Answer To Tutorial 1 MA1102R Calculus Semester 2 2005/06

1. (b) Elements 2, 3 and 8 of *B* are also elements of *A*.
   Yes, every element of *B* is also an element of *A*. Therefore *B* is a subset of *A*.
   (c) Elements 2 , 5 and 10 are also elements of *A*.
   Yes. *C* is a subset of *A*.
   (d) Yes, *D* is a subset of *A*. No, B is not a subset of *C* because 3 and 8 are B but not C.
   C is not a subset of *B* because 5 and 10 are in *C* but they are not elements of *B*.

2. (a) False; (b) False; (c) True; (d) True; (e) True; (f) False; (g) False; (h) True; (i) False.
   (a) and (f) are meant to provoke some thoughts. Justification is not expected.
   [Answer to (a) has a story to tell. First of all what is $\varnothing$? and what is 0? Consider the following. They are supposed to be mathematical objects. Both are impossible to define, to be exact. If we take the view all objects in mathematics must be constructs of set theory, then we will need a set theory to begin with. Set theory begins with the postulate of existence of empty set. Using this set we can construct a model for the set of natural numbers. In this way $0 = \varnothing$ can be taken to be a definition and therefore (a) is true. But this can only be supported by a definition of the natural number system according to VON NEUMANN starting from the empty set $\varnothing$ so that the number concept is complete with zero 0 being defined as the empty set. On the other hand if 0 is taken as in the definition of the natural number system (which include 0 and which has 0 as the starting number) using the existence of a set satisfying the Peano Axioms, a kind of formal language describing the natural number systems, it is not clear then that $\varnothing = 0$. Note that in this consideration the number 0 is just the concept of a counting number as in "there are no elements in the empty set". No particular choice of a model for the natural number system only properties characterising the natural number system. Then just by these differing paths of consideration we may say that the answer is false. A further twist is as follows: if 0 is the number 0 in the integers, then a standard construction of the integers will place 0 as an equivalence class in $\mathbf{N}\times\mathbf{N}$, with **N** as the VON NEUMANN's natural numbers ($0 = \varnothing$, $1 = \{\varnothing\}$, $2 = \{\varnothing,\{\varnothing\}\}$, $3 =\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$, ..., $n+1 = n\cup\{n\}$...) and so it is not empty and so (a) is false). What kind of mathematical object do we view the number 0 as? 0 as in the integers? 0 as in the rational numbers? or 0 as in the real numbers? If we take the point of view that all mathematical objects are constructs in an appropriate kind of set theory, then we can have the different answers above. See the reference *The Concept of Number by Artman*]
   (f) False. Again the comparison "=" has to have appropriate interpretation. If we separate the symbol 4 from the set {4}, then we can argue that they are not the same. We can base this on the axiomatic definition of 4, 4 being the successor of 3, simply as formal language describing the property of being 4. If mathematics were to be described as constructs of set theory (even this proves to be difficult, requiring for example Zermelo Fraenkel axiomatisation of set theory), then we can put both 4 and {4} in a particular model starting from a model that can describe 4. If we use Von Neumann's natural numbers, then 4 is defined to be $\{\varnothing, \{\varnothing\}, \{\varnothing,\{\varnothing\}\}, \{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}\}$ and {4} is then equal to $\{ \{\varnothing, \{\varnothing\}, \{\varnothing,\{\varnothing\}\}, \{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}\} \}$. Note that the former has 4 elements and the later has only one elements. Therefore, the two sets are not the same. Here we are taking the point of view the meaning of 4 as a natural number, a number in an inductive set. On the other hand if 4 were to be equal to {4} since $4 \in \{4\} = 4$ this would imply $4 \in 4$. But this can not happen in Zermelo-Fraenkel set theory, where the axiom of regularity does not allow any statement of the form $x \in x$.

3. (a) $\{x \in \boldsymbol{N} : x^2 - 2x - 24 = 0\} = \{x \in \boldsymbol{N} : (x-6)(x+4) = 0\} = \{6\}$.
   (b) $\{x \in \boldsymbol{N} : x^2 - 6x + 10 = 0\} = \{x \in \boldsymbol{N} : (x-3)^2 + 1 = 0\} = \phi$ because for any real x $(x-3)^2 + 1 \ge 0 + 1 \ge 1 > 0$.
   (c) $\{x \in \boldsymbol{N} : x^2 + 7x + 10 = 0\} = \{x \in \boldsymbol{N} : (x+5)(x+2) = 0\} = \phi$.
4. (a) $A \cup B = \{x \in \boldsymbol{R} : 3 \le x \le 9\} = [3, 9]$.
   (b) $A' = \{x \in \boldsymbol{R} : 0 < x < 3 \text{ or } 11 \ge x > 6\} = (0, 3) \cup (6, 11]$.
   (c) $A \cap B' = A - B = \{x \in \boldsymbol{R} : 3 \le x < 4\} = [3, 4)$.
   (d) $B \cap C' = B - C = \{x \in \boldsymbol{R} : 4 \le x < 5 \text{ or } 8 < x \le 9\} = [4, 5) \cup (8, 9]$.
   (e) $B - A' = B \cap (A')' = B \cap A = \{x \in \boldsymbol{R} : 4 \le x \le 6\} = [4, 6]$.
   (f) $A \cup (B \cap C') = \{x \in \boldsymbol{R} : 3 \le x \le 6 \text{ or } 8 < x \le 9\} = [3, 6] \cup (8, 9]$.

5. $A = \{-2, -1, 0, 1, 2\}$. Therefore $A \times A$ has 25 points and is given by

$$\begin{aligned} A \times A = \{&(-2,-2), (-2,-1), (-2,0), (-2,1), (-2,2), (-1,-2), (-1,-1), (-1,0), (-1,1), (-1,2), \\ &(0,-2), (0,-1), (0,0), (0,1), (0,2), (1,-2), (1,-1), (1,0), (1,1), (1,2), \\ &(2,-2), (2,-1), (2,0), (2,1), (2,2)\} . \end{aligned}$$

$S = A \times A - \{(-2,-2), (-2,2), (2,-2), (2,2)\}$.

$T = \{(-2,-1), (-2,0), (-1,-1), (-1,0), (0,0), (0,1), (0,-1), (1,0), (1,1),$ $(2,0), (2,1)\}$. (Note: $x/4 - 1 \le y \le x/4 + 1$.)

$S \cap T = T$.

S and T can also be described geometrically.
S = All points on or inside the circle centred at (0,0) of radius $\sqrt{5}$.
T = All points bounded between the parallel lines $y = x/4 + 1$ and $y = x/4 - 1$.

6. The subsets of $A = \{a, \{a\}\}$ are $\phi, \{a\}, \{\{a\}\}, \{a, \{a\}\}$.

7. (a) False. Counterexample: Take $A = \{1\}, B = \{1, 2\}, C = \{1, 3\}$. Then $A \subseteq B, B \not\subseteq C$ but $A \subseteq C$.
   (b) False. Counterexample: Take $A = \{1, 2\}, B = \{1\}$ and $C = \{2\}$. Then $A \cup B = A \cup C = C$ but $B \ne C$.
   (c) False. Counterexample: Take $A = \{1\}, B = \{1, 2\}, C = \{1, 3\}$. Then $A \cap B = A \cap C = A$ but $B \ne C$.
   (d) True.

$$\begin{aligned} B &= B \cap (A \cup B) = B \cap (A \cup C) \text{ since } A \cup B = A \cup C \\ &= (B \cap A) \cup (B \cap C) = (A \cap C) \cup (B \cap C) \text{ since } B \cap A = A \cap C \\ &= (A \cup B) \cap C = (A \cup C) \cap C \text{ since } A \cup B = A \cup C \\ &= C \end{aligned}$$

<u>OR.</u>

$$\begin{aligned} x \in B &\Rightarrow x \in A \cup B \Rightarrow x \in A \cup C \text{ since } A \cup B = A \cup C \\ &\Rightarrow x \in A \text{ or } x \in C \Rightarrow x \in A \cap B \text{ or } x \in C \\ &\Rightarrow x \in A \cap C \text{ or } x \in C \text{ since } A \cap B = A \cap C \\ &\Rightarrow x \in C \text{ or } x \in C \Rightarrow x \in C. \end{aligned}$$

Thus $B \subseteq C$. In exactly the same way we can show that $C \subseteq B$. Thus $B = C$.

(e) True. $(a, c) \in A \times C \Rightarrow a \in A$ and $c \in C \Rightarrow a \in B$ and $c \in C$ since $A \subseteq B$.
$\Rightarrow (a, c) \in B \times C$.
(f) False. Take $A = \{1\}, B = \{2\}$. Then $A \times B = \{(1, 2)\} \ne \{(2, 1)\} = B \times A$.

8. First note that we may express $A \oplus B = (A \cap B') \cup (A' \cap B)$.
   (1) $A \oplus B = (A \cap B') \cup (A' \cap B) = (A' \cap B) \cup (A \cap B') = (B \cap A') \cup (A' \cap B) = B \oplus A$.
   OR $A \oplus B = A \cup B - A \cap B = B \cup A - B \cap A = B \oplus A$.
   (2) $A \oplus A = A \cup A - A \cap A = A - A = \phi$.

(3) First show that

$$(A\oplus B)\oplus C=(A'\cap B'\cap C)\cup(A\cap B'\cap C')\cup(A'\cap B\cap C')\cup(A\cap B\cap C).$$

Then $A\oplus(B\oplus C)=(B\oplus C)\oplus A$ by part (1)

$$\begin{aligned}&=(B'\cap C'\cap A)\cup(B\cap C'\cap A')\cup(B'\cap C\cap A')\cup(B\cap C\cap A)\\&=(A'\cap B'\cap C)\cup(A\cap B'\cap C')\cup(A'\cap B\cap C')\cup(A\cap B\cap C)\\&=(A\oplus B)\oplus C.\end{aligned}$$

4.$$\begin{aligned}(A\cap C)\oplus(B\cap C)&=[(A\cap C)\cap(B\cap C)']\cup[(A\cap C)'\cap(B\cap C)]\\&=(A\cap C\cap B')\cup(A'\cap B\cap C)\\&=(A\cap B'\cap C)\cup(A'\cap B\cap C)\\&=[(A\cap B')\cup(A'\cap B)]\cap C\\&=(A\oplus B)\cap C\end{aligned}$$

OR $$\begin{aligned}(A\cap C)\oplus(B\cap C)&=(A\cap C)\cup(B\cap C)-(A\cap C)\cap(B\cap C)\\&=(A\cup B)\cap C-(A\cap B)\cap C\\&=((A\cup B)-(A\cap B))\cap C\\&=(A\oplus B)\cap C\end{aligned}$$

9. $(x, y)\in(A\cup B)\times(C\cup D)\Leftrightarrow x\in A\cup B$ and $y\in C\cup D$

$\Leftrightarrow$ ($x\in$ A or $x\in$ B) and ($y\in$ C or $y\in$ D)

$\Leftrightarrow$ ($x\in$ A and ($y\in$ C or $y\in$ D)) or ($x\in$ B and ($y\in$ C or $y\in$ D))

$\Leftrightarrow$ (($x\in$ A and $y\in$ C) or ($x\in$ A and $y\in$ D)) or (($x\in$ B and $y\in$ C) or ($x\in$ B and $y\in$ D))

$\Leftrightarrow$ ($x\in$ A and $y\in$ C) or ($x\in$ A and $y\in$ D) or (($x\in$ B and $y\in$ C) or ($x\in$ B and $y\in$ D)

$\Leftrightarrow$ $(x, y)\in A\times C$ or $(x, y)\in A\times D$ or $(x, y)\in B\times C$ or $(x, y)\in B\times D$

$\Leftrightarrow$ $(x, y)\in(A\times C)\cup(A\times D)\cup(B\times C)\cup(B\times D)$.