

Expanding and removing brackets

In this unit we see how to expand an expression containing brackets. By this we mean to rewrite the expression in an equivalent form without any brackets in. Fluency with this sort of algebraic manipulation is an essential skill which is vital for further study.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature. To help you to achieve this, the unit includes a substantial number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- expand expressions involving brackets

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1. Introduction

In this unit we will see how expressions involving brackets can be written in equivalent forms without any brackets. This process is called ‘expanding’ or ‘removing’ brackets, and is an important algebraic skill. We motivate the study of this topic by first recalling an investigation that you may have met before at school.

2. Frogs

There is a well-known investigation at GCSE level called ‘frogs’. Some of you might well have attempted this particular investigation. We want to start this section by having a look at this investigation because it will show us why we might want to use brackets.

Study Figure 1 which shows three 10 pence coins and three pounds coins.

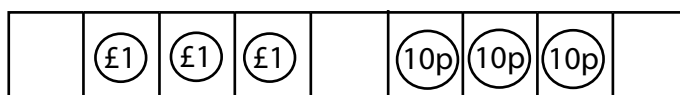


Figure 1.

We want to try and interchange the pound coins on the left side with the 10-pence coins on the right side. We have to do this by using one of two kinds of move. We can either slide into an empty space or we can hop, or jump, over a coin of the opposite kind. We shall do this in stages, and keep count of the number of hops, the number of slides, and the total number of moves.

We can slide the right-most £1 coin into the vacant space on its right. Then we can hop with the left-most 10p coin into the space vacated by the £1 coin, as shown in Figure 2.

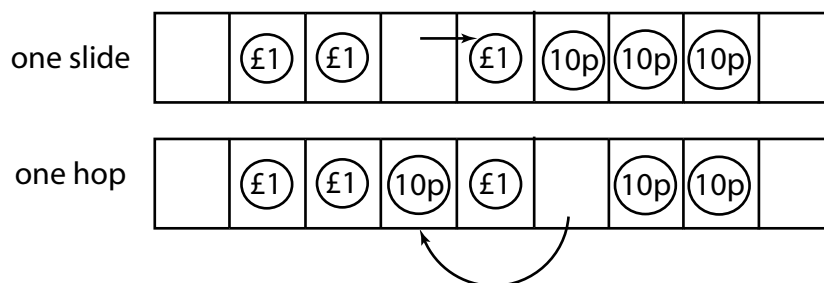


Figure 2.

We now continue in this fashion and at each stage record the number of slides and hops. The details are summarised in Figure 3. If we add up the total numbers of slides and hops we find 6 slides and 9 hops, a total of 15 moves.

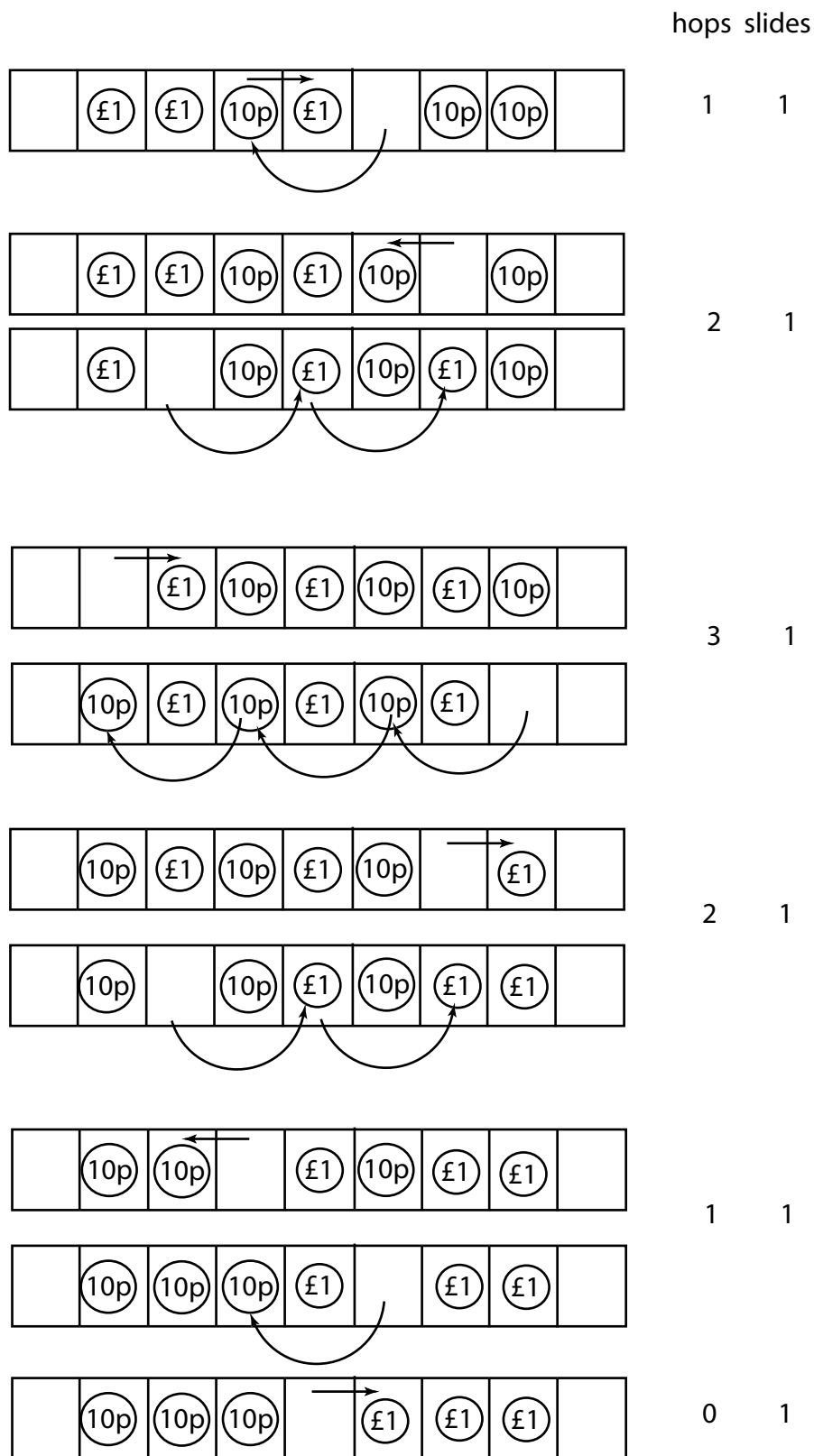


Figure 3.

If we were to repeat this game with a different number of coins on each side we would find the number of slides and hops as given in Table 1.

Table 1

number of coins on each side (n)	Hops	Slides	Moves
1	1	2	3
2	4	4	8
3	9	6	15
4	16	8	24
5	25	10	35

Now the object of most investigations is to try and arrive at a prediction. Can we say what the result would be if we had 10 coins on each side, or 50 coins on each side? This is the power that mathematics gives us - the power to model in symbols and to be able to use those symbols to make our predictions. Can we make that prediction?

Look at the numbers in the column headed 'Moves'. Notice that $3 = 1 \times 3$, $8 = 2 \times 4$, $15 = 3 \times 5$ and so on. It would appear that the total number of moves is equal to the number of coins, multiplied by the number of coins plus 2.

So if we let the number of coins be n , then the total number of moves appears to be $n \times (n + 2)$, or simply $n(n + 2)$.

From Table 1, notice also that the number of hops appears to be n^2 . The number of slides appears to be $2n$. This information is summarised in Table 2.

Table 2

number of coins (n)	Hops	Slides	Moves	
1	$1 = 1^2$	$2 = 2 \times 1$	3	$= 1 \times 3$
2	$4 = 2^2$	$4 = 2 \times 2$	8	$= 2 \times 4$
3	$9 = 3^2$	$6 = 2 \times 3$	15	$= 3 \times 5$
4	$16 = 4^2$	$8 = 2 \times 4$	24	$= 4 \times 6$
5	$25 = 5^2$	$10 = 2 \times 5$	35	$= 5 \times 7$
n	n^2	$2n$	$n(n + 2)$	

Now the sum of the number of hops and the number of slides must equal the total number of moves and so we have

$$n^2 + 2n = n(n + 2)$$

So, this shows us how we can remove brackets. On the right hand side, the n outside must multiply both terms inside the bracket in order to give us the result $n^2 + 2n$.

3. Examples of expanding brackets

Example 1

Expand $3(x + 2)$.

The 3 outside must multiply both terms inside the brackets:

$$3(x + 2) = 3x + 6$$

Example 2

Expand $x(x - y)$.

The x outside must multiply both terms inside the brackets:

$$x(x - y) = x^2 - xy$$

Example 3

Expand $-3a^2(3 - b)$.

Both terms inside the brackets must be multiplied by $-3a^2$:

$$-3a^2(3 - b) = -9a^2 + 3a^2b$$

Example 4

Expand $-2x(x - y - z)$.

All terms inside the brackets must be multiplied by $-2x$:

$$-2x(x - y - z) = -2x^2 + 2xy + 2xz$$



Key Point

The term outside the brackets multiplies each term inside the brackets.

$$a(\overset{\curvearrowright}{b+c}) = ab + ac \quad (\overset{\curvearrowright}{b+c})a = ab + ac$$

Exercises

1. Remove the brackets from the following expressions.

- a) $5(x + 4)$ b) $2(y - 3)$ c) $4(3 - a)$ d) $x(2 + x)$
e) $p(q + 3)$ f) $-3(2 + a)$ g) $s(t - s)$ h) $-2(b - 3)$
i) $5a(2b + 3c)$ j) $-y(2x - 5y)$ k) $4(x + 2y - 3z)$ l) $-2a(3a - 5b + 2c)$

2. Simplify the following expressions.

- a) $5 + 2(x + 1)$ b) $3x + 4(2x - 1)$ c) $6a - 3(a + 2)$ d) $x^2 - x(1 + x)$
e) $12c - 5(2c - 1)$ f) $4(y + 3) - 2y$ g) $5pq - p(2 - p)$ h) $7r + 2(3 + 4r)$
i) $5(2a - b) + 3(4b - 3a)$

4. Multiplying together two bracketed terms

Let us now have a look at what happens if we want to multiply out expressions where there are two brackets multiplying each other.

Suppose we wish to expand $(x + 5)(x + 10)$.

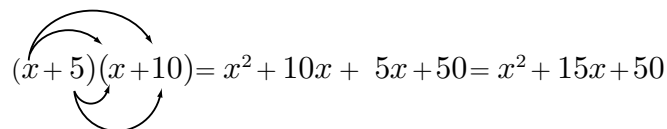
We imagine that the term $(x + 5)$ is a single quantity and use it to multiply both the x and the 10 in the second pair of brackets:

$$\begin{aligned}(x + 5)(x + 10) &= (x + 5)x + (x + 5)10 \\ &= x^2 + 5x + 10x + 50 \\ &= x^2 + 15x + 50\end{aligned}$$

Having seen how to do this, we can shorten the process:

To find $(x + 5)(x + 10)$:

We must ensure that each term in the first bracket multiplies each term in the second. The arrows in the figure below help us to see that all terms have been taken into account:

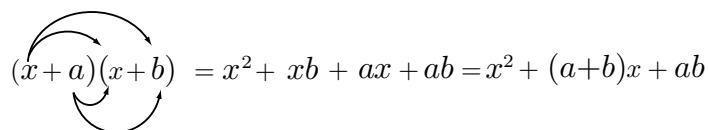

$$(x + 5)(x + 10) = x^2 + 10x + 5x + 50 = x^2 + 15x + 50$$

We can use this process for bracketed expressions like these that lead us to quadratic expressions, and for multiplying pairs of brackets together.



Key Point

When multiplying out two brackets, each containing two terms, we must ensure that each term in the first bracket multiplies each term in the second:


$$(x + a)(x + b) = x^2 + xb + ax + ab = x^2 + (a+b)x + ab$$

Example 5

Expand $(x - 7)(x - 10)$.

$$\begin{aligned}(x - 7)(x - 10) &= x^2 - 10x - 7x + 70 \\ &= x^2 - 17x + 70\end{aligned}$$

Example 6Expand $(x + 6)(x - 6)$.

$$\begin{aligned}(x + 6)(x - 6) &= x^2 - 6x + 6x - 36 \\ &= x^2 - 36\end{aligned}$$

Example 7Expand $(2x - 3)(x + 1)$.

$$\begin{aligned}(2x - 3)(x + 1) &= 2x^2 + 2x - 3x - 3 \\ &= 2x^2 - x - 3\end{aligned}$$

Example 8Expand $(3x - 2)(3x + 2)$.

$$\begin{aligned}(3x - 2)(3x + 2) &= 9x^2 + 6x - 6x - 4 \\ &= 9x^2 - 4\end{aligned}$$

Example 9Expand $(x^2 - 2x^3 + 8)(x + 2)$.

$$\begin{aligned}(x^2 - 2x^3 + 8)(x + 2) &= x^3 + 2x^2 - 2x^4 - 4x^3 + 8x + 16 \\ &= -2x^4 - 3x^3 + 2x^2 + 8x + 16\end{aligned}$$

Example 10Expand $(x^2 + x - 2)(x^2 + x - 6)$.

$$\begin{aligned}(x^2 + x - 2)(x^2 + x - 6) &= x^4 + x^3 - 6x^2 + x^3 + x^2 - 6x - 2x^2 - 2x + 12 \\ &= x^4 + 2x^3 - 7x^2 - 8x + 12\end{aligned}$$

Exercises

3. Expand each of the following.

- | | | |
|-------------------------|-----------------------------------|------------------------------------|
| a) $(x + 2)(x + 3)$ | b) $(a + b)(c + 3)$ | c) $(y - 3)(y + 2)$ |
| d) $(2x + 1)(3x - 2)$ | e) $(3x - 1)(3x + 1)$ | f) $(5x - 1)(x - 5)$ |
| g) $(2p + 3q)(5p - 2q)$ | h) $(x + 2)(2x^2 - x - 1)$ | i) $(4p + 3)(2p - q - 5)$ |
| j) $(2z + 3)(2z + 3)$ | k) $(x^2 - 2x + 1)(x^2 + 4x + 3)$ | l) $(3x^2 - 2x + 1)(x^2 - 4x - 5)$ |

5. Dealing with nested brackets

Sometimes we have collections of expressions nested in various sets of brackets.

Example 11

Simplify $a - (b - c) + a + (b - c) + b - (c - a)$.

$$\begin{aligned}a - (b - c) + a + (b - c) + b - (c - a) &= a - b + c + a + b - c + b - c + a \\ &= 3a + b - c\end{aligned}$$

Example 12

In this Example we have a look at some nested brackets.

Simplify $-\{5x - (11y - 3x) - [5y - (3x - 6y)]\}$.

$$\begin{aligned}-\{5x - (11y - 3x) - [5y - (3x - 6y)]\} &= -\{5x - (11y - 3x) - [5y - 3x + 6y]\} \\ &= -\{5x - 11y + 3x - 5y + 3x - 6y\} \\ &= -\{11x - 22y\} \\ &= -11x + 22y\end{aligned}$$

Example 13

Simplify $3b - \{5a - [6a + 2(10a - b)]\}$.

$$\begin{aligned}3b - \{5a - [6a + 2(10a - b)]\} &= 3b - \{5a - [6a + 20a - 2b]\} \\ &= 3b - \{5a - [26a - 2b]\} \\ &= 3b - \{5a - 26a + 2b\} \\ &= 3b - \{-21a + 2b\} \\ &= 3b + 21a - 2b \\ &= b + 21a\end{aligned}$$

Exercises

4. Simplify the following expressions:

$$\text{a) } 3a - 2(a + b) + 5b + 3(2b - a) \quad \text{b) } 4x - 2[5y - x + 3(2x - y)]$$

Answers

- a) $5x + 20$ b) $2y - 6$ c) $12 - 4a$ d) $2x + x^2$
1. e) $pq + 3p$ f) $-6 - 3a$ g) $st - s^2$ h) $-2b + 6$
i) $10ab + 15ac$ j) $-2xy + 5y^2$ k) $4x + 8y - 12z$ l) $-6a^2 + 10ab - 4ac$

- a) $7 + 2x$ b) $11x - 4$ c) $3a - 6$ d) $-x$
 2. e) $2c + 5$ f) $2y + 12$ g) $5pq - 2p + p^2$ h) $15r + 6$
 i) $a + 7b$
- a) $x^2 + 5x + 6$ b) $ac + 3a + bc + 3b$ c) $y^2 - y - 6$
 d) $6x^2 - x - 2$ e) $9x^2 - 1$ f) $5x^2 - 26x + 5$
 3. g) $10p^2 + 11pq - 6q^2$ h) $2x^3 + 3x^2 - 3x - 2$ i) $8p^2 - 4pq - 14p - 3q - 15$
 j) $4z^2 + 12z + 9$ k) $x^4 + 2x^3 - 4x^2 - 2x + 3$ l) $3x^4 - 14x^3 - 6x^2 + 6x - 5$
4. a) $-2a + 9b$ b) $-6x - 4y$