National University of Singapore

Department of Mathematics

level 1000 Semester 2 (2005/2006) MA1102R Calculus Tutorial set 5

"Bolzano (1817) was the first to give the definition of derivative and recognised the distinction between continuity and differentiability. He in 1834 gave an example of a continuous function which does not possess a finite derivative at any point, in a work that was edited and published by Rychlik in Prague much later in 1930. For 50 years or so mathematicians believed all continuous function to be differentiable except perhaps for isolated points. This example of Bolzano was not noticed then. Weierstrass in 1872 in his lecture to the Berlin Academy gave the example of a continuous function which is nowhere differentiable and the example was published in 1875 in the Journal für Mathematik 79. Many more of this type of pathological example followed. From then on mathematicians became all the more fearful of trusting intuition and geometrical thinking."

Definition 1. Let I be an open interval and x_0 be a point in I. We say a function $f: I \to \mathbf{R}$ defined on I is *differentiable at* x_0 in I if the limit $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists. (To check this we usually show that the left and the right limits exist and are the same.) If this limit exists, it is called the *derivative* of f at x_0 and is written as $f'(x_0)$ or $\frac{d}{dx} f(x)|_{x=x_0}$ or $D_x f(x_0)$, i.e., $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.

The derivative may also be defined by $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

Geometrically, the tangent line to the curve $y = f(x)$ at x_0 has gradient $f'(x_0)$ and the tangent line is given by $\frac{y-f(x_0)}{x-x_0} = f'(x_0)$.

If $f: I \to \mathbf{R}$ is differentiable at x for all x in I, then we say f is *differentiable on an open interval I*.

Activity 1. Let $f: \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = 9x + 5$. Show that f is differentiable at $x = 3$ and find the derivative there.

Theorem 1. Suppose f is defined on an open interval I containing a point a. If f is *differentiable* at a, then f is *continuous* at a.

(It follows from this that if f is differentiable on I then f is continuous on I. The converse of this statement, I fear is false, not even if you try to modify it to say if f is continuous on I , then f is differentiable at some point in I. See my note on the Calculus Web site: http://www.math.nus.edu.sg/~matngtb/Calculus/Continuity_Differential/Continuity_Differential.htm)

Activity 2. 1. Explain why the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases}$ is *not* differentiable at $x = 0$. (You may use the logic in Theorem 1.)

2. Let $f(x) = \begin{cases} 2x+1, & x \ge 1 \\ 5x^2, & x < 1 \end{cases}$. Explain why f is differentiable on $(-\infty, 1) \cup (1, \infty)$..

Theorem 2. Let f and g be defined on an open interval containing x_0. L et λ and μ be any real numbers. Then if f and g are differentiable at x_0 ,

1. $\frac{d}{dx}\{\lambda f + \mu g\}|_{x=x_0} = \lambda \frac{d}{dx} f\,|_{x=x_0} + \mu \frac{d}{dx} g|_{x=x_0}$

2. $\frac{d}{dx}\{f \cdot g\}|_{x=x_0} = (\frac{d}{dx} f\,|_{x=x_0}) \cdot g(x_0) + f(x_0) \cdot (\frac{d}{dx} g\,|_{x=x_0})$, (Product rule)

3. if $g(x_0) \neq 0$, $\frac{d}{dx}\left\{\frac{f}{g}\right\}|_{x=x_0} = \dfrac{\left(\frac{d}{dx} f\,|_{x=x_0}\right) \cdot g(x_0) - f(x_0) \cdot \left(\frac{d}{dx} g\,|_{x=x_0}\right)}{(g(x_0))^2}$, (Quotient rule) .

Theorem 3 (Chain Rule). Let $f : I \to \mathbf{R}$ be a function defined on an open interval I. Suppose $f(I) \subseteq J$, where J is an open interval. Let $g: J \to \mathbf{R}$ be a function defined on J. Then we have the composite $g \circ f : I \to \mathbf{R}$ defined by $g \circ f(x) = g(f(x))$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and $\mathbf{(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)}$ (or $\frac{d}{dx}(g \circ f)|_{x=x_0} = (\frac{d}{dx} g|_{f(x_0)}) \cdot (\frac{d}{dx} f|_{x_0})$ or $D(g \circ f)(x_0) = Dg(f(x_0))\, Df(x_0) = Dg(y_0)\, Df(x_0)$. This is usually remembered in the form $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$, where $z(x) = g \circ f(x)$ and $y(x) = f(x)$.

Activity 3. Let $h(x) = 3 + 7x^2$ and $g(x) = x^5$. Use the *Chain Rule* to determine the derivative of $f(x) = g \circ h(x) = (3 + 7x^2)^5$.

Activity 4. Suppose f and g are two differentiable functions such that $f(0) = 2$, $g(0) = 5$, $f'(0) = 3$, $f'(5) = g'(0) = 2$. Find the values of (a) $(3f + g)'(0)$; (b) $(fg)'(0)$; (c) $\left(\frac{3f}{g}\right)'(0)$; (d) $(f \circ g)'(0)$.

Theorem 4. $\frac{d}{dx}\sin(x) = \cos(x)$; $\frac{d}{dx}\cos(x) = -\sin(x)$; $\frac{d}{dx}\tan(x) = \sec^2(x)$.

Activity 5. 1. Show that $\frac{d}{dx}\cot(x) = -\csc^2(x)$.

2. Find the derivative of $F(x) = \sin(5x^3 - x)$ ..

3. Consider the equation $y^3 + y = x$. Find $\frac{dy}{dx}$ implicitly.

MA1102R Calculus **Assignment 5**

1. Determine if the function f defined below is differentiable at $x = 0$. If it is, what is $f'(0)$?

$$f(x) = \begin{cases} x^3 \sin(\frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Determine the derived function f'. Is f' continuous at $x = 0$?

2. Each of the following limits is the derivative of a function f at some point a. Find the function f and the point a.

a. $\lim\limits_{x \to 2} \dfrac{1/(x-1)^2 - 1}{x-2}$. b. $\lim\limits_{h \to 0} \dfrac{\sin(h^2)}{h}$. c. $\lim\limits_{h \to 0} \dfrac{\sqrt{h+1} - (1)}{h}$.

[Hint: Think of the definition $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x-a}$ and come up with a possible function f and the point a.]

For Question 3 and 4 you might want to check out my note on differentiability,
http://www.math.nus.edu.sg/~matngtb/Calculus/Derivedfunction/Derivative.htm

3. Find, if it exists, the derivative f' of each of the following functions.

a. $f(x) = |x|^3$. b. $f(x) = \begin{cases} x^4 + 3, & x \leq 1 \\ 3x^2, & x > 1 \end{cases}$.

4. Let $f(x) = \begin{cases} ax^2 + 3, & x < 1 \\ bx^3, & x \geq 1 \end{cases}$. Find the values of a and b for which the derivative $f'(1)$ exists.

5. Compute the derivative of each of the following functions.

a. $f(x) = \cot(x^2) + \sec(x)$. b. $f(x) = \dfrac{\cos(3x) + 1}{\cos(2x) - 1}$. c. $f(x) = \sin^2(7x)\cos^3(x)$.

d. $f(x) = \left(\dfrac{7x+1}{x^2+x+1}\right)^4$. e. $f(x) = \cos^3(\cos(5x))$. f. $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

6. If g is a twice differentiable function and suppose $f(x) = x\, g(x^2)$, find f'' in terms of g' and g''.

7. Find $\dfrac{d^2y}{dx^2}$ in its simplest terms when

a. $x^6 + y^6 = 1$. b. $y^3 + x = \sin(y)$.

8. **(Optional)** A model for the anticipated population growth of a small town development is as follows (p is the number of people after t years) $p(t) = \dfrac{50000e^{0.6t}}{9 + e^{0.6t}}$

A conservation consultant uses a pollution index to help predict the effect of urbanisation on environment. The consultant uses a simple model to estimate the value of the pollution index, I units.

$$I(p) = \sqrt{p}\ .$$

(a) What is the upper limit on the population? How long will it take to reach 90% of this limit?

(b) What would be the pollution index when the population reaches its upper limit? What population would give a pollution index of 100? How long does it take for the pollution index to reach 100?

(c) What is the formula for the rate at which the pollution index grows with time, $I'(t)$.?