Factorising quadratics

An essential skill in many applications is the ability to factorise quadratic expressions. In this unit you will see that this can be thought of as reversing the process used to 'remove' or 'multiply-out' brackets from an expression.

You will see a number of worked examples followed by a discussion of special cases which occur frequently with which you must become familiar.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that all this becomes second nature, and you eventually carry out the process of factorising simply by inspection. To help you to achieve this, the unit includes a substantial number of such exercises.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- factorise quadratic expressions

- understand what is meant by, and factorise, a 'difference of two squares'

- understand what is meant by, and factorise, a 'complete square'

Contents

1. Introduction

In this unit you will learn how many quadratic expressions can be factorised. Essentially, this is the reverse process of removing brackets from expressions such as $(x + 2)(x + 3)$.

2. Multiplying-out brackets, and quadratic expressions

You will be familiar already with the well-known process of multiplying-out brackets. For example, you will have seen expressions like $(x + 2)(x + 3)$ and then expanded these terms to arrive at a quadratic expression. Let us see how this works. The arrows in the figure below show how each term in the first pair of brackets multiplies each term in the second pair.

$$(x + 2)(x + 3) = x^2 + 3x + 2x + 6$$
$$= x^2 + 5x + 6$$

Then the like-terms, $3x$ and $2x$, are collected together to give $5x$. The result of multiplying-out the brackets is the **quadratic expression** $x^2 + 5x + 6$.

Key Point

A **quadratic expression** has the general form $ax^2 + bx + c$ where a, b and c are numbers. Note that in a quadratic expression the highest power of x is 2. The number a is called the **coefficient** of x^2, b is called the **coefficient** of x, and c is called the **constant term**. These numbers can be positive or negative. The numbers b and c can also be zero.

Some examples of quadratic expressions:

$$x^2 - 7x + 11, \qquad 4x^2 + 3x - 1, \qquad x^2 + 8x, \qquad 3x^2 + 2$$

Like many processes in mathematics, it is useful to be able to go the other way. That is, starting with the quadratic expression $x^2 + 5x + 6$, can we carry out a process which will result in the form $(x + 2)(x + 3)$? The answer is: yes we can! This process is called **factorising the quadratic expression**. This would help, for example, if we wanted to solve a quadratic equation. Such an equation is formed when we set a quadratic expression equal to zero, as in

$$x^2 + 5x + 6 = 0$$

This is because the equation can then be written

$$(x + 2)(x + 3) = 0$$

and if we have two expressions multiplied together resulting in zero, then one or both of these must be zero. So, either $x + 2 = 0$, or $x + 3 = 0$, from which we can conclude that $x = -2$, or $x = -3$. We have found the solutions of the quadratic equation $x^2 + 5x + 6 = 0$.

So, the ability to factorise a quadratic is a useful basic skill, which you will learn about in this unit. However, be warned, not all quadratics will factorise, but a lot do and so this is a process you have got to get to know!

Exercises 1

Prepare yourself for factorising quadratic expressions by multiplying out the brackets in each of the following cases:

a) $(x+1)(x+2)$ b) $(x+2)(x+3)$ c) $(x+1)(x-3)$
d) $(x+2)(x-4)$ e) $(x-3)(x+7)$ f) $x(x-7)$
g) $(x-3)(x-2)$ h) $(x-7)(x-7)$ i) $(x-2)(x-2)$
j) $(x+2)(x-2)$ k) $(x-3)(x+3)$ l) $(2x+1)(x+1)$
m) $(x-5)(2x-1)$ n) $(3x-1)(3x+1)$ o) $4x(x-2)$

3. Factorising quadratics

To learn how to factorise let us study again the removal of brackets from $(x+3)(x+2)$.

$$(x+3)(x+2) = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$$

Clearly the number 6 in the final answer comes from *multiplying* the numbers 3 and 2 in the brackets. This is an important observation. The term $5x$ comes from *adding* the terms $2x$ and $3x$.

So, if we were to begin with $x^2 + 5x + 6$ and we were going to reverse the process we need to look for two numbers which multiply to give 6 and add to give 5.

$$? \times ? = 6$$
$$? + ? = 5$$

What are these numbers ? Well, we know that they are 3 and 2, and you will learn with practise to find these simply by inspection.

Using the two numbers which add to give 5 we split the $5x$ term into $3x$ and $2x$. We can set the calculation out as follows.

$$
\begin{aligned}
x^2 + 5x + 6 &= x^2 + 3x + 2x + 6 \\
&= x(x+3) + 2x + 6 && \text{by factorising the first two terms} \\
&= x(x+3) + 2(x+3) && \text{by factorising the last two terms} \\
&= (x+3)(x+2) && \text{by noting the common factor of } x+3
\end{aligned}
$$

The quadratic has been factorised. Note that you should never get these wrong, because the answer can always be checked by multiplying-out the brackets again!

Example

Suppose we want to factorise the quadratic expression $x^2 - 7x + 12$.

Starting as before we look for two numbers which multiply together to give 12 and add together to give -7. Think about this for a minute and you will realise that the two numbers we seek are -3 and -4 because

$$-4 \times -3 = 12, \qquad \text{and} \qquad -4 + -3 = -7$$

So, using the two numbers which add to give -7 we split the $-7x$ term into $-4x$ and $-3x$. We set the calculation out like this:

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$$\begin{aligned}
x^2 - 7x + 12 \quad &= \quad x^2 - 4x - 3x + 12 \\
&= \quad x(x-4) - 3x + 12 \qquad \text{by factorising the first two terms} \\
&= \quad x(x-4) - 3(x-4) \qquad \text{by factorising the last two terms extracting} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{a factor of } -3 \text{ in order to leave } x - 4 \\
&= \quad (x-4)(x-3) \qquad\quad \text{by noting the common factor of } x - 4
\end{aligned}$$

Once again, note that the answer can be checked by multiplying out the brackets again.

Example

Suppose we wish to factorise the quadratic expression $x^2 - 5x - 14$.

Starting as before we look for two numbers which multiply together to give -14 and add together to give -5. Think about this for a minute and you will realise that the two numbers we seek are -7 and 2 because

$$-7 \times 2 = -14, \qquad \text{and} \qquad -7 + 2 = -5$$

So, using the two numbers which add to give -5 we split the $-5x$ term into $-7x$ and $+2x$. We set the calculation out like this:

$$\begin{aligned}
x^2 - 5x - 14 \quad &= \quad x^2 - 7x + 2x - 14 \\
&= \quad x(x-7) + 2x - 14 \qquad \text{by factorising the first two terms} \\
&= \quad x(x-7) + 2(x-7) \qquad \text{by factorising the last two terms} \\
&= \quad (x-7)(x+2) \qquad\qquad \text{by noting the common factor of } x - 7
\end{aligned}$$

So the factorisation of $x^2 - 5x - 14$ is $(x-7)(x+2)$.

4. Factorising by inspection

Clearly, when you have some experience of this method, it is possible to avoid writing out all these stages. This is rather long-winded. What you need to do now is carry out the process **by inspection**. That is, simply by looking at the given expression, decide in your head which numbers need to be placed in the brackets to do the job. Look at the following two examples and see if you can do this.

Example

Suppose we want to factorise the quadratic expression $x^2 - 9x + 20$ by inspection.

We write

$$x^2 - 9x + 20 = (\qquad)(\qquad)$$

and try to place the correct quantities in brackets. Clearly we will need an x in both terms:

$$x^2 - 9x + 20 = (x \qquad)(x \qquad)$$

We want two numbers which multiply to give 20 and add to give -9. With practice you will be able to do this in your head. The two numbers are -4 and -5.

$$x^2 - 9x + 20 = (x-4)(x-5)$$

The answer should always be checked by multiplying-out the brackets again.

Example

Suppose we want to factorise the quadratic expression $x^2 - 9x - 22$ by inspection.

We write

$$x^2 - 9x - 22 = (\quad)(\quad)$$

and try to place the correct quantities in brackets. Clearly we will need an x in both terms:

$$x^2 - 9x - 22 = (x \quad)(x \quad)$$

We want two numbers which multiply to give -22 and add to give -9. The two numbers are -11 and $+2$.

$$x^2 - 9x - 22 = (x - 11)(x + 2)$$

The answer should always be checked by multiplying-out the brackets again.

If you can't manage to do these by inspection yet do not worry. Do it the way we did it before. Your ability to do this will improve with practice and experience.

All of these examples have involved quadratic expressions where the coefficient of x^2 was 1. When the coefficient is a number other than 1 the problem is more difficult. We will look at examples like this in the next section.

Exercises 2.

Factorise the following.

a) $x^2 + 8x + 15$ b) $x^2 + 10x + 24$ c) $x^2 + 9x + 8$ d) $x^2 + 15x + 36$

e) $x^2 + 2x - 3$ f) $x^2 + 2x - 8$ g) $x^2 - 2x - 3$ h) $x^2 + 7x - 18$

i) $x^2 - 12x + 35$ j) $x^2 - 11x + 10$ k) $x^2 - 13x + 22$ l) $x^2 + 12x + 27$

5. Expressions where the coefficient of x^2 is not 1.

Suppose we wish to factorise the expression $3x^2 + 5x - 2$.

As we did before we look for two numbers which add to give the coefficient of x; so we seek two numbers which add to give 5.

However, instead of looking for two numbers which multiply to give -2 we must look for two numbers which multiply to give -6, (that is, the coefficient of x^2 multiplied by the constant term, 3×-2). This is entirely consistent with the method we applied before because in those examples the coefficient of x^2 was always 1.

So

$$? \quad \times \quad ? = -6 \qquad ? \quad + \quad ? = 5$$

By inspection, or trial and error, two such numbers are 6 and -1.

$$6 \times -1 = -6 \qquad 6 + -1 = 5$$

We use these two numbers to split the $5x$ term into $6x$ and $-1x$.

$$
\begin{aligned}
3x^2 + 5x - 2 \quad &= \quad 3x^2 + 6x - x - 2 \\
&= \quad 3x(x + 2) - x - 2 \qquad \text{by factorising the first two terms} \\
&= \quad 3x(x + 2) - (x + 2) \qquad \text{factorising the last two terms by extracting } -1 \\
&= \quad (x + 2)(3x - 1) \qquad \text{by noting the common factor of } x + 2
\end{aligned}
$$

Example

Suppose we wish to factorise $2x^2 + 5x - 7$.

As we did before we look for two numbers which add to give the coefficient of x; so we seek two numbers which add to give 5.

We look for two numbers which multiply to give -14, (that is, the coefficient of x^2 multiplied by the constant term, 2×-7).

So

$$? \quad \times \quad ? = -14 \qquad\qquad ? \quad + \quad ? = 5$$

By inspection, or trial and error, two such numbers are 7 and -2.

$$7 \times -2 = -14 \qquad\qquad 7 + -2 = 5$$

We use these two numbers to split the $5x$ term into $7x$ and $-2x$.

$$
\begin{aligned}
2x^2 + 5x - 7 \;&=\; 2x^2 + 7x - 2x - 7 \\
&=\; x(2x + 7) - 2x - 7 && \text{by factorising the first two terms} \\
&=\; x(2x + 7) - (2x + 7) && \text{factorising the last two terms by extracting } -1 \\
&=\; (2x + 7)(x - 1) && \text{by noting the common factor of } 2x + 7
\end{aligned}
$$

Example

Suppose we wish to factorise $6x^2 - 5x - 4$.
We look for two numbers which multiply to give -24, (that is, the coefficient of x^2 multiplied by the constant term, 6×-4).

We look for two numbers which add to give the coefficient of x; so we seek two numbers which add to give -5.

So

$$? \quad \times \quad ? = -24 \qquad\qquad ? \quad + \quad ? = -5$$

By inspection, or trial and error, two such numbers are -8 and 3.

$$3 \times -8 = -24 \qquad\qquad 3 + -8 = -5$$

We use these two numbers to split the $-5x$ term into $3x$ and $-8x$.

$$
\begin{aligned}
6x^2 - 5x - 4 \;&=\; 6x^2 + 3x - 8x - 4 \\
&=\; 3x(2x + 1) - 8x - 4 && \text{by factorising the first two terms} \\
&=\; 3x(2x + 1) - 4(2x + 1) && \text{factorising the last two terms by extracting } -4 \\
&=\; (2x + 1)(3x - 4) && \text{by noting the common factor of } 2x + 1
\end{aligned}
$$

Example

Suppose we wish to factorise $15x^2 - 3x - 18$.

In this example note that the coefficients share a common factor of 3. This can be extracted:

$$3(5x^2 - x - 6)$$

Now we are looking at factorising $5x^2 - x - 6$.

We look for two numbers which multiply to give -30, (that is, the coefficient of x^2 multiplied by the constant term, 5×-6).

We look for two numbers which add to give the coefficient of x; so we seek two numbers which add to give -1.

So
$$? \quad \times \quad ? = -30 \qquad\qquad ? \quad + \quad ? = -1$$

By inspection, or trial and error, two such numbers are 5 and -6.

$$5 \times -6 = -30 \qquad\qquad 5 + -6 = -1$$

We use these two numbers to split the $-x$ term into $5x$ and $-6x$.

$$
\begin{aligned}
3(5x^2 - x - 6) &= 3(5x^2 + 5x - 6x - 6) \\
&= 3\left(5x(x+1) - 6x - 6)\right) && \text{by factorising the first two terms} \\
&= 3\left(5x(x+1) - 6(x+1)\right) && \text{factorising the last two terms by extracting } -6 \\
&= 3\left((x+1)(5x-6)\right) && \text{by noting the common factor of } x+1
\end{aligned}
$$

Exercises 3

Factorise the following.

a) $2x^2 + 11x + 5$ b) $3x^2 + 19x + 6$ c) $3x^2 + 17x - 6$ d) $6x^2 + 7x + 2$

e) $7x^2 - 6x - 1$ f) $12x^2 + 7x + 1$ g) $8x^2 + 6x + 1$ h) $8x^2 - 6x + 1$

6. Special case 1 - the difference of two squares

A special case concerns what is known as **the difference of two squares**. What does this look like ? Typically, $x^2 - 9$. Note that there is no x term and that the number 9 is itself a square number. A square number is one which has resulted from squaring another number. In this case 9 is the result of squaring 3, $(3^2 = 9)$, and so 9 is a square number.

Hence $x^2 - 9$ is the difference of two squares, $x^2 - 3^2$.

When we try to factorise $x^2 - 9$ we are looking for two numbers which add to zero (because there is no term in x), and which multiply to give -9. Two such numbers are -3 and 3 because

$$-3 \ \times \ 3 \ = \ -9, \qquad \text{and} \qquad -3 \ + \ 3 \ = \ 0$$

We use these two numbers to split the $0x$ term into $-3x$ and $3x$.

$$
\begin{aligned}
x^2 - 9 &= x^2 - 3x + 3x - 9 \\
&= x(x-3) + 3x - 9 && \text{by factorising the first two terms} \\
&= x(x-3) + 3(x-3) && \text{factorising the last two terms by extracting } +3 \\
&= (x-3)(x+3) && \text{by noting the common factor of } x-3
\end{aligned}
$$

So
$$x^2 - 9 = (x-3)(x+3)$$

Recall that 9 is a square number, $(9 = 3^2)$. So this example is a difference of two square. The result of this example is true in more general cases of the difference of two squares.

It is always the case that $x^2 - a^2$ factorises to $(x-a)(x+a)$.

The difference of two squares, $x^2 - a^2$, always factorises to

$$x^2 - a^2 = (x - a)(x + a)$$

Example

Suppose we want to factorise $x^2 - 25$.

Note that $x^2 - 25$ is the difference of two squares because 25 is a square number ($25 = 5^2$). So we need to factorise $x^2 - 5^2$. We can do this by inspection using the formula above.

$$x^2 - 5^2 = (x - 5)(x + 5)$$

Example

Suppose we want to factorise $2x^2 - 32$.

This time there is a common factor of 2 which can be extracted first.

$$2x^2 - 32 = 2(x^2 - 16) = 2(x - 4)(x + 4)$$

Example

A slightly different form occurs if we now include a square number in front of the x^2 term. For example, suppose we wish to factorise $9x^2 - 16$. Note that 9 is a square number, and so the term $9x^2$ can be written $(3x)^2$. So we still have a difference of two squares

$$(3x)^2 - 4^2$$

To factorise this we write

$$9x^2 - 16 = (3x - 4)(3x + 4)$$

Exercises 4

Factorise the following.

a) $x^2 - 144$ b) $x^2 - 36$ c) $x^2 - 1$ d) $x^2 - 121$

e) $9x^2 - 1$ f) $16x^2 - 9$ g) $49x^2 - 1$ h) $25x^2 - 16$

7. Special case 2 - complete squares

A second important special case is when we need to deal with complete squares. This happens when the answer can be written in the form (   )^2, that is as a single term, squared.

Consider the following example.

Example

Suppose we wish to factorise $x^2 + 10x + 25$.

As before we want two numbers which multiply to give 25 and add to give 10. Two such numbers are 5 and 5 because
$$5 \times 5 = 25, \quad \text{and} \quad 5 + 5 = 10$$
We use these two numbers to split the $10x$ term into $5x$ and $5x$.

$$
\begin{aligned}
x^2 + 10x + 25 &= x^2 + 5x + 5x + 25 \\
&= x(x+5) + 5x + 25 && \text{by factorising the first two terms} \\
&= x(x+5) + 5(x+5) && \text{factorising the last two terms by extracting } +5 \\
&= (x+5)(x+5) && \text{by noting the common factor of } x+5
\end{aligned}
$$

The result can be written as $(x+5)^2$, a **complete square**.

Example

Suppose we want to factorise $x^2 - 8x + 16$.

We seek two numbers which add to give -8 and multiply to give 16. The two required numbers are -4 and -4 and so, by inspection,

$$x^2 - 8x + 16 = (x-4)(x-4)$$

The result can be written as $(x-4)^2$, a **complete square**.

More complicated examples can occur, for example when there is a number in front of the x^2. Work through the following example.

Example

Suppose we want to factorise $25x^2 - 20x + 4$.
Note that $25x^2$ can be written as $(5x)^2$, a squared term. Note also that $4 = 2^2$. In this case, by inspection,
$$25x^2 - 20x + 4 = (5x-2)(5x-2)$$

The result can be written as $(5x-2)^2$, a **complete square**.

Do not worry if you have difficulty with this last example. The skill will come with practice. Even if you did not recognise the complete square, you could still use the previous method:

To factorise $25x^2 - 20x + 4$

As before we want two numbers which multiply to give 100 (the coefficient of x^2 multiplied by the constant term) and add to give -20. Two such numbers are -10 and -10 because

$$-10 \times -10 = 100, \quad \text{and} \quad -10 + -10 = -20$$

We use these two numbers to split the $-20x$ term into $-10x$ and $-10x$.

$$
\begin{aligned}
25x^2 - 20x + 4 &= 25x^2 - 10x - 10x + 4 \\
&= 5x(5x-2) - 10x + 4 && \text{by factorising the first two terms} \\
&= 5x(5x-2) - 2(5x-2) && \text{factorising the last two terms by extracting } -2 \\
&= (5x-2)(5x-2) && \text{by noting the common factor of } 5x-2
\end{aligned}
$$

So $25x^2 - 20x + 4 = (5x-2)^2$ as before.

8. Special case 3 - the constant term is missing

There is one more special case that you ought to be familiar with. This arises in examples where the constant term is absent, as in $3x^2 - 8x$. How do we proceed ?

The way forward is to look for common factors. In the case of $3x^2 - 8x$ note that each of the two terms has a common factor of x. (Note that this is not the case when dealing with a quadratic expression containing a constant term).

The common factor, x, is written outside a bracket, and the contents of the bracket are chosen by inspection so that they multiply-out to give the correct terms.

$$3x^2 - 8x = x(3x - 8)$$

Example

Suppose we wish to factorise $10x^2 + 5x$.

This time we note the common factor of $5x$. This is extracted and we adjust the contents of the brackets accordingly:

$$10x^2 + 5x = 5x(2x + 1)$$

Exercises 5.
Factorise the following.

a) $x^2 + 18x + 81$ b) $x^2 - 4x + 4$ c) $x^2 - 22x + 121$ d) $25x^2 + 40x + 16$
e) $64x^2 + 16x + 1$ f) $x^2 + 6x$ g) $3x^2 - x$ h) $5x^2 - 20x$

Answers

Exercises 1

a) $x^2 + 3x + 2$ b) $x^2 + 5x + 6$ c) $x^2 - 2x - 3$ d) $x^2 - 2x - 8$
e) $x^2 + 4x - 21$ f) $x^2 - 7x$ g) $x^2 - 5x + 6$ h) $x^2 - 14x + 49$
i) $x^2 - 4x + 4$ j) $x^2 - 4$ k) $x^2 - 9$ l) $2x^2 + 3x + 1$
m) $2x^2 - 11x + 5$ n) $9x^2 - 1$ o) $4x^2 - 8x$.

Exercises 2

a) $(x + 3)(x + 5)$ b) $(x + 4)(x + 6)$ c) $(x + 1)(x + 8)$ d) $(x + 3)(x + 12)$
e) $(x + 3)(x - 1)$ f) $(x + 4)(x - 2)$ g) $(x - 3)(x + 1)$ h) $(x + 9)(x - 2)$
i) $(x - 7)(x - 5)$ j) $(x - 1)(x - 10)$ k) $(x - 11)(x - 2)$ l) $(x + 9)(x + 3)$

Exercises 3

a) $(2x + 1)(x + 5)$ b) $(3x + 1)(x + 6)$ c) $(3x - 1)(x + 6)$ d) $(2x + 1)(3x + 2)$
e) $(7x + 1)(x - 1)$ f) $(3x + 1)(4x + 1)$ g) $(2x + 1)(4x + 1)$ h) $(2x - 1)(4x - 1)$

Exercises 4

a) $(x - 12)(x + 12)$ b) $(x - 6)(x + 6)$ c) $(x - 1)(x + 1)$ d) $(x - 11)(x + 11)$
e) $(3x + 1)(3x - 1)$ f) $(4x + 3)(4x - 3)$ g) $(7x + 1)(7x - 1)$ h) $(5x + 4)(5x - 4)$

Exercises 5

a) $(x + 9)^2$ b) $(x - 2)^2$ c) $(x - 11)^2$ d) $(5x + 4)^2$
e) $(8x + 1)^2$ f) $x(x + 6)$ g) $x(3x - 1)$ h) $5x(x - 4)$