NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2 EXAMINATION 1999 – 2000

MA1102 CALCULUS

April/May 2000 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO (2)** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} -x^3, & x > 1 \\ x^3 \cos(\frac{\pi}{x}), & -1 \le x \le 1 \text{ and } x \ne 0 \\ x^2 + 1, & x < -1 \\ 0, & x = 0 \end{cases}.$$

(a) Find the *range* of the function f.

(b) Determine if f is surjective.

(c) Find the values of x (if any) where
(i) $f(x) = 1$,
(ii) $f(x) = -2$.

(d) Determine all x in $\mathbf{R}$ at which the function f is *continuous*.

(e) Is the function f *differentiable* at $x = 1$? Justify your answer.

(f) Compute $\int_{-1}^{1} f(x)dx$.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\lim_{x \to -\infty} \dfrac{73|x|^5 + 5x + 1000}{62x^5 + x^2 + 3}$.

(b) $\lim_{x \to 0} \dfrac{5x^4}{(\sqrt{5x^2 + 169} - 13)\sin(x^2)}$.

(c) $\lim_{x \to 0} \dfrac{e^{(e^x)} + 3x^2 - ex - e}{x^2}$.

(d) $\lim_{x \to 0} \dfrac{\sin(\sin(x^2))}{\sin(3x) + x}$.

(e) $\lim_{x \to \infty} \dfrac{\ln(\ln(x))}{\ln(x)}$.

Question 3 [**20** marks]

Evaluate the following integrals.

(a) $\int \frac{1}{(x^2+2x+2)(x^2+2x+3)}dx.$

(b) $\int \cos^{-1}(7x)dx.$

(c) $\int_0^{\frac{\pi}{7}} e^{2x}\cos(7x)dx.$

(d) $\int_0^3 \frac{x+4}{x^2+6x+9}dx.$

SECTION B

Answer not more than ***TWO (2)*** *questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Suppose $g : [0, 3] \to \mathbf{R}$ is a real valued function defined on the interval [0, 3] by $g(x) = 2x^3 - 9x^2 + 12x + 1$. Determine the absolute maximum and absolute minimum of the function g.

(b) Differentiate the following functions.

(i) $h(x) = \sin^{-1}(\cos(x)).$

(ii) $j(x) = \ln\left(\frac{2+e^x}{1+e^x}\right).$

(iii) $k(x) = \ln(\ln(x^2+2)).$

(c) Suppose f is a continuous function defined on the closed interval $[-1, 2]$ such that

$$-1 \le f(x) \le 2 \text{ for all } x \text{ in } [-1, 2].$$

Prove that there exists a point c in $[-1, 2]$ such that $f(c) = c$.

Question 5 [**20** marks]

Let the function f be defined on **R** by

$$f(x) = 3x^5 + 15x^4 + 35x^3 + 90x^2 + 32.$$

(a) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(b) Find the intervals on which the graph of f is *concave upward* or *concave downward*.

(c) Find the *relative extrema* of f.

(d) Find the *points of inflection* of the graph of f.

(e) Sketch the graph of f.

Question 6 [**20** marks]

(a) State clearly the Fundamental Theorem of Calculus.

Hence use it or otherwise to differentiate the function

$$g(x) = \int_{-\ln(x)}^{x^3} \frac{1}{1 + e^{2t} + t^2}\, dt.$$

(b) Let the function k be defined on **R** by

$$k(x) = \int_2^x \frac{1}{\sqrt{1 + t^2 + t^4}}\, dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(c) Find the following limit.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i}{n^2} \cdot \sqrt{4 + \left(\frac{i}{n}\right)^2}\,.$$

END OF PAPER

Answer To MA1102 Calculus

SECTION A (Compulsory)

1. The function f is defined by $f(x) = \begin{cases} -x^3, & x > 1 \\ x^3\cos(\frac{\pi}{x}), & -1 \le x \le 1 \text{ and } x \ne 0 \\ x^2+1, & x < -1 \\ 0, & x = 0 \end{cases}$.

(a) For $x < -1$, $f(x) = x^2 + 1 > 2$. Also, for $x < -1$, $x^2 + 1 > 2 \Leftrightarrow x < -1$.
Thus f maps $(-\infty, -1)$ onto $(2, \infty)$. (Because for any $y > 2$, we can take $x = -\sqrt{y-1}$ so that $f(x) = y$) Also, for $-1 \le x \le 1$, $-1 \le f(x) \le 1$. This is seen as follows. For $-1 \le x \le 1$ and $x \ne 0$, $|f(x)| = \left|x^3\cos(\frac{\pi}{x})\right| \le |x|^3 \le 1$. Now $f(0) = 0$. Thus $-1 \le f(x) \le 1$. Therefore, $f(-1) = 1$ is the absolute maximum of f on [–1, 1] and $f(1) = -1$ is the absolute minimum of f on [–1, 1]. Assuming that f is continuous on [–1, 1] (as we shall show in part (d) below), by the Intermediate Value Theorem) f maps the interval [–1,1] onto [–1,1]. [We can also use the fact that $x^3\cos(\frac{\pi}{x})$ maps [2/3, 1] onto [–1,0] and [-1, -3/2] onto [0,1].]
Finally for $x > 1$, $f(x) = -x^3 < -1$. And for any $y < -1$, we can take $x = \sqrt[3]{-y} > 1$ so that $f(x) = y$. Hence f maps $(1, \infty)$ onto $(-\infty, -1)$.
Hence the range of f is $(2, \infty) \cup [-1, 1] \cup (-\infty, -1) = (-\infty, 1] \cup (2, \infty)$.

(b) By part (a) Range(f) = $(-\infty, 1] \cup (2, \infty) \ne \mathbf{R}$ = codomain(f), therefore f is not surjective.

(c) (i) By part (a) 1 is in the image of [–1, 1] under f. Thus, to find the preimage we need to solve the equation $x^3\cos(\frac{\pi}{x}) = 1$ for x *in* [–1, 1]-{0}.
For $x \ne 0$ and $-1 < x < 1$, $|f(x)| = \left|x^3\cos(\frac{\pi}{x})\right| \le |x|^3 < 1$. Since we know $f(-1) = 1$ and $f(1) < 0$, $x = -1$.
(ii) From part (a) -2 is in the image of $(1, \infty)$ under of f. Thus, the solution of $f(x) = -2$ does gives $x = \sqrt[3]{2}$.

(d) When $x < -1$, $f(x) = x^2 - 1$, which is a polynomial function, therefore f is continuous on $(-\infty, -1)$, since any polynomial function is continuous on the real numbers and so is continuous on any open interval. When $-1 < x < 1$ and $x \ne 0$, $f(x) = x^3\cos(\frac{\pi}{x})$ and since $x^3\cos(\frac{\pi}{x})$ is continuous on (–1, 0) and on (0, 1), f is continuous on the union of these two intervals. Finally when $x > 1$, $f(x)$ is a polynomial function and so it is continuous for $x > 1$. Thus it remains to check if f is continuous at $x = -1$, 0 or 1. Consider the left limit at $x = 1$,
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3\cos(\frac{\pi}{x}) = 1^3\cos(\pi) = -1 = f(1)$ and the right limit at $x = 1$
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -x^3 = -1$. Thus
$\lim_{x\to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.

Now consider the left limit of f at $x = -1$,
$\lim_{x\to(-1)^-} f(x) = \lim_{x\to(-1)^-} x^2 + 1 = 2$ and the right limit at $x = -1$,
$\lim_{x\to(-1)^+} f(x) = \lim_{x\to(-1)^+} x^3\cos(\frac{\pi}{x}) = -1\cos(-\pi) = 1$.

Thus the left and the right limits of f at $x = -1$ are not the same and so f is not continuous at $x = -1$.
Now $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^3 \cos(\frac{\pi}{x}) = 0$ by the Squeeze Theorem and $f(0) = 0$. Therefore, f is continuous at $x = 0$.
Hence f is continuous at x for all $x \neq -1$.

(e) f is differentiable at $x = 1$. This is seen as follows.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x^3\cos(\frac{\pi}{x})+1}{x-1} = \lim_{x\to 1^-} \frac{3x^2\cos(\frac{\pi}{x})+x\pi\sin(\frac{\pi}{x})}{1}$$

by L' Hôpital's Rule

$$= -3.$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{-x^3+1}{x-1} = -\lim_{x\to 1^+} (x^2+x+1) = -3.$$

Thus f is differentiable at $x = 1$ and $f'(1) = -3$.

(f) Note that f is an odd function since $f(-x) = -f(x)$.

$\int_{-1}^{0} f(x)dx = -\int_{1}^{0} f(-t)dt$ where $t = -x$

$= \int_{1}^{0} f(t)dt = -\int_{0}^{1} f(t)dt.$

Therefore, $\int_{-1}^{1} f(x)dx = \int_{0}^{1} f(x)dx + \int_{-1}^{0} f(x)dx = 0.$

2. (a) $\lim_{x\to-\infty} \dfrac{73|x|^5+5x+1000}{62x^5+x^2+3} = \lim_{x\to-\infty} \dfrac{-73x^5+5x+1000}{62x^5+x^2+3} = -\dfrac{73}{62}.$

(b) $\lim_{x\to 0} \dfrac{5x^4}{(\sqrt{5x^2+169}-13)\sin(x^2)}$

$=\lim_{x\to 0} 5\dfrac{x^2(\sqrt{5x^2+169}+13)}{(\sqrt{5x^2+169}-13)(\sqrt{5x^2+169}+13)} \cdot \dfrac{x^2}{\sin(x^2)}$

$=\lim_{x\to 0} 5\dfrac{x^2(\sqrt{5x^2+169}+13)}{5x^2} \cdot \lim_{x\to 0} \dfrac{x^2}{\sin(x^2)} = 26.$

(c) $\lim_{x\to 0} \dfrac{e^{(e^x)}+3x^2-ex-e}{x^2} = \lim_{x\to 0} \dfrac{e^{(e^x)}e^x+6x-e}{2x} = \lim_{x\to 0} \dfrac{e^{(e^x)}e^x+e^{(e^x)}e^{2x}+6}{2} = e+3$
by repeated use of L' Hôpital's rule.

(d) $\lim_{x\to 0} \dfrac{\sin(\sin(x^2))}{\sin(3x)+x} = \lim_{x\to 0} \dfrac{\sin(\sin(x^2))}{\sin(x^2)} \cdot \dfrac{\sin(x^2)}{x^2} \cdot \dfrac{1}{\frac{\sin(3x)}{3x}+\frac{x}{3x}} \cdot \frac{x}{3} = 1 \cdot 1 \cdot \frac{1}{1+\frac{1}{3}} \cdot 0 = 0$

by L' Hôpital's rule.

(e) $\lim_{x\to\infty} \dfrac{\ln(\ln(x))}{\ln(x)} = \lim_{x\to\infty} \dfrac{\frac{1}{x\ln(x)}}{\frac{1}{x}} = \lim_{x\to\infty} \dfrac{1}{\ln(x)} = 0$ by L' Hôpital's rule and the last equality is because $\lim_{x\to\infty} \ln(x) = \infty$.

3. (a)

$$\int \frac{dx}{(x^2+2x+2)(x^2+2x+3)} = \int \left(\frac{1}{x^2+2x+2} - \frac{1}{x^2+2x+3}\right)dx$$

$$= \int \left(\frac{1}{(x+1)^2+1} - \frac{1}{(x+1)^2+2}\right)dx = \tan^{-1}(x+1) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

(b)

$$\int \cos^{-1}(7x)dx = x\cos^{-1}(7x) - \int x\frac{-7}{\sqrt{1-49x^2}}dx = x\cos^{-1}(7x) - \tfrac{1}{14}\int \frac{-98x}{\sqrt{1-49x^2}}dx$$

$$= x\cos^{-1}(7x) - \tfrac{1}{7}\sqrt{1-49x^2} + C\ .$$

(c)

$$\int e^{2x}\cos(7x)dx = \tfrac{1}{2}e^{2x}\cos(7x) + \tfrac{1}{2}\int e^{2x}7\sin(7x)dx$$

$$= \tfrac{1}{2}e^{2x}\cos(7x) + \tfrac{7}{2}[\tfrac{1}{2}e^{2x}\sin(7x) - \tfrac{1}{2}\int e^{2x}7\cos(7x)dx]$$

$$= \tfrac{1}{2}e^{2x}\cos(7x) + \tfrac{7}{4}e^{2x}\sin(7x) - \tfrac{49}{4}\int e^{2x}\cos(7x)dx$$

by integration by parts.

Therefore, $\tfrac{53}{4}\int e^{2x}\cos(7x)dx = \tfrac{1}{2}e^{2x}\cos(7x) + \tfrac{7}{4}e^{2x}\sin(7x) + C.$

Thus $\int e^{2x}\cos(7x)dx = \tfrac{2}{53}e^{2x}\cos(7x) + \tfrac{7}{53}e^{2x}\sin(7x) + C'.$

Therefore, $\int_0^{\frac{\pi}{7}} e^{2x}\cos(7x)dx = \frac{1}{53}[2e^{2x}\cos(7x) + 7e^{2x}\sin(7x)]_0^{\frac{\pi}{7}}$

$$= \frac{1}{53}\left[-2e^0\cos(0) + 2e^{\frac{2\pi}{7}}\cos(\pi)\right]$$

$$= -\frac{2}{53}(1 + e^{\frac{2\pi}{7}}).$$

(d)

$$\int \frac{x+4}{x^2+6x+9}dx = \int\left(\tfrac{1}{2}\frac{2x+6}{x^2+6x+9} + \frac{1}{x^2+6x+9}\right)dx$$

$$= \tfrac{1}{2}\ln(x^2+6x+9) + \int \frac{1}{(x+3)^2}dx$$

$$= \tfrac{1}{2}\ln(x^2+6x+9) - \frac{1}{(x+3)} + C\ .$$

Therefore, $\int_0^3 \frac{x+4}{x^2+6x+9}dx = \left[\tfrac{1}{2}\ln(x^2+6x+9) - \frac{1}{(x+3)}\right]_0^3$

$$= \tfrac{1}{2}((\ln(36) - \ln(9)) - \frac{1}{6} + \frac{1}{3}$$

$$= \ln(2) + \frac{1}{6}.$$

Question 4.

(a) Note that g, being defined by a polynomial, is continuous on the closed interval [0, 3] Therefore the *Extreme value Theorem* says that g has an absolute maximum value and an absolute minimum value.
Since $g(x) = 2x^3 - 9x^2 + 12x + 1$, its derivative is given by

$$g'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2) .$$

Thus $g'(x) = 0$ if and only if $x = 1$, and $x = 2$. Therefore, the critical points of g in (0, 3) are 1 and 2. Now $g(0) = 1$, $g(1) = 6$, $g(2) = 5$ and $g(3) = 10$. Therefore the absolute maximum value of g is 10 and the absolute minimum value of g is 1.

(b)

(i) $h(x) = \sin^{-1}(\cos(x))$. Therefore

$$h'(x) = (\sin^{-1})'(\cos(x))\ \cos'(x) = \frac{1}{\sqrt{1-\cos^2(x)}}(-\sin(x))$$

$$= \frac{1}{\sqrt{\sin^2(x)}}(-\sin(x)) = -\frac{\sin(x)}{|\sin(x)|} = -sign(\sin(x)).$$

(ii) $j(x) = \ln\left(\frac{2+e^x}{1+e^x}\right) = \ln(2+e^x) - \ln(1+e^x)$.

Therefore, $j'(x) = \ln'(2+e^x)e^x - \ln(1+e^x)e^x = \frac{e^x}{2+e^x} - \frac{e^x}{1+e^x}$

$$= \frac{e^x + e^{2x} - 2e^x - e^{2x}}{(2+e^x)(1+e^x)} = -\frac{e^x}{(1+e^x)(2+e^x)} \ .$$

(iii) $k(x) = \ln(\ln(x^2+2))$.

Therefore, $k(x) = \ln'(\ln(x^2+2))\ \ln'(x^2+2)\cdot\frac{d}{dx}(x^2+2)$.

$$= \frac{1}{\ln(x^2+2)}\cdot\frac{1}{(x^2+2)}\cdot 2x = \frac{2x}{(x^2+2)\ln(x^2+2)}.$$

(c)

Let $g(x) = f(x) - x$. Then g is a continuous function on [−1, 2] since f is continuous on [−1, 2] and x is a continuous function and we know the difference of two continuous function is a continuous function.
Because $-1 \le f(x) \le 2$ for all x in [−1, 2], $f(2) - 2 \le 0$ and $f(-1) - (-1) \ge 0$.
I.e. $g(2) = f(2) - 2 \le 0 \le f(-1) - (-1) = g(-1)$. Therefore, by the *Intermediate Value Theorem*, there exists a point c in [−1, 2] such that $g(c) = 0$. That is, $f(c) = c$.

5. $f(x) = 3x^5 + 15x^4 + 35x^3 + 90x^2 + 32$. Note that f is continuous and differentiable on **R**.

$f'(x) = 15x^4 + 60x^3 + 105x^2 + 180x = 15x(x^3 + 4x^2 + 7x + 12)$.
Now we know that the cubic polynomial function $g(x) = x^3 + 4x^2 + 7x + 12 = 0$ has a real root. We can try to use the Intermediate Value Theorem to locate the root. (Of Course we can use Cardano's formula for the cubic but we can do this first.) We compute some value of g. $g(-1) = -1 + 4 - 7 + 12 = 8 > 0$, $g(-2) = -8 + 16 - 14 + 12 = 6 > 0$ and $g(-3) = -3\cdot 3^2 + 4\cdot 3^2 - 21 + 12 = 0$. Therefore, $(x + 3)$ is a factor of $x^3 + 4x^2 + 7x + 12$. Perform a long division to obtain $x^3 + 4x^2 + 7x + 12 = (x + 3)(x^2 + x + 4)$. Thereofore

$$f'(x) = 15x(x^3 + 4x^2 + 7x + 12) = 15x(x+3)(x^2 + x + 4)$$
$$= 15x(x+3)((x+\tfrac{1}{2})^2 - \tfrac{1}{4} + 4) = 15x(x+3)((x+\tfrac{1}{2})^2 + \tfrac{15}{4}) \qquad (1)$$

$f''(x) = 60x^3 + 180x^2 + 210x + 180 = 30(2x^3 + 6x^2 + 7x + 6)$. So f'' is given by a cubic polynomial function. Again we know it must have a real root. So we shall try to locate the root by the use of the Intermediate Value Theorem. Consider the function $k(x) = 2x^3 + 6x^2 + 7x + 6$. Then $k(0) = 6 > 0$, $k(-1) = 3 > 0$ and $k(-2) = 0$. Thus factoring out $(x + 2)$, we have

$$k(x) = (x+2)(2x^2 + 2x + 3) = (x+2)(2(x+\tfrac{1}{2})^2 - \tfrac{1}{2} + 3)$$

and so

$$f''(x) = 30k(x) = 30(x+2)(2(x+\tfrac{1}{2})^2 + \tfrac{5}{2})). \qquad (2)$$

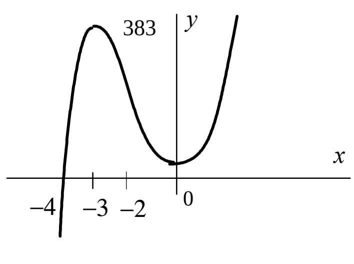
a. From (1), $f'(x) = 0$ if and only if $x = 0$ and $x = -3$ Note that $x^2 + x + 4 > 0$ and so from (1) we have: $x < -3 \Rightarrow f'(x) > 0$ so that f is increasing on$(-\infty, -3]$; $-3 < x < 0 \Rightarrow f'(x) < 0$ so that f is decreasing on $[-3, 0]$ and $x > 0 \Rightarrow f'(x) > 0$ so that f is increasing on $[0, \infty)$.

b. From (2), $f''(x) = 0 \Leftrightarrow x = -2$. Now $x < -2 \Rightarrow f''(x) < 0$. Therefore, the graph of f is concave downward on the interval $(-\infty, -2)$. Likewise from (2), $x > -2 \Rightarrow x + 2 > 0$ so that $f''(x) > 0$ when $x > -2$. Thus the graph of f is concave upward on $(-2, \infty)$.

c. From part a, by the first derivative test, $f(-3) = 383$ is a relative maximum and $f(0) = 32$ is a relative minimum.

d. From part b, $(-2, f(-2)) = (-2, 256)$ is a point of inflection of the graph of f.

e.



6. (a) **Fundamental Theorem of Calculus**.
Let f be a continuous function defined on $[a, b]$. For the function F defined on $[a, b]$

by $\int_a^x f(t)dt$, $F'(x) = f(x)$. I.e. F is a special anti-derivative of f given via the definition of Riemann integral. Moreover, for any anti-derivative G of f, $\int_a^b f(t)dt = G(b) - G(a)$.

$$g(x) = \int_{-\ln(x)}^{x^3} \frac{1}{1+e^{2t}+t^2}dt = \int_0^{x^3} \frac{1}{1+e^{2t}+t^2}dt + \int_{-\ln(x)}^{0} \frac{1}{1+e^{2t}+t^2}dt$$
$$= \int_0^{x^3} \frac{1}{1+e^{2t}+t^2}dt - \int_0^{-\ln(x)} \frac{1}{1+e^{2t}+t^2}dt.$$
$$= F(x^3) - F(-\ln(x)) \quad \text{where } F(x) = \int_0^x \frac{1}{1+e^{2t}+t^2}dt.$$

Therefore,

$g'(x) = F'(x^3)\cdot 3x^2 - F'(-\ln(x))\cdot(-\frac{1}{x})$ by the *Chain Rule*

$$= \frac{3x^2}{1+e^{2x^3}+x^6} + \frac{1}{x(1+\frac{1}{x^2}+(\ln(x))^2)} \text{ by the FTC.}$$
$$= \frac{3x^2}{1+e^{2x^3}+x^6} + \frac{x}{(x^2+1+x^2(\ln(x))^2)}.$$

(b) (i) Since $k(x) = \int_2^x \frac{1}{\sqrt{1++t^2+t^4}}dt$, by the FTC,

$$k'(x) = \frac{1}{\sqrt{1+x^2+x^4}} > 0 \text{ since } 1+x^2+x^4 > 0.$$

Therefore, k is increasing on the whole of **R.** Thus k is injective.

(ii) $(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))}$. So we need to know the value of $k^{-1}(0)$. Now $k^{-1}(0) = x \Leftrightarrow k(x) = 0 \Leftrightarrow \int_2^x \frac{1}{\sqrt{1+t^4}}dt = 0$. Since $k(2) = \int_2^2 \frac{1}{\sqrt{1+t^2+t^4}}dt = 0$ and k is injective, $x = 2$.

Therefore, $(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))} = \frac{1}{k'(2)} = \frac{1}{\frac{1}{\sqrt{21}}} = \sqrt{21}$.

(c) Try to write the following as a Riemann sum

$$\sum_{i=1}^{n} \frac{i}{n^2}\sqrt{4+\left(\frac{i}{n}\right)^2} = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \cdots < x_n$ is a regular partition and $\Delta x = \Delta x_i = x_i - x_{i-1}$.
Therefore, we can take $x_i = \frac{i}{n}$ so that $\Delta x = \frac{1}{n}, x_0 = 0$ and $x_n = 1$. Thus by comparing

$f(x_i)\Delta x$ with $\frac{i}{n^2}\sqrt{4+\left(\frac{i}{n}\right)^2} = \frac{i}{n}\sqrt{4+\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n}$

we would want $f(x_i) = f(\frac{i}{n}) = \frac{i}{n}\sqrt{4+\left(\frac{i}{n}\right)^2}$. Thus $f(x) = x\sqrt{4+x^2}$.

Therefore $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2}\sqrt{4+\left(\frac{i}{n}\right)^2} = \int_0^1 x\sqrt{4+x^2}\,dx = \frac{1}{3}\left[(4+x^2)^{\frac{3}{2}}\right]_0^1$
$= \frac{1}{3}(5^{\frac{3}{2}} - 8).$