Transposition or rearranging formulae

It is often useful to rearrange, or transpose, a formula in order to write it in a different, but equivalent form. This unit explains the procedure for doing this.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- rearrange, or transpose, a formula by reversing the operations involved
- rearrange, or transpose, a formula by performing the same operations on both sides of an equals sign

Contents

1. Introduction

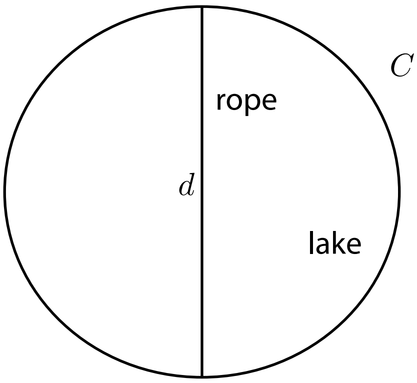
Standard formulae are presented in a particular way. It is often necessary to rewrite them in alternative forms. This unit shows the ways in which this can be done.

2. The formula for the circumference of a circle

The formula for working out the length of the circumference, C, of a circle if we are given the diameter, d, is $C = \pi d$. This is the way in which the formula is usually presented and the way we remember it. But what if we know the circumference of a circle and want to find the diameter. Consider the following problem.

Example

Suppose we know the circumference of a circular lake, and wish to know the length of rope required to stretch across the lake to divide it into two equal parts as shown.



Suppose the circumference C is 250m. To work out the length of the rope we need to find the diameter, d, of this circle. We shall now investigate the way in which a formula for d can be derived.

To use the formula $C = \pi d$ note that we start with a diameter, d, and multiply it by π to give the circumference, C. We can depict the steps required to do this as follows:

$$d \quad \rightarrow \quad \times\pi \quad \rightarrow \quad C$$

To work the other way around we must reverse the process. The reverse process of multiplication by π is division by π. So, starting with C we need to divide it by π to give d, as shown:

$$d \quad \leftarrow \quad \div\pi \quad \leftarrow \quad C$$

Therefore the formula for finding d is

$$d = \frac{C}{\pi}$$

We have rearranged the original formula to make d the **subject**. This means that d is on its own on one side of the formula.

Applying this to the lake problem, with $C = 250$, we find

$$d = \frac{C}{\pi} = \frac{250}{\pi}$$

Using a calculator this equals 79.6 metres.

Another formula for the circumference of a circle is $C = 2\pi r$. Suppose we are given C and want a formula for finding r.

To use the formula $C = 2\pi r$ note that we start with r and multiply it by 2π to obtain C. We can think of this process as follows:

$$r \quad \rightarrow \quad \times 2\pi \quad \rightarrow \quad C$$

To work the other way around we must reverse the process. The reverse process of multiplication by 2π is division by 2π. So, starting with C we need to divide it by 2π to give r, as shown:

$$r \quad \leftarrow \quad \div 2\pi \quad \leftarrow \quad C$$

Therefore the formula for finding r is

$$r = \frac{C}{2\pi}$$

We have rearranged the original formula to make r the subject. This means that r is on its own on one side of the formula.

3. Examples using the equations of motion

Consider the equation of motion $v = u + at$ where u is the initial speed, v is the final speed, a is the acceleration, and t is the time.

Suppose a sports car accelerates from rest to 100 $\mathrm{km\,h^{-1}}$ in 6.4 seconds, and suppose we want to find its acceleration a. So $u = 0$, $v = 100$ and $t = 6.4$.

To use the equations of motion we need to ensure we work with a consistent set of units. 100$\mathrm{km\,h^{-1}}$ is approximately the same as 28 $\mathrm{ms^{-1}}$.

To use the formula $v = u + at$ note that, to obtain v, we start with a, then multiply it by t and add u. We can think of this process as follows:

$$a \quad \rightarrow \quad \times t \quad \rightarrow \quad +u \quad \rightarrow \quad v$$

To work the other way around we must reverse the process. The reverse process of adding u is to subtract u. The reverse process of multiplying by t is to divide by t. So, starting with v we need to subtract u and then divide the result by t to give a, as shown:

$$a \quad \leftarrow \quad \div t \quad \leftarrow \quad -u \quad \leftarrow \quad v$$

Therefore the formula for finding a is

$$a = \frac{v - u}{t}$$

We have rearranged the original formula to make a the subject. Now we can substitute the given values:

$$a = \frac{28 - 0}{6.4} = 4.4\,\mathrm{ms^{-2}}$$

4. The formula to convert temperatures in $^\circ$F to $^\circ$C

The formula for converting a temperature given in degrees Fahrenheit, F, to one in degrees Celsius, C is

$$C = \frac{5}{9}(F - 32)$$

Note the processes used in applying this formula. Starting with F we subtract 32 and then multiply by 5 and finally divide by 9 to give C. Schematically,

$$F \quad \rightarrow \quad -32 \quad \rightarrow \quad \times 5 \quad \rightarrow \quad \div 9 \quad \rightarrow \quad C$$

To work the other way around we must reverse the process. The reverse process of dividing by 9 is to multiply by 9. The reverse process of multiplying by 5 is to divide by 5. The reverse process of subtracting 32 is to add 32. So, starting with C we need to multiply by 9, and then divide the result by 5, and finally add 32 to give F, as shown:

$$F \quad \leftarrow \quad +32 \quad \leftarrow \quad \div 5 \quad \leftarrow \quad \times 9 \quad \leftarrow \quad C$$

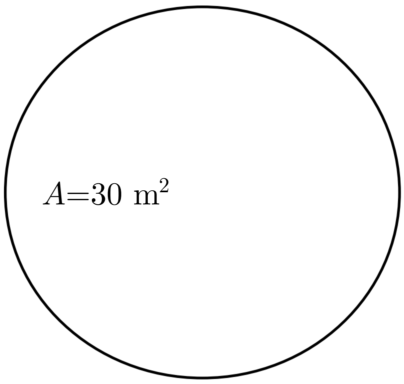
Therefore the formula for finding F is

$$F = \frac{9C}{5} + 32$$

We have rearranged the original formula to make F the subject. This means that F is on its own on one side of the formula.

5. The formula for the area of a circle

Imagine we want to make a circular lawn, and to do this we have bought 30m^2 of turf. We know the area of lawn, and to construct it we will need to calculate the radius of the circle.



The formula for the area of a circle is $A = \pi r^2$. Let us examine the processes involved in applying this formula. Starting with the radius r, we square it and then multiply the result by π to give A. Schematically,

$$r \quad \rightarrow \quad \text{square} \quad \rightarrow \quad \times\pi \quad \rightarrow A$$

To apply the reverse process, starting with A, the inverse of multiplying by π is to divide by π. The inverse of squaring is square-rooting. So we have

$$r \quad \leftarrow \quad \surd \quad \leftarrow \quad \div\pi \quad \leftarrow A$$

This produces the formula

$$r = \sqrt{\frac{A}{\pi}}$$

In this example $A = 30$ and so

$$r = \sqrt{\frac{30}{\pi}} = 3.09\text{m}$$

6. A formula used in designing roads

Road designers sometimes make use of the formula $R = \dfrac{V^2}{15}$ to calculate a safe minimum value for the radius of a bend in a road given the speed V at which cars are likely to be travelling along the road. This formula has been devised so that the speed is measured in kilometres per hour, and the radius in metres.

Suppose because of space constraints the radius of the bend cannot be more than 100m; we need to determine the maximum safe speed for cars going round this bend.

We shall make V the subject of the given formula so that we can calculate the corresponding speed.

Note that in applying $R = \dfrac{V^2}{15}$ we square V and then divide by 15 as shown schematically:

$$V \quad \rightarrow \quad \text{square} \quad \rightarrow \quad \div 15 \quad \rightarrow \quad R$$

To reverse the process note that the inverse of dividing by 15 is to multiply by 15. The inverse of squaring is to take the square-root. So we have

$$V \quad \leftarrow \quad \sqrt{\ } \quad \leftarrow \quad \times 15 \quad \leftarrow \quad R$$

giving the formula

$$V = \sqrt{15R}$$

So, with $R = 100$ we find $V = \sqrt{15 \times 100} = 38.7\ \text{km}\,\text{h}^{-1}$.

7. Another example involving an equation of motion

Consider the formula $s = ut + \dfrac{1}{2}at^2$ where s = distance travelled, a = acceleration, t = time. Suppose we wish to make a the subject of the formula.

If we start with a, we multiply it by $\frac{1}{2}$ then multiply it by t^2, then add ut, as shown:

$$a \quad \rightarrow \quad \times\frac{1}{2} \quad \rightarrow \quad \times t^2 \quad \rightarrow \quad +ut \quad \rightarrow s$$

Reversing the processes:

$$a \quad \leftarrow \quad \div\frac{1}{2} \quad \leftarrow \quad \div t^2 \quad \leftarrow \quad -ut \quad \leftarrow s$$

giving the formula

$$a = \frac{\frac{s-ut}{t^2}}{\frac{1}{2}}$$

Dividing by $\frac{1}{2}$ is equivalent to multiplying by 2 and so this can be written as

$$a = \frac{2(s-ut)}{t^2}$$

and we have now made a the subject of the original formula.

8. The formula for the period of a pendulum

The formula for the period, T, of a pendulum is $T = 2\pi\sqrt{\frac{\ell}{g}}$. Here, ℓ is the length of the pendulum and g is the acceleration due to gravity.

Suppose we want to make ℓ the subject of this formula.

In the formula $T = 2\pi\sqrt{\frac{\ell}{g}}$ observe that, starting with ℓ we divide by g, then square-root the result, and finally multiply by 2π. Schematically,

$$\ell \quad \rightarrow \quad \div g \quad \rightarrow \quad \surd \quad \rightarrow \quad \times 2\pi \quad \rightarrow \quad T$$

Reversing the processes:

$$\ell \quad \leftarrow \quad \times g \quad \leftarrow \quad \text{square} \quad \leftarrow \quad \div 2\pi \quad \leftarrow \quad T$$

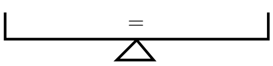
so the required formula for ℓ is

$$\ell = g\left(\frac{T}{2\pi}\right)^2$$

9. As your confidence grows....

When you are confident with the method outlined above for rearranging formulae, you should also learn how to rearrange formulae in the following way.

Think of the equals sign in a formula as a pivot on a balance. To keep the formula balanced, whatever we do to one side we must do the same to the other.



Example

Consider the formula for the circumference of a circle, $C = \pi d$.

Suppose we want to make d the subject of this formula. On the right-hand side d is multiplied by π. We shall do the inverse operation and divide the right-hand side by π. But to keep the balance we must do exactly the same to both sides.

$$\begin{aligned} C &= \pi d \\ \frac{C}{\pi} &= \frac{\pi d}{\pi} && \text{dividing both sides by } \pi \\ \frac{C}{\pi} &= \frac{\not{\pi} d}{\not{\pi}} && \text{cancelling the } \pi\text{'s on the right} \\ \frac{C}{\pi} &= d \end{aligned}$$

and so $d = \dfrac{C}{\pi}$ as before.

Example

Consider the formula $v = u + at$ and suppose we wish to make a the subject.

Note that in applying this formula a is multiplied by t, and then we add u. So to reverse, or undo this process, we will need to subtract u and then divide by t. So first of all we will subtract u from both sides.

$$\begin{aligned} v &= u + at \\ v - u &= u + at - u && \text{subtracting } u \text{ from both sides} \\ v - u &= at && \text{so that the } u\text{'s on the right cancel out} \\ \frac{v-u}{t} &= \frac{at}{t} && \text{dividing both sides by } t \\ \frac{v-u}{t} &= \frac{a\not{t}}{\not{t}} && \text{cancelling the } t\text{'s on the right} \\ \frac{v-u}{t} &= a \end{aligned}$$

and so $a = \dfrac{v-u}{t}$.

Example

Consider the formula used to convert temperatures in degrees Fahrenheit to temperatures in degrees Celsius.

$$C = \frac{5}{9}(F - 32)$$

We will perform a sequence of operations, doing the same to both sides, in order to get F on its own.

$$
\begin{aligned}
C &= \frac{5}{9}(F-32) \\
\frac{9C}{5} &= \frac{5}{9}(F-32)\times\frac{9}{5} && \text{multiplying both sides by } \frac{9}{5} \\
\frac{9C}{5} &= \frac{\not{5}}{\not{9}}(F-32)\times\frac{\not{9}}{\not{5}} \\
\frac{9C}{5} &= F-32 \\
\frac{9C}{5}+32 &= F-32+32 && \text{adding 32 to both sides} \\
\frac{9C}{5}+32 &= F
\end{aligned}
$$

and so $F = \dfrac{9C}{5} + 32$.

Example

Let us revisit the formula for the period of a pendulum, $T = 2\pi\sqrt{\dfrac{\ell}{g}}$. We want to make ℓ the subject.

$$
\begin{aligned}
T &= 2\pi\sqrt{\frac{\ell}{g}} \\
\frac{T}{2\pi} &= \sqrt{\frac{\ell}{g}} && \text{dividing both sides by } 2\pi \\
\left(\frac{T}{2\pi}\right)^2 &= \frac{\ell}{g} && \text{squaring both sides} \\
g\left(\frac{T}{2\pi}\right)^2 &= \ell && \text{multiplying both sides by } g
\end{aligned}
$$

and so finally we have $\ell = g\left(\dfrac{T}{2\pi}\right)^2$.

10. Exercises

Exercises 1

Given below are a selection of formulae relating to areas, perimeters and volumes of geometric shapes.

Circles: $A = \pi r^2 = \pi\left(\dfrac{d^2}{4}\right)$ and $C = 2\pi r = \pi d$

where A is the area, C is the circumference, r is the radius and d is the diameter.

Rectangles: $A = wh$ and $P = 2(w + h)$

where A is the area, P is the perimeter, w is the width and h is the height.

Spheres: $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$

where V is the volume, A is the surface area and r is the radius.

Cylinders: $V = \pi r^2 h$, $C = 2\pi rh$ and $A = 2\pi r(r + h)$

where V is the volume, C is the curved surface area, A is the surface area of a solid cylinder (including the ends), r is the radius and h is the height.

In the following exercises give answers which are not whole numbers correct to 1 decimal place.

1. What is the radius of a circle with area 16π cm^2?
2. What is the diameter of a circle with circumference 5π cm?
3. What is the width of a rectangle with area 22cm^2 and height 8 cm?
4. What is the height of a rectangle with perimeter 23 cm and width 3.2 cm?
5. What is the radius of a sphere with volume 20 cm^3?
6. What is the radius of a cylinder with curved surface area 100 cm^2 and height 24 cm?
7. What is the radius of a sphere with surface area 24.7 cm^2?
8. The total surface area of a cylindrical can is 54 cm^2 and its radius is 2.3 cm. What is its height?
9. What is the radius of a solid hemisphere with total surface area 19.6 cm^2?
10. A large sphere has radius twice that of a small sphere. If the total volume of both spheres is 96π cm^3, what is the radius of the small sphere?

Exercises 2

Some formulae from mechanics are given below.

Constant Acceleration

$v = u + at$, $v^2 = u^2 + 2as$ and $s = ut + \frac{1}{2}at^2$

where u is the initial speed, v is the final speed, t is time, s is distance travelled and a is acceleration. In the SI system of units u and v are measured in metres/second (m/s or ms^{-1}), t in seconds (s), s in metres (m) and a in metres per second squared (m/s^2 or ms^{-2}).

Work and Energy

$W = Fd$, $K.E. = \frac{1}{2}mv^2$, $P.E. = mgh$

where W is work done, $K.E.$ is kinetic energy, $P.E.$ is potential energy, F is force, d is distance moved by the force, m is mass, v is speed, g is acceleration due to gravity and h is height. In the SI system of units W, $K.E.$ and $P.E.$ are measured in Joules (J), F in Newtons (N), d in metres (m), m in kilograms (kg), v in metres per second (m/s or ms^{-1}), g in metres per second squared (m/s^2 or ms^{-2}) and h in metres (m).

Throughout these exercises you should take the acceleration due to gravity, g, as 9.8 m/s^2 and give your answers correct to 2 decimal places.

1. A car accelerates from 2 m/s to 8 m/s in 1.2 seconds. What is its acceleration?
2. A car accelerating at 3 m/s^2 travels 220 m in 10 seconds. What is the car's initial speed?
3. A ball is thrown vertically upward at a speed of 10 m/s. Its acceleration is -9.8 m/s^2. How high does the stone go? [Hint: its speed at its highest point is 0.]
4. A ball thrown vertically upwards reaches a maximum height of 24m. What was its initial speed?
5. The kinetic energy of a stone travelling at 2 m/s is 0.36 J. What is the mass of the stone?
6. The kinetic energy of a missile of mass 500 kg is 1,000,000 J. What is the speed of the missile?
7. How high must an object of mass 2.1 kg be placed in order to have potential energy of 100 J?
8. If a force of 23.4 N does 84.1 J work on an object, how far does the object move in the direction of the force.

Exercises 3
Formulae can be used to generate the terms of a sequence. For example, the sequence 2, 4, 6, 8, ... is generated by the formula $u_n = 2n$ where u_n is the nth term of the sequence.

In the exercises below term values which are not whole numbers are given correct to 2 decimal places. In each case you should determine which term is given. For example, in the sequence generated by the formula $u_n = 2n$, we see that 10 is term number 5 in the sequence.

1. In the sequence generated by the formula $u_n = 3n + 1$ which term is 37?
2. In the sequence generated by the formula $u_n = 54 - 2n$ which term is 36?
3. In the sequence generated by the formula $u_n = n^2 - 5$ which term is 116?
4. In the sequence generated by the formula $u_n = 5\sqrt{n-4}$ which term is 40?

5. In the sequence generated by the formula
$u_n = \sqrt{\dfrac{1}{2n+1}}$ which term is 0.30?

Answers

Exercise 1

1)	4 cm	2)	5 cm	3)	2.8 cm	4)	8.3 cm
5)	1.68 cm	6)	0.66 cm	7)	1.40 cm	8)	1.44 cm
9)	1.44 cm	10)	2 cm				

Exercise 2

1)	$5\,\mathrm{m/s^2}$	2)	7 m/s	3)	10.2 m	4)	21.69 m/s
5)	0.18 kg	6)	63.25 m/s	7)	4.86 m	8)	3.59 m

Exercise 3

1)	12	2)	9	3)	11	4)	68	5)	5

