NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1 EXAMINATION 1999 – 2000

MA1102 CALCULUS

November 1999 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO (2)** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f: \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \frac{2}{3}x^3 + \frac{1}{3}, & x > 1 \\ x^2 \sin(\frac{\pi}{2x}), & -1 \le x \le 1 \text{ and } x \ne 0 \\ x^3 - 1, & x < -1 \\ 0, & x = 0 \end{cases}.$$

(a) Find the *range* of the function f.

(b) Determine if f is surjective.

(c) Find the values of x (if any) where
(i) $f(x) = 1$,
(ii) $f(x) = -2$.

(d) Determine all x in $\mathbf{R}$ at which the function f is *continuous*.

(e) Is the function f *differentiable* at $x = 1$? Justify your answer.

(f) Compute $\int_{-1}^{1} f(x)dx$.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\lim\limits_{x\to\infty} \dfrac{61x^7 + 2x^3 + 1}{907x^7 + 7x^3 + 5x^2 + 7}$.

(b) $\lim\limits_{x\to 0} \dfrac{\sqrt{7x^2 + 121} - 11}{14x^2}$.

(c) $\lim\limits_{x\to\infty} \dfrac{x^5}{e^{(x^2)}}$.

(d) $\lim\limits_{x\to 0} \dfrac{\sin(\tan(x))}{\tan(\sin(x))}$.

(e) $\lim_{x\to 0} (e^{(x^3)} + 3x^2)^{(1/x^2)}$.

Question 3 [**20** marks]

Evaluate the following integrals.

(a) $\int \frac{1}{(x^2+2)(x^2+3)}dx$.

(b) $\int \sin^{-1}(4x)dx$.

(c) $\int_0^{\frac{\pi}{5}} e^{2x}\sin(5x)dx$.

(d) $\int_0^2 \frac{x+3}{x^2+2x+4}dx$.

SECTION B

*Answer not more than **TWO (2)** questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Using the definition of limit, prove that $\lim_{x\to 1} 3x - 2 = 1$.

(b) Find the values of a and b so that the function

$$f(x) = \begin{cases} ax^3 + bx, -\pi \le x \le \pi \\ \sin(x), x < -\pi \text{ or } x > \pi \end{cases}$$

is differentiable on the whole of $\mathbf{R}$.

(c) Show that the equation $2x^3 + 3x + 1 = 3\sin(x)\cos(x)$ has exactly one real root.

Question 5 [**20** marks]

Let the function f be defined on $\mathbf{R}$ by

$$f(x) = \begin{cases} \frac{2x|x|}{1+x^2}, & x < 1 \\ \frac{1}{x}, & x \ge 1 \end{cases}.$$

(a) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(b) Find the *horizontal asymptotes* of the graph of f .

(c) Find the intervals on which the graph of f is *concave upward* or *concave downward*.

(d) Find the *relative extrema* of f .

(e) Find the *points of inflection* of the graph of f.

(f) Sketch the graph of f.

Question 6 [**20** marks]

(a) Differentiate the function $g(x) = \int_{-x}^{x^3} \frac{1}{1+\sin^2(2t)+t^2} dt.$

(b) Let the function k be defined on **R** by

$$k(x) = \int_1^x \frac{1}{\sqrt{1+t^4}} dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(c) Let $h : [a,b] \to \mathbf{R}$ be any continuous and increasing function.

(i) Prove that $\int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds = xh(x) + C$ for some constant C.

Hence, or otherwise, deduce that

$$\int_a^b h(x)dx = bh(b) - ah(a) - \int_{h(a)}^{h(b)} h^{-1}(s)ds$$

(ii) Using part (i) or otherwise, evaluate $\int_0^1 \sqrt{1+(x-1)^{\frac{1}{3}}}\, dx$

END OF PAPER

Answer To MA1102 Calculus

SECTION A (Compulsory)

1. The function f is defined by $f(x) = \begin{cases} \frac{2}{3}x^3 + \frac{1}{3}, & x > 1 \\ x^2 \sin(\frac{\pi}{2x}), & -1 \le x \le 1 \text{ and } x \ne 0 \\ x^3 - 1, & x < -1 \\ 0, & x = 0 \end{cases}$.

(a) For $x < -1$, $f(x) = x^3 - 1 < -2$. Also, for $x < -1$, $x^3 - 1 < -2 \Leftrightarrow x < -1$. Thus f maps $(-\infty, -1)$ onto $(-\infty, -2)$. (Because for any $y < -2$, we can take $x = \sqrt[3]{y+1}$ so that $f(x) = y$) Also, for $-1 \le x \le 1$, $-1 \le f(x) \le 1$. This is seen as follows. For $-1 \le x \le 1$ and $x \ne 0$, $|f(x)| = \left|x^2 \sin(\frac{\pi}{2x})\right| \le x^2 \le 1$. Now $f(0) = 0$. Thus $-1 \le f(x) \le 1$. Therefore, $f(1) = 1$ is the absolute maximum of f on [-1, 1] and $f(-1) = -1$ is the absolute minimum of f on [-1, 1]. Assuming that f is continuous on [-1, 1] (as we shall show in part (d) below), by the Intermediate Value Theorem) f maps the interval [-1,1] onto [-1,1]. [We can also use the fact that $\sin(\frac{\pi}{2x})$ maps [1/2, 1] onto [0,1] and [-1, -1/2] onto [-1,0].]

Finally for $x > 1$, $f(x) = \frac{2}{3}x^3 + \frac{1}{3} > 1$. And for any $y > 1$, we can take $x = \sqrt[3]{\frac{3y-1}{2}} > 1$ so that $f(x) = y$. Hence f maps $(1, \infty)$ onto $(1, \infty)$.

Hence the range of f is $(-\infty, -2) \cup [-1, 1] \cup (1, \infty) = (-\infty, -2) \cup [-1, \infty)$.

(b) By part (a) Range(f) = $(-\infty, -2) \cup [-1, \infty) \ne \mathbf{R}$ = codomain(f), therefore f is not surjective.

(c) (i) By part (a) 1 is in the image of $[-1, 1]$ under f. Thus, to find the preimage we need to solve the equation $x^2 \sin(\frac{\pi}{2x}) = 1$ for x *in* [-1, 1]-{0}.

For $x \ne 0$ and $-1 < x < 1$, $\left|x^2 \sin(\frac{\pi}{2x})\right| \le x^2 < 1$. Since we know $f(1) = 1$, and $f(-1) < 0$, $x = 1$.

(ii) From part (a) -2 is not in the range of f. Thus, the solution of $f(x) = -2$ does not exist. Therefore, there is no value of x such that $f(x) = -2$.

(d) When $x < -1$, $f(x) = x^3 - 1$, which is a polynomial function, therefore f is continuous on $(-\infty, -1)$, since any polynomial function is continuous on the real numbers and so is continuous on any open interval. When $-1 < x < 1$ and $x \ne 0$, $f(x) = x^2 \sin(\frac{\pi}{2x})$ and since $x^2 \sin(\frac{\pi}{2x})$ is continuous on $(-1, 0)$ and on $(0, 1)$, f is continuous on the union of these two intervals. Finally when $x > 1$, $f(x)$ is a polynomial function and so it is continuous for $x > 1$. Thus it remains to check if f is continuous at $x = -1$, 0 or 1. Consider the left limit at $x = 1$,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 \sin(\frac{\pi}{2x}) = 1^2 \sin(\frac{\pi}{2}) = 1$ and the right limit at $x = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2}{3}x^3 + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = 1 = f(1)$. Thus
$\lim_{x\to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.

Now consider the left limit of f at $x = -1$,
$\lim_{x\to(-1)^-} f(x) = \lim_{x\to(-1)^-} x^3 - 1 = -2$ and the right limit at $x = -1$,
$\lim_{x\to(-1)^+} f(x) = \lim_{x\to(-1)^+} x^2 \sin(\frac{\pi}{2x}) = 1^2 \sin(-\frac{\pi}{2}) = -1$.
Thus the left and the right limits of f at $x = -2$ are not the same and so f is not continuous at $x = -1$.
Now $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin(\frac{\pi}{2x}) = 0$ by the Squeeze Theorem and $f(0) = 0$. Therefore, f is continuous at $x = 0$.
Hence f is continuous at x for all $x \neq -1$.

(e) f is differentiable at $x = 1$. This is seen as follows.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x^2\sin(\frac{\pi}{2x})-1}{x-1} = \lim_{x\to 1^-} \frac{2x\sin(\frac{\pi}{2x}) - \frac{\pi}{2}\cos(\frac{\pi}{2x})}{1}$$

by L' Hôpital's Rule

$= 2.$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{\frac{2}{3}x^3 + \frac{1}{3} - 1}{x-1} = \frac{2}{3} \lim_{x\to 1^+} \frac{x^3-1}{x-1} = 2.$$

Thus f is differentiable at $x = 1$ and $f'(1) = 2$.

(f) Note that f is an odd function since $f(-x) = -f(x)$.
$\int_{-1}^{0} f(x)dx = -\int_{1}^{0} f(-t)dt$ where $t = -x$
$= \int_{1}^{0} f(t)dt = -\int_{0}^{1} f(t)dt.$
Therefore, $\int_{-1}^{1} f(x)dx = \int_{0}^{1} f(x)dx + \int_{-1}^{0} f(x)dx = 0.$

2. (a) $\lim\limits_{x\to\infty} \dfrac{61x^7+2x^3+1}{907x^7+7x^3+5x^2+7} = \lim\limits_{x\to\infty} \dfrac{61+\frac{2}{x^4}+\frac{1}{x^7}}{907+\frac{7}{x^4}+\frac{5}{x^5}+\frac{7}{x^7}} = \dfrac{61+0+0}{907+0+0+0} = \dfrac{61}{907}.$

(b) $\lim\limits_{x\to 0} \dfrac{\sqrt{7x^2+121}-11}{14x^2} = \lim\limits_{x\to 0} \dfrac{7x^2}{14x^2(\sqrt{7x^2+121}+11)} = \lim\limits_{x\to 0} \dfrac{1}{2(\sqrt{7x^2+121}+11)}$
$= \dfrac{1}{44}.$

(c) $\lim\limits_{x\to\infty} \dfrac{x^5}{e^{x^2}} = \lim\limits_{x\to\infty} \dfrac{5x^4}{2xe^{x^2}} = \frac{5}{2}\lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} = \frac{5}{2}\lim\limits_{x\to\infty} \dfrac{3x^2}{2xe^{x^2}} = \frac{15}{4}\lim\limits_{x\to\infty} \dfrac{x}{e^{x^2}} = \frac{15}{4}\lim\limits_{x\to\infty} \dfrac{1}{2xe^{x^2}} = 0$
by repeated use of L' Hôpital's rule and the last equality comes about because $\lim\limits_{x\to\infty} 2xe^{x^2} = \infty$ so that the limit of the reciprocal function $\lim\limits_{x\to\infty} \dfrac{1}{2xe^{x^2}}$ is 0.

(d) $\lim\limits_{x\to 0} \dfrac{\sin(\tan(x))}{\tan(\sin(x))} = \lim\limits_{x\to 0} \dfrac{\cos(\tan(x))\,\sec^2(x)}{\sec^2(\sin(x))\,\cos(x)} = \dfrac{\cos(\tan(0))\,\sec^2(0)}{\sec^2(\sin(0))\,\cos(0)} = 1$
by L' Hôpital's rule.

(e) Let $y = (e^{(x^3)}+3x^2)^{(1/x^2)}$. Then $\ln(y) = \dfrac{1}{x^2}\ln(e^{(x^3)}+3x^2)$.
Now $\lim\limits_{x\to 0} \ln(y) = \lim\limits_{x\to 0} \dfrac{\ln(e^{(x^3)}+3x^2)}{x^2} = \lim\limits_{x\to 0} \dfrac{3x^2e^{(x^3)}+6x}{2x(e^{(x^3)}+3x^2)} = \frac{3}{2}\,\dfrac{0+2}{1+0} = 3.$
Therefore, $\lim\limits_{x\to 0} y = \lim\limits_{x\to 0} e^{\ln(y)} = e^{\lim_{x\to 0}\ln(y)} = e^3.$

3. (a) $\int \frac{dx}{(x^2+2)(x^2+3)} = \int(\frac{1}{x^2+2} - \frac{1}{x^2+3})dx = \frac{1}{\sqrt{2}}\tan^{-1}(\frac{x}{\sqrt{2}}) - \frac{1}{\sqrt{3}}\tan^{-1}(\frac{x}{\sqrt{3}}) + C.$

(b) $\int \sin^{-1}(4x)dx = x\sin^{-1}(4x) - \int 4x\frac{1}{\sqrt{1-16x^2}}dx = x\sin^{-1}(4x) + \frac{1}{8}\int \frac{-32x}{\sqrt{1-16x^2}}dx$

$= x\sin^{-1}(4x) + \frac{1}{4}\sqrt{1-16x^2} + C$.

(c) $\int e^{2x}\sin(5x)dx = \frac{1}{2}e^{2x}\sin(5x) - \frac{1}{2}\int e^{2x}5\cos(5x)dx$

$= \frac{1}{2}e^{2x}\sin(5x) - \frac{5}{2}[\frac{1}{2}e^{2x}\cos(5x) + \frac{1}{2}\int e^{2x}5\sin(5x)dx]$

$= \frac{1}{2}e^{2x}\sin(5x) - \frac{5}{4}e^{2x}\cos(5x) - \frac{25}{4}\int e^{2x}\sin(5x)dx]$

by integration by parts.

Therefore, $\frac{29}{4}\int e^{2x}\sin(5x)dx = \frac{1}{2}e^{2x}\sin(5x) - \frac{5}{4}e^{2x}\cos(5x) + C.$

Thus $\int e^{2x}\sin(5x)dx = \frac{2}{29}e^{2x}\sin(5x) - \frac{5}{29}e^{2x}\cos(5x) + C'$.

Therefore, $\int_0^{\frac{\pi}{5}} e^{2x}\sin(5x)dx = \frac{1}{29}[2e^{2x}\sin(5x) - 5e^{2x}\cos(5x)]_0^{\frac{\pi}{5}}$

$= \frac{1}{29}\left[5e^0\cos(0) - 5e^{\frac{2\pi}{5}}\cos(\pi)\right]$

$= \frac{5}{29}(1 + e^{\frac{2\pi}{5}}).$

(d) $\int \frac{x+3}{x^2+2x+4}dx = \int\left(\frac{1}{2}\frac{2x+2}{x^2+2x+4} + \frac{2}{x^2+2x+4}\right)dx$

$= \frac{1}{2}\ln(x^2+2x+4) + 2\int \frac{1}{(x+1)^2+3}dx$

$= \frac{1}{2}\ln(x^2+2x+4) + \frac{2}{\sqrt{3}}\tan^{-1}(\frac{x+1}{\sqrt{3}}) + C$.

Therefore, $\int_0^2 \frac{x+3}{x^2+2x+4}dx = \left[\frac{1}{2}\ln(x^2+2x+4) + \frac{2}{\sqrt{3}}\tan^{-1}(\frac{x+1}{\sqrt{3}})\right]_0^2$

$= \frac{1}{2}((\ln(12) - \ln(4)) + \frac{2}{\sqrt{3}}(\tan^{-1}(\sqrt{3}) - \tan^{-1}(\frac{1}{\sqrt{3}}))$

$= \frac{1}{2}\ln(3) + \frac{2}{\sqrt{3}}(\frac{\pi}{3} - \frac{\pi}{6}) = \frac{1}{2}\ln(3) + \frac{\pi}{3\sqrt{3}} = \frac{1}{2}\ln(3) + \frac{\sqrt{3}\pi}{9}.$

Question 4.

(a) Let $f(x) = 3x - 2$. First note that $|f(x) - 1| = |3x - 2 - 1| = 3|x - 1|$. Therefore, given any $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{3}$. Thus,

$$0 < |x - 1| < \delta \Rightarrow |f(x) - 1| = 3|x - 1| < 3\delta = \varepsilon.$$

Therefore by the definition of limit, $\lim_{x\to 1} f(x) = 1$.

(b) First note that f is differentiable at x for x in $(-\infty, -\pi)$ or (π, ∞) since on these intervals the function is the same as sine(x) and sine (x) is differentiable on these intervals.
Now for x such that $-\pi < x < \pi$, $f(x)$ is given by a polynomial and any polynomial is differentiable on the interval $(-\pi, \pi)$. Thus we need only concern ourselves with the differentiability of f at $-\pi$ and at π.
Then a necessary condition for f to be differentiable at π is that f be continuous at π.
That is,
$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x) = f(\pi).$$

Now $\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} ax^3 + bx = a\pi^3 + b\pi = f(\pi)$ and
$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} \sin(x) = \sin(\pi) = 0.$
and so our first condition is $a\pi^3 + b\pi = 0$.
I.e. $a\pi^2 + b = 0$ -- (1)
Since we know the derivative of sin(x) is cos(x), that is

$$\lim_{y\to x} \frac{\sin(y) - \sin(x)}{y - x} = \cos(x),$$

$$\lim_{x\to\pi^+} \frac{f(x) - f(\pi)}{x - \pi} = \lim_{x\to\pi^+} \frac{\sin(x) - 0}{x - \pi} = \lim_{x\to\pi} \frac{\sin(x) - \sin(\pi)}{x - \pi} = \cos(\pi) = -1.$$

Similarly,

$$\lim_{x\to\pi^-} \frac{f(x) - f(\pi)}{x - \pi} = \lim_{x\to\pi^-} \frac{ax^3 + bx - (a\pi^3 + b\pi)}{x - \pi} = 3a\pi^2 + b.$$

Therefore, in addition to equation (1), for differentiability at π, we must have
$3a\pi^2 + b = -1$ -- (2)
Solving (1) and (2) gives $b = \frac{1}{2}$ and $a = -\frac{1}{2\pi^2}$.

For differentiability at $-\pi$, we get equations (1) and (2) above. Thus the same values for a and b above will guaranteed differentiability at $-\pi$ too.

(c) Let $f(x) = 2x^3 + 3x + 1 - 3\sin(x)\cos(x) = 2x^3 + 3x + 1 - \frac{3}{2}\sin(2x)$.
Then $f'(x) = 6x^2 + 3 - 3\cos(2x) = 6x^2 + 3(1 - \cos^2(x) + \sin^2(x)) = 6(x^2 + \sin^2(x))$.
Therefore, $f'(x) > 0$ for $x \neq 0$.
Since f is continuous on **R**, in particular f is continuous at $x = 0$.
Thus f is increasing on $(-\infty, 0]$ and on $[0, \infty)$ and so it is increasing on **R**.
Therefore f is injective.
Now $f(0) = 1 > 0$ and $f(-\pi) = -2\pi^2 - 3\pi + 1 < 0$. Therefore, by the *Intermediate Value Theorem*, there exists a point c in **R** such that $f(c) = 0$. That is, f has a root in **R**. Since f is injective, it has exactly one real root.

5. Observe that $f(x)=\begin{cases}\dfrac{2x|x|}{1+x^2}, & x<1\\ \dfrac{1}{x}, & x\ge 1\end{cases} = \begin{cases}-\dfrac{2x^2}{1+x^2}, & x<0\\ \dfrac{2x^2}{1+x^2}, & 0\le x<1\\ \dfrac{1}{x}, & x\ge 1\end{cases}$. We note that f is continuous on $(1,\infty)$ because f is a rational function on $(1,\infty)$. There are a number of ways to show that f is continuous on $(-\infty, 1)$. The following is one way. $|x|$ is a continuous function on **R** because x is continuous on **R** and that the modulus of any continuous function is also continuous. $\dfrac{2x}{1+x^2}$ is a rational function with domain **R** and so it is continuous on **R.** Therefore, $\dfrac{2x|x|}{1+x^2}$ being the product of two continuous functions is therefore continuous on **R**. In particular, $\dfrac{2x|x|}{1+x^2}$ is continuous on $(-\infty, 1)$.

Now $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}\dfrac{1}{x} = 1 = f(1)$ and

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}\dfrac{2x|x|}{1+x^2} = 1.$

Therefore, $\lim\limits_{x\to 1} f(x) = f(1)$ and so f is continuous at $x = 1$. Thus f is continuous on **R**.

Then

$$f'(x)=\begin{cases}-\dfrac{4x}{(1+x^2)^2}, & x<0\\ \dfrac{4x}{(1+x^2)^2}, & 0<x<1\\ -\dfrac{1}{x^2}, & x>1\end{cases} \quad \text{---} \quad (1)$$

$$f''(x)=\begin{cases}4\dfrac{(3x^2-1)}{(1+x^2)^3}, & x<0\\ -4\dfrac{(3x^2-1)}{(1+x^2)^3}, & 0<x<1\\ \dfrac{2}{x^3}, & x>1\end{cases} = \begin{cases}12\dfrac{(x^2-\frac{1}{3})}{(1+x^2)^3}, & x<0\\ -12\dfrac{(x^2-\frac{1}{3})}{(1+x^2)^3}, & 0<x<1\\ \dfrac{2}{x^3}, & x>1\end{cases} \quad \text{--------------} \quad (2)$$

(a)

For $x < 0$, $-4x > 0$ and so from (1), $f'(x) = \dfrac{-4x}{(1+x^2)^2} > 0$ for x in $(-\infty, 0)$ since $(1 + x^2) > 0$. Thus f is increasing on the interval $(-\infty, 0]$ since f is continuous at $x = 0$. Now for x in (0, 1) . $f'(x) = \dfrac{4x}{(1+x^2)^2} > 0$. Therefore, f is increasing on [0, 1] since f is continuous at $x = 0$ and at $x = 1$. Thus f is increasing on the interval $(-\infty, 1]$.

For $x > 1$, $f'(x) = -\dfrac{1}{x^2} < 0$. Thus f is decreasing on the interval $[1,\infty)$ since f is continuous at $x = 1$.

(b) Now $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\dfrac{1}{x} = 0$ and so the line $y = 0$ is a horizontal asymptote of the graph of f. Next we check the following limit.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} -\dfrac{2x^2}{1+x^2} = \lim\limits_{x\to-\infty} -\dfrac{2}{1+\frac{1}{x^2}} = -2$.

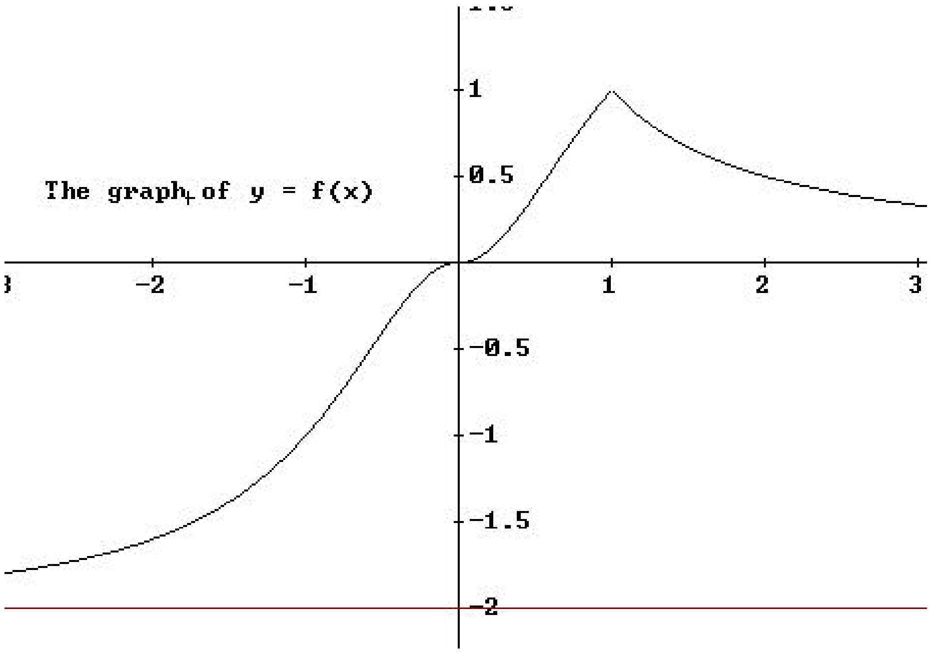
Therefore, the line $y = -2$ is another horizontal asymptote of the graph of f.

(c) When $x < -\frac{1}{\sqrt{3}}$, from (2), $f''(x) = 12\dfrac{(x^2 - \frac{1}{3})}{(1+x^2)^3} > 0$ since $(x^2 - \frac{1}{3}) > 0$. Hence the graph of f is concave upward on the interval $(-\infty, -\dfrac{1}{\sqrt{3}})$. Also from (2), when $-\frac{1}{\sqrt{3}} < x < 0$, $f''(x) = 12\dfrac{(x^2 - \frac{1}{3})}{(1+x^2)^3} < 0$. Therefore, the graph of f is concave downward on the interval $(-\dfrac{1}{\sqrt{3}}, 0)$. Again from (2), for $0 < x < \frac{1}{\sqrt{3}} (< 1)$, $f''(x) = -12\dfrac{(x^2 - \frac{1}{3})}{(1+x^2)^3} > 0$ since $(x^2 - \frac{1}{3}) < 0$. Therefore, the graph of f is concave upward on $(0, \frac{1}{\sqrt{3}})$. Now for $\frac{1}{\sqrt{3}} < x < 1$, $f''(x) = -12\dfrac{(x^2 - \frac{1}{3})}{(1+x^2)^3} < 0$ and therefore the graph of f is concave downward on $(\frac{1}{\sqrt{3}}, 1)$. Finally for $x > 1$, $f''(x) = \dfrac{2}{x^3} > 0$and so the graph of f is concave upward on $(1, \infty)$.

(d) Since from part (a) f is increasing on $(-\infty, 1]$ and decreasing on $[1, \infty)$, f has a relative maximum value at $x = 1$. Indeed the relative maximum value is $f(1) = 1$. Since f is increasing on $(-\infty, 1]$ f has no relative minimum in $(-\infty, 1]$. Likewise since f is decreasing on $[1, \infty)$ f has no relative minimum value in $[1, \infty)$. Therefore f has no relative minimum value.

(e) From part (c), there are changes of concavity before and after the following points in the graph: $(-\dfrac{1}{\sqrt{3}}, f(-\dfrac{1}{\sqrt{3}})) = (-\dfrac{1}{\sqrt{3}}, -\dfrac{1}{2})$, $(0, f(0)) = (0, 0)$, $(\dfrac{1}{\sqrt{3}}, f(\dfrac{1}{\sqrt{3}})) = (\dfrac{1}{\sqrt{3}}, \dfrac{1}{2})$ and $(1, f(1)) = (1, 1)$. Therefore these are the points of inflection

(f) ***The graph of f (not drawn to scale)***



6. (a) $g(x)=\int_{-x}^{x^3}\frac{1}{1+\sin^2(2t)+t^2}dt=\int_0^{x^3}\frac{1}{1+\sin^2(2t)+t^2}dt+\int_{-x}^{0}\frac{1}{1+\sin^2(2t)+t^2}dt$

$=\int_0^{x^3}\frac{1}{1+\sin^2(2t)+t^2}dt-\int_0^{-x}\frac{1}{1+\sin^2(2t)+t^2}dt.$

$=F(x^3)-F(-x)$ where $F(x)=\int_0^{x}\frac{1}{1+\sin^2(2t)+t^2}dt.$

Therefore, $g'(x)=F'(x^3)\cdot 3x^2-F'(-x)\cdot(-1)$ by the *Chain Rule*

$=\frac{3x^2}{1+\sin^2(2x^3)+x^6}+\frac{1}{1+\sin^2(-2x)+x^2}$ by the FTC.

(b) (i) Since $k(x)=\int_1^x\frac{1}{\sqrt{1+t^4}}dt$, by the FTC,

$k'(x)=\frac{1}{\sqrt{1+x^4}}>0$ since $1+x^4>0$.

Therefore, k is increasing on the whole of **R.** Thus k is injective.

(ii) $(k^{-1})'(0)=\frac{1}{k'(k^{-1}(0))}$. So we need to know the value of $k^{-1}(0)$. Now

$k^{-1}(0)=x\Leftrightarrow k(x)=0\Leftrightarrow\int_1^x\frac{1}{\sqrt{1+t^4}}dt=0$. Since

$k(1)=\int_1^1\frac{1}{\sqrt{1+t^4}}dt=0$ and k is injective, $x=1$.

Therefore, $(k^{-1})'(0)=\frac{1}{k'(k^{-1}(0))}=\frac{1}{k'(1)}=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}$.

(c) (i) Let $f(x)=\int_a^x h(t)dt+\int_{h(a)}^{h(x)}h^{-1}(s)ds-xh(x)$.

We want to show that this is a constant function. At this point, it is reasonable to make some assumption that allows us to proceed to show that this is true under this assumption. We then assume that h is differentiable. This is to make sure that the function f is differentiable. Notice that f is continuous on $[a, b]$ - a fact borne out by the Fundamental Theorem of Calculus and the continuity of h. We can then apply the method of differential calculus. With this assumption, by the Fundamental Theorem of Calculus, f is indeed differentiable and

$f'(x)=h(x)+h^{-1}(h(x))h'(x)-(h(x)+xh'(x))$

(by the Fundamental Theorem of Calculus and the Chain Rule)

$=0.$

Therefore, $f(x)=C$ for some constant C. Thus $C=f(a)=-a\,h(a)$.

Hence $\int_a^x h(t)d=xh(x)-ah(a)-\int_{h(a)}^{h(x)}h^{-1}(s)ds$. In particular

$\int_a^b h(t)d=bh(b)-ah(a)-\int_{h(a)}^{h(b)}h^{-1}(s)ds.$

(ii) Let $h(x)=\sqrt{1+(x-1)^{\frac{1}{3}}}$ for x in [0,1].

$h(x)=y$ if and only if $1+(x-1)^{\frac{1}{3}}=y^2\Leftrightarrow(x-1)^{\frac{1}{3}}=y^2-1$ so that

$x=(y^2-1)^3+1=y^6-3y^4+3y^2$. Therefore $h^{-1}(y)=y^6-3y^4+3y^2$.

Now $h(0)=0$ and $h(1)=1$.

Before we use part (i), note that in part (i) we only require that h be differentiable on (a, b). Hence by part (i),

$\int_0^1\sqrt{1+(x-1)^{\frac{1}{3}}}\,dx=h(1)-\int_0^1(y^6-3y^4+3y^2)dy=1-[\tfrac{1}{7}y^7-\tfrac{3}{5}y^5+y^3]_0^1$

$=1-(1+\tfrac{1}{7}-\tfrac{3}{5})=\tfrac{16}{35}.$

Or use substitution $u=1+(x-1)^{\frac{1}{3}}$. Then $x=u^3-3u^2+3u$.

$\int_0^1\sqrt{1+(x-1)^{\frac{1}{3}}}\,dx=\int_0^1(3u^{\frac{5}{2}}-6u^{\frac{3}{2}}+3u^{\frac{1}{2}})du=3\left[\tfrac{2}{7}u^{\frac{7}{2}}-\tfrac{4}{5}u^{\frac{5}{2}}+\tfrac{2}{3}u^{\frac{3}{2}}\right]_0^1$

$=\tfrac{6}{7}-\tfrac{12}{5}+2=\tfrac{16}{35}.$

Alternative proof of part (i). (This is not the expected answer for the exam. But it is the kind of answer we hope to see. The kind that will show your understanding and maturity in mathematical thought.)

This is an elegant proof that requires some deep understanding of the meaning of the Riemann integrals. First the following observation will make the proof a little easier to follow. The function h is continuous and so it is (Riemann) integrable on the interval $[a, x]$ for any $x > a$. (A deep result we are invoking.) Also since h is continuous and increasing, by a Theorem in the lectures, it then has an inverse function h^{-1} defined on $[h(a), h(b)]$ which is also continuous and increasing. (Another subtle result here.) Therefore, h^{-1} is Riemann integrable on the interval $[h(a), h(b)]$. We shall show that for a fixed $x > a$,

$$\int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds = xh(x) - ah(a).$$

We shall show that the integrals in the left hand side can be approximated by Riemann sums whose sum can be made to be as close to $x\,h(x) - a\,h(a)$ as we like. (What does this mean? It means the difference can be observed to be as small as we like.) We are going to use the following simple test of two quantities c and d being equal:

If for any $\varepsilon > 0$, we have $c \le d + \varepsilon$ and $d \le c + \varepsilon$, then $c = d$.

Take a regular partition of the interval $[a, x]$,

$$P : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = x,$$

where $x_i = a + i\dfrac{x-a}{n}$ for i =0, 1, ... , n.

Then this gives rise to a partition

$$Q : h(a) = h(x_0) < h(x_1) < h(x_2) < \cdots < h(x_{n-1}) < h(x_n) = h(x)$$

for the interval $[h(a), h(b)]$. Note that the strict inequality is preserved because h is increasing.

(The above partition for $[h(a), h(b)]$ need not necessary be uniform.)

The Lower Riemann Sum for the function h with respect to the partition P is

$$\sum_{i=1}^n m_i(h)(x_i - x_{i-1}) = \sum_{i=1}^n h(x_{i-1})(x_i - x_{i-1}) = h(x_0)(x_1 - x_0) + \sum_{i=2}^n h(x_{i-1})(x_i - x_{i-1}),$$

where $m_i(h) = \min\{h(k) : k \in [x_{i-1}, x_i]\} = h(x_{i-1})$ because h is an increasing and continuous function. (We are using here the Extreme Value Theorem.)

Thus

$$\int_a^x h(t)dt \ge h(x_0)(x_1 - x_0) + \sum_{i=2}^n h(x_{i-1})(x_i - x_{i-1}) = h(x_0)(x_1 - x_0) + \Delta x \sum_{i=2}^n h(x_{i-1})$$

where $\Delta x = \dfrac{x-a}{n}$.

(We are using here the fact that if the Riemann integral exists, then it is always greater than any Lower Riemann Sum.)

Likewise the Lower Riemann Sum for the function h^{-1} with respect to the partition Q is

$$\sum_{i=1}^n m_i(h^{-1})(h(x_i) - h(x_{i-1})) = \sum_{i=1}^n h^{-1}(h(x_{i-1}))(h(x_i) - h(x_{i-1})) = \sum_{i=1}^n x_{i-1}(h(x_i) - h(x_{i-1})),$$

where $m_i(h^{-1}) = \min\{h^{-1}(k) : k \in [h(x_{i-1}), h(x_i)]\} = h^{-1}(h(x_{i-1})) = x_{i-1}$ because h^{-1} is an increasing and continuous function.

Therefore $\displaystyle\int_{h(a)}^{h(x)} h^{-1}(s)ds \ge \sum_{i=1}^n x_{i-1}(h(x_i) - h(x_{i-1})) = \sum_{i=1}^n x_{i-1}h(x_i) - \sum_{i=1}^n x_{i-1}h(x_{i-1})$

$$= x_{n-1}h(x_n) + \sum_{i=1}^{n-1} x_{i-1}h(x_i) - x_0h(x_0) - \sum_{i=1}^{n-1} x_i h(x_i)$$

$$= x_{n-1}h(x_n) - x_0h(x_0) - \sum_{i=1}^{n-1}(x_i - x_{i-1})h(x_i) = x_{n-1}h(x_n) - x_0h(x_0) - \Delta x \sum_{i=1}^{n-1} h(x_i).$$

Thus, $\displaystyle\int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds \ge h(x_0)(x_1 - x_0) + x_{n-1}h(x_n) - x_0h(x_0)$

$$= x_nh(x_n) - x_0h(x_0) + h(x_0)(x_1 - x_0) - (x_n - x_{n-1})h(x_n)$$

$$= xh(x) - ah(a) - h(x_n)\Delta x + h(x_0)\Delta x$$
$$= xh(x) - ah(a) - (h(x) - h(a))\Delta x.$$

Hence $xh(x) - ah(a) \le \int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds + (h(x) - h(a))\Delta x.$

Now for any ε > 0, we can choose an integer $n > 0$ such that
$$(h(x) - h(a))\Delta x = (h(x) - h(a))\frac{x-a}{n} \le \varepsilon.$$

Hence for any ε > 0, $xh(x) - ah(a) \le \int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds + \varepsilon.$ Therefore
$$xh(x) - ah(a) \le \int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds \quad \text{---------------------------} \quad (1)$$

Now we shall use the Upper Riemann Sum of h with respect to the partition P in a similar way. The Upper Riemann Sum of h with respect to the partition P is
$$\sum_{i=1}^{n} h(x_i)(x_i - x_{i-1}).$$

Thus we have
$$\int_a^x h(t)dt \le \Delta x \sum_{i=1}^{n} h(x_i).$$

(Here we are using the fact that if the Riemann integral exists, then it is always less than any Upper Riemann Sum.)

The Upper Riemann sum of the function h^{-1} with respect to the partition Q is
$$\sum_{i=1}^{n} M_i(h^{-1})(h(x_i) - h(x_{i-1})) = \sum_{i=1}^{n} h^{-1}(h(x_i))(h(x_i) - h(x_{i-1})) = \sum_{i=1}^{n} x_i(h(x_i) - h(x_{i-1})),$$

where $M_i(h^{-1}) = \max\{h^{-1}(k) : k \in [h(x_{i-1}), h(x_i)]\} = h^{-1}(h(x_i)) = x_i$ because h^{-1} is an increasing and continuous function.

Therefore, $\int_{h(a)}^{h(x)} h^{-1}(s)ds \le \sum_{i=1}^{n} x_i(h(x_i) - h(x_{i-1})) = \sum_{i=1}^{n} x_i h(x_i) - \sum_{i=1}^{n} x_i h(x_{i-1})$
$$= x_n h(x_n) - x_0 h(x_0) + \sum_{i=1}^{n} x_{i-1} h(x_{i-1}) - \sum_{i=1}^{n} x_i h(x_{i-1})$$
$$= x_n h(x_n) - x_0 h(x_0) - \sum_{i=1}^{n} (x_i - x_{i-1}) h(x_{i-1})$$
$$= xh(x) - ah(a) - \Delta x \sum_{i=1}^{n} h(x_{i-1}) = xh(x) - ah(a) - \Delta x \sum_{i=0}^{n-1} h(x_i).$$

Thus $\int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds \le xh(x) - ah(a) + \Delta x \sum_{i=1}^{n} h(x_i) - \Delta x \sum_{i=0}^{n-1} h(x_i)$
$$= xh(x) - ah(a) + \Delta x h(x_n) - \Delta x h(x_0)$$
$$= xh(x) - ah(a) + (h(x) - h(a))\Delta x.$$

Since for any ε > 0, we can choose an integer $n > 0$ such that
$$(h(x) - h(a))\Delta x = (h(x) - h(a))\frac{x-a}{n} \le \varepsilon,$$

we have that $\int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds \le xh(x) - ah(a) + \varepsilon$ for any ε > 0. Thus
$$\int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds \le xh(x) - ah(a). \quad \text{------------------------} \quad (2)$$

Therefore (1) and (2) imply that
$$\int_a^x h(t)dt + \int_{h(a)}^{h(x)} h^{-1}(s)ds = xh(x) - ah(a).$$

This is a proof that demonstrates the precision and thoroughness of mathematical reasoning. The use of some important theorems and results along the way shows how intricate the solution is woven to invoke the rhythm of the mathematics to deduce the answer in this resounding manner. Notice how the theorems and results are calling out to be used.

Tze Beng Ng November 1999