mathbf{R}: a < x \le b\}$ by $(a, b]$ and $\{x \in \mathbf{R}: a \le x \le b\}$by $[a, b]$. So the set $\mathcal{U}$ is (0, 11]. Try to express your answer in the interval notation as far as possible.

5. The Cartesian product $X \times Y$ of two sets X and Y is defined by $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$, where (x, y) is an ordered pair. Let $A = \{n \in Z : -2 \le n \le 2\}$, $S = \{(x, y) \in A \times A : x^2 + y^2 \le 5\}$ and $T = \{(x, y) \in A \times A : \left|\frac{1}{4}x - y\right| \le 1\}$. List the elements of $A \times A$, S, T and $S \cap T$.

6. List all the subsets of $A = \{a, \{a\}\}$. (Hint: Recall the meaning of subsets)

Optional Questions

7. Which of the following statements are true? If you think it is false, give a counter-example. If you think it is true, then prove it.

   A. If $A \subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.

   b. If $A \cup B = A \cup C$, then $B = C$.

   c. If $A \cap B = A \cap C$, then $B = C$.

   d. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.

   e. If $A \subseteq B$, then for any set C, $A \times C \subseteq B \times C$.

   f. For any sets A and B, $A \times B = B \times A$.

8. For any sets A and B define the 'symmetric sum' $A \oplus B = A \cup B - A \cap B$. Draw a Venn diagram to represent $A \oplus B$. Prove that

   1. $A \oplus B = B \oplus A$

   2. $A \oplus A = \phi$

   3. $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

   4. $(A \oplus B) \cap C = (A \cap C) \oplus (B \cap C)$

9. Prove that $(A \cup B)\times(C \cup D) = (A\times C) \cup (A\times D) \cup (B\times C) \cup (B\times D)$.