NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2001 – 2002

MA1102R CALCULUS

May 2002 – Time Allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SIX (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

3. Answer not more than **TWO (2)** questions from Section B. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

*Answer **ALL** questions in this section.*

Question 1 [**20** marks]

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 14, & x < -3 \\ 5, & x = -3 \\ 6x + 13, & -3 < x < 1 \\ 2x^3 + 17, & x \geq 1 \end{cases} .$$

(a) Find the *range* of the function f .

(b) Determine if f is surjective.

(c) Determine all x in **R** at which the function f is *continuous*.

(d) Find all x in **R** at which the function f is *differentiable*. Justify your answer.

(e) Compute $\int_{-4}^{0} f(x)dx$.

Question 2 [**20** marks]

Evaluate, if it exists, each of the following limits.

(a) $\lim\limits_{x \to -\infty} \dfrac{3x^3 - 7|x^5| + 9}{49x^5 + 5x^2 + 1}$.

(b) $\lim\limits_{x \to \infty} (1 + x^2)^{\frac{1}{x^2}}$.

(c) $\lim\limits_{x \to 0} \dfrac{\sin(\sin(x^3))}{\sin(3x^2) + x^2}$.

(d) $\lim\limits_{x \to 4} \dfrac{\sqrt{20 + x^2} - 6}{x - 4}$.

(e) $\lim\limits_{x \to 0^+} (\sin(x^2))^{\sin(x^2)}$.

Question 3 [**20** marks]

(a) Evaluate $\int \dfrac{15x^2e^{5x^3}+\sin(2x)}{e^{5x^3}+\sin^2(x)+3}dx.$

(b) Compute $\int_{-1}^{1}\sin(|x|+3)dx.$

(c) Find an antiderivative of $g(x)$, which is defined by

$$g(x)=\begin{cases} 2x^3+4, & x\geq 1 \\ x^4+5, & x<1 \end{cases}.$$

(d) Evaluate $\int_0^2 \dfrac{x+3}{x^2+4x+4}dx.$

(e) Evaluate $\int x^2\cos(2x)dx.$

SECTION B

*Answer not more than **TWO (2)** questions from this section. Each question in this section carries 20 marks.*

Question 4 [**20** marks]

(a) Suppose g: [0, 5] $\to$ **R** is a real valued function defined on the interval [0, 5] by $g(x) = x^3 - 6x^2 + 9x + 12$. Determine the absolute maximum and absolute minimum of the function g.

(b) Differentiate the following functions.

(i) $h(x)=\cos^{-1}(\sin(2x))$.

(ii) $j(x)=\ln\left(\dfrac{x^2+e^x}{1+e^{(x^2)}}\right)$.

(iii) $k(x)=\ln(\ln(e^x+2)+x^2)$.

(c) Suppose f is a continuous function defined on the closed interval [1, 3] such that

$$1\leq f(x)\leq 3 \text{ for all } x \text{ in } [1, 3].$$

Prove that there exists a point c in [1, 3] such that $f(c) = c$.

Question 5 [**20** marks]

Let the function f be defined on **R** by

$$f(x) = x^5 - 10x^2 + 1 \ .$$

(a) Find the intervals on which f is (i) *increasing*, and (ii) *decreasing*.

(b) Find the intervals on which the graph of f is *concave upward* or *concave downward.*

(c) Find the *relative extrema* of f .

(d) Find the *points of inflection* of the graph of f.

(e) Sketch the graph of f.

Question 6 [**20** marks]

(a) State the Fundamental Theorem of Calculus.

Use it ,or otherwise, to differentiate the function

$$g(x) = \int_{\ln(x)}^{x^2} \frac{1}{1 + e^{3t} + \sin^2(t)} dt.$$

(b) Let the function k be defined on **R** by

$$k(x) = \int_1^x \frac{1}{1 + t^3 + t^6} dt.$$

(i) Without integrating, show that the function k is injective.

(ii) Determine $(k^{-1})'(0)$.

(c) Find the following limit.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i}{n^2} \cdot \sin\left(2 + 4\left(\frac{i}{n}\right)^2\right).$$

END OF PAPER

Answer To MA1102 Calculus

SECTION A (Compulsory)

1. The function f is defined by $f(x) = \begin{cases} x^2 - 14, & x < -3 \\ 5\,, & x = -3 \\ 6x + 13\,, & -3 < x < 1 \\ 2x^3 + 17, & x \geq 1 \end{cases}$.

(a) For $x < -3$, $f(x) = x^2 - 14 > -5$. Also, for $x < -3$,$x^2 - 14 > -5 \Leftrightarrow x < -3$. Thus f maps $(-\infty, -3)$ onto $(-5, \infty)$. (Because for any $y > -5$, we can take $x = -\sqrt{y+14}$ (< -3) so that $f(x) = y$) Also, for$-3 < x < 1$, $f(x) = 6x + 13$. Therefore, $-5 < f(x) < 19$. This is because $-3 < x < 1 \Leftrightarrow -5 < 6x + 13 < 19$. For any y with $-5 < y < 19$ we can take $x = \dfrac{y-13}{6}$ and for this value of x, $-3 < x < 1$. It follows that f maps $(-3, 1)$ onto $(-5, 19)$. Now for $x \geq 1$, $f(x) = 2x^3 + 17 \geq 19$. Also for any $y \geq 19$, we can take $x = \sqrt[3]{\dfrac{y-17}{2}} \geq 1$. Therefore, f maps $[1, \infty)$ onto $[19, \infty)$. Hence the range of f is $f((-\infty, -3)) \cup \{f(-3)\} \cup f((-3, 1)) \cup f([1,\infty)) = (-5, \infty) \cup \{5\} \cup (-5, 19) \cup [19, \infty) = (-5, \infty)$.

(b) By part (a) Range(f) = $(-5, \infty) \neq \mathbf{R}$ =codomain of f . Therefore, f is not surjective.

(c) and (d)

When $x < -3$, $f(x) = x^2 - 14$, which is a polynomial function. Therefore, f is differentiable on $(-\infty, -3)$, since any polynomial function is differentiable on the real numbers and so is differentiable on any open interval. When $-3 < x < 1$, $f(x) = 6x + 13$ and is a polynomial function and so f is differentiable on $(-3, 1)$. Likewise f is differentiable on $(1, \infty)$ since $f(x) = 2x^3 + 17$, a polynomial function. Thus we can conclude that f is differentiable at x for $x \neq -3, 1$. Since differentiability implies continuity we conclude that f is continuous at x in $\mathbf{R}$ for $x \neq -3, 1$. Thus it remains to check if f is continuous at $x = -3$ or 1. Consider the left limit at $x = -3$,

$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} x^2 - 14 = -5$ and the right limit at $x = -3$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} 6x + 13 = -5.$$

Thus, since $\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x)$, $\lim_{x \to -3} f(x) = -5$. But $f(-3) = 5$ and so $\lim_{x \to -3} f(x) \neq f(-3)$ and it follows that f is not continuous at $x = -3$. Now consider the left limit of f at $x = 1$,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 6x + 13 = 19$ and the right limit at $x = 1$,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x^3 + 17 = 19 = f(1).$$

Therefore, $\lim_{x \to 1} f(x) = f(1)$ and so f is continuous at $x = 1$.

Hence f is continuous at x for all $x \neq -3$.

(d) From above since f is not continuous at $x = -3$, f is not differentiable at $x = -3$. Since we have already shown that f is differentiiable everywhere except for $x = -3$ or

1, it remains now to check the differentiability of f at $x = 1$.

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{2x^3 + 17 - 19}{x - 1} = \lim_{x \to 1^+} 2(x^2 + x + 1) = 6$$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{6x + 13 - 19}{x - 1} = \lim_{x \to 1^-} 6 = 6.$$

Therefore, f is differentiable at $x = 1$ since $\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$.

Hence, f is differentiable at x for x not equal to -3 .

e) $$\int_{-4}^{0} f(x)dx = \int_{-4}^{-3} f(x)dx + \int_{-3}^{0} f(x)dx$$

$$= \int_{-4}^{-3} (x^2 - 14))dx + \int_{-3}^{0} (6x + 13)dx$$

$$= \left[\frac{x^3}{3} - 14x \right]_{-4}^{-3} + [3x^2 + 13x]_{-3}^{0}$$

$$= \left[\frac{4^3 - 3^3}{3} - 14 \right] - 27 + 39 = 10\ 1/3.$$

2. (a) $\lim_{x\to-\infty} \frac{3x^3-7|x^5|+9}{49x^5+5x^2+1} = \lim_{x\to-\infty} \frac{3x^3+7x^5+9}{49x^5+5x^2+1}$
$= \lim_{x\to-\infty} \frac{3/x^2+7+9/x^5}{49+5/x^3+1/x^5} = \frac{0+7+0}{49+0+0} = \frac{1}{7}$.

(b) $\lim_{x\to\infty} (1+x^2)^{1/x^2}$. Let $y=(1+x^2)^{1/x^2}$. Since
$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{1}{x^2}\ln(1+x^2) = \lim_{x\to\infty} \frac{2x/(1+x^2)}{2x} = \lim_{x\to\infty} \frac{1}{1+x^2} = 0$ by L' Hôpital's Rule.
Therefore,

$$\lim_{x\to\infty} y = e^{\lim_{x\to\infty}\ln(y)} = e^0 = 1.$$

(c) $\lim_{x\to 0} \frac{\sin(\sin(x^3))}{\sin(3x^2)+x^2} = \lim_{x\to 0} \frac{\cos(\sin(x^3))\cos(x^3)3x^2}{6x\cos(3x^2)+2x} = \lim_{x\to 0} \frac{\cos(\sin(x^3))\cos(x^3)3x}{6\cos(3x^2)+2} = \frac{0}{8} = 0$

by L' Hôpital's Rule.

(d) $\lim_{x\to 4} \frac{\sqrt{20+x^2}-6}{x-4} = \lim_{x\to 4} \frac{\frac{1}{2}(20+x^2)^{-1/2}2x}{1} = \frac{4}{\sqrt{36}} = \frac{2}{3}$ by L' Hôpital's rule.

(e)

$\lim_{x\to 0^+} \sin(x^2)\ln(\sin(x^2)) = \lim_{x\to 0^+} \frac{\ln(\sin(x^2))}{\csc(x^2)} = \lim_{x\to 0^+} -\frac{\cot(x^2)2x}{\csc(x^2)\cot(x^2)2x} = \lim_{x\to 0^+} -\sin(x^2) = 0$
by L' Hôpital's rule and the last equality is because $\lim_{x\to 0^+} \sin(x^2) = \sin(0) = 0$.
Therefore, $\lim_{x\to 0^+} (\sin(x^2))^{\sin(x^2)} = e^{\lim_{x\to\infty} \sin(x^2)\ln(\sin(x^2))} = e^0 = 1$.

3. (a) $\int \frac{15x^2e^{5x^3}+\sin(2x)}{e^{5x^3}+\sin^2(x)+3}dx = \int \frac{1}{e^{5x^3}+\sin^2(x)+3}\frac{dy}{dx}dx,$

where $y = e^{5x^3}+\sin^2(x)+3, \frac{dy}{dx} = 15x^2e^{5x^3}+\sin(2x),$

$= \int \frac{1}{y}dy$ by substitution or change of variable

$= \ln|y| + C = \ln(e^{5x^3}+\sin^2(x)+3)+C.$

(b) $\int_{-1}^{1}\sin(|x|+3)dx = \int_{-1}^{0}\sin(-x+3)dx + \int_{0}^{1}\sin(x+3)dx$

$= [\cos(-x+3)]_{-1}^{0} + [-\cos(x+3)]_{0}^{1}$

$= 2(\cos(3)-\cos(4)).$

(c)

$$g(x) = \begin{cases} 2x^3+4, x \geq 1 \\ x^4+5, x < 1 \end{cases}.$$

Note that $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}(2x^3+4) = 6 = g(1)$ and

$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}(x^4+5) = 6.$

Therefore, $\lim_{x\to 1} g(x) = g(1)$ and so g is continuous at $x = 1$. Since g is a polynomial function on the open interval (-∞, 1) and also on (1, ∞), g is continuous on these two intervals. Thus g is continuous on the whole of **R**. Therefore, we can use the Fundamental Theorem of Calculus to obtain an antiderivative. By the FTC,

$G(x) = \int_1^x g(t)dt$ is an antiderivative of g(x).

Now $$G(x) = \int_1^x g(t)dt = \begin{cases} \int_1^x g(t)dt, x \geq 1 \\ \int_1^x g(t)dt, x < 1 \end{cases} = \begin{cases} \int_1^x (2t^3+4)dt, x \geq 1 \\ \int_1^x (t^4+5)dt, x < 1 \end{cases}$$

$$= \begin{cases} \left[\frac{t^4}{2}+4t\right]_1^x, x \geq 1 \\ \left[\frac{t^5}{5}+5t\right]_1^x, x < 1 \end{cases} = \begin{cases} \frac{x^4}{2}+4x-\frac{9}{2}, x \geq 1 \\ \frac{x^5}{5}+5x-\frac{26}{5}, x < 1 \end{cases}.$$

(Any antiderivative of g(x) of the form G(x) + C is acceptable.).

(d)

$$\int_0^2 \frac{x+3}{x^2+4x+4}dx = \int_0^2 \frac{1}{x+2}dx + \int_0^2 \frac{1}{(x+2)^2}dx = [\ln(x+2)]_0^2 + \left[\frac{-1}{x+2}\right]_0^2 = \ln(2)+\frac{1}{4}.$$

(e) $\int x^2\cos(2x)dx = x^2\frac{1}{2}\sin(2x) - \int x\sin(2x)dx$ by integration by parts

$= \frac{1}{2}x^2\sin(2x) - \left(x(-\frac{1}{2}\cos(2x)) + \int \frac{1}{2}\cos(2x)dx\right)$

by integration by parts applied to $\int x^2\sin(x)dx$

$= \frac{1}{2}(x^2\sin(2x) + x\cos(2x)) - \frac{1}{4}\sin(2x) + C$

Question 4.

(a) Note that g, being defined by a polynomial, is continuous on the closed interval [0, 5]. Therefore, the *Extreme value Theorem* says that g has an absolute maximum value and an absolute minimum value.

Since $g(x) = x^3 - 6x^2 + 9x + 12$, its derivative is given by

$$g'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) .$$

Thus $g'(x) = 0$ if and only if $x = 1$, and $x = 3$. Therefore, the critical points of g in (0, 5) are 1 and 3. Now $g(0) = 12$, $g(1) = 16$, $g(3) = 12$ and $g(5) = 32$. Therefore, the absolute maximum value of g is 32 and the absolute minimum value of g is 12.

(b) (i) $h(x) = \cos^{-1}(\sin(2x))$. Therefore, for x such that $\sin(2x) \neq \pm 1$, that is for $2x \neq 2n\pi \pm \pi/2$ or equivalently $x \neq n\pi \pm \pi/4$ any integer n,

$$h'(x) = (\cos^{-1})'(\sin(2x))2\cos(2x) = \frac{-2\cos(2x)}{\sqrt{1-\sin^2(2x)}}$$

$$= \frac{1}{\sqrt{\cos^2(2x)}}(-2\cos(2x)) = -2\frac{\cos(2x)}{|\cos(2x)|} = -2\, sign(\cos(2x)).$$

The left and right limits of $h'(x)$ at $x = n\pi + \pi/4$ or $n\pi - \pi/4$ exist but are not the same by the above (one is 2 and the other −2 or the other way round). Therefore, h is not differentiable at these points. (For the reason for this see the artcle derived function and derivative.)

(ii) $j(x) = \ln\left(\frac{e^x + x^2}{1 + e^{(x^2)}}\right) = \ln(e^x + x^2) - \ln(1 + e^{(x^2)})$.

Therefore, $j'(x) = \ln'(e^x + x^2)(e^x + 2x) - \ln'(1 + e^{(x^2)})2xe^{(x^2)} = \frac{e^x + 2x}{e^x + x^2} - \frac{2xe^{(x^2)}}{1 + e^{(x^2)}}$.

$$= \frac{e^{x^2+x}(1-2x) + 2xe^{x^2}(1-x^2) + e^x + 2x}{(e^x + x^2)(1 + e^{(x^2)})}$$

.

(iii) $k(x) = \ln(\ln(e^x + 2) + x^2)$.

Therefore, $k(x) = \ln'(\ln(e^x + 2) + x^2) \cdot (\ln'(e^x + 2)e^x + 2x)$.

$$= \frac{1}{\ln(e^x + 2) + x^2} \cdot (\frac{e^x}{(e^x + 2)} + 2x)$$

$$= \frac{e^x + 2xe^x + 4x}{(\ln(e^x + 2) + x^2)(e^x + 2)}).$$

(c) Let $g(x) = f(x) - x$. Then g is a continuous function on the interval [1, 3] since f is continuous on [1, 3] and x is a continuous function and that we know that the difference of two continuous function is a continuous function.

Because $1 \leq f(x) \leq 3$ for all x in [1, 3], $f(3) - 3 \leq 0$ and $f(1) - 1 \geq 0$.

I.e. $g(3) = f(3) - 3 \leq 0 \leq f(1) - 1 = g(1)$. Therefore, by the *Intermediate Value Theorem*, there exists a point c in [1, 3] such that $g(c) = 0$. That is, $f(c) = c$.

5. $f(x) = x^5 - 10x^2 + 1$. Note that f is continuous and differentiable on **R**.

$$f'(x) = 5x^4 - 20x = 5x(x^3 - 4) = 5x(x - 4^{(1/3)})(x^2 + 4^{(1/3)}x + 4^{(2/3)}).$$

Now we know that the cubic $g(x) = x^3 - 4 = 0$ has a real root. We have used the identy

$(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$ to obtain the above factorisation Notice that $x^2 + 4^{(1/3)}x + 4^{(2/3)} = (x + 4^{(1/3)}/2)^2 + 4^{(2/3)} - \frac{1}{4}4^{(2/3)}) > 0$.

Thereofore,

$$f'(x) = 5x(x - 4^{(1/3)})((x + 4^{(1/3)}/2)^2 + \tfrac{3}{4}4^{(2/3)}) \qquad (1)$$

$$f''(x) = 20x^3 - 20 = 20(x^3 - 1) = 20(x-1)(x^2 + x + 1) = 20(x-1)((x + \tfrac{1}{2})^2 + \tfrac{3}{4}) \text{ ---- } (2).$$

So f'' is given by a cubic polynomial function. Again we know it must have a real root. The root is easily obtained by the above factorisation.

a.

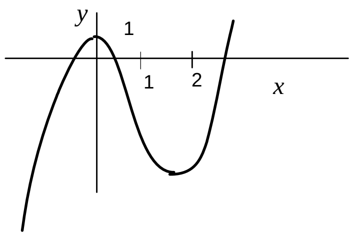
From (1), $f'(x) = 0$ if and only if $x = 0$ and $x = 4^{(1/3)}$. From (1) the sign of $f'(x)$ is the same as the sign of $x(x - 4^{(1/3)})$ because $(x + 4^{(1/3)}/2)^2 + \frac{3}{4}4^{(2/3)} > 0$. Thus we have: $x < 0 \Rightarrow x < 4^{(1/3)} \Rightarrow x - 4^{(1/3)} < 0 \Rightarrow x(x - 4^{(1/3)}) > 0 \Rightarrow f'(x) > 0$ so that f is increasing on$(-\infty, 0]$. Now $0 < x < 4^{(1/3)} \Rightarrow x - 4^{(1/3)} < 0 \Rightarrow x(x - 4^{(1/3)}) < 0 \Rightarrow f'(x) < 0$ so that f is decreasing on $[0, 4^{(1/3)}]$ and $x > 4^{(1/3)} \Rightarrow x(x - 4^{(1/3)}) > 0 \Rightarrow f'(x) > 0$ so that f is increasing on $[4^{(1/3)}, \infty)$. Now that the end points of the interval are included by virtue of continuity there.

b. From (2), $f''(x) = 0 \Leftrightarrow x = 1$ and that the sign of $f''(x)$ is the same as that of $x - 1$. Now $x < 1 \Rightarrow x - 1 < 0 \Rightarrow f''(x) < 0$. Therefore, the graph of f is concave downward on the interval $(-\infty, 1)$. Likewise from (2), $x > 1 \Rightarrow x - 1 > 0$ so that $f''(x) > 0$ when $x > 1$. Thus the graph of f is concave upward on $(1, \infty)$.

c. From part a, by the first derivative test, $f(0) = 1$ is a relative maximum and $f(4^{(1/3)}) = 1 - 12 * 2^{(1/3)}$ is a relative minimum.

d. From part b, since at $x = 1$, there is a change of concavity before and after $x = 1$, $(1, f(1)) = (1, -8)$ is a point of inflection of the graph of f. There are no other points of inflection.

e.



6. (a)

Fundamental Theorem of Calculus.

Let f be a continuous function defined on $[a, b]$. For the function F defined on $[a, b]$ by $F(x) = \int_a^x f(t)dt$, F is differentiable at each x in $[a, b]$ and $F'(x) = f(x)$. I.e. F is a special anti-derivative of f given via the definition of Riemann integral. Moreover, for any anti-derivative G of f, $\int_a^b f(t)dt = G(b) - G(a)$.

$$g(x) = \int_{\ln(x)}^{x^2} \frac{1}{1 + e^{3t} + \sin^2(t)} dt = \int_0^{x^2} \frac{1}{1 + e^{3t} + \sin^2(t)} dt + \int_{\ln(x)}^{0} \frac{1}{1 + e^{3t} + \sin^2(t)} dt$$

$$= \int_0^{x^2} \frac{1}{1 + e^{3t} + \sin^2(t)} dt - \int_0^{\ln(x)} \frac{1}{1 + e^{3t} + \sin^2(t)} dt$$

$$= F(x^2) - F(\ln(x)) \quad \text{where } F(x) = \int_0^x \frac{1}{1+e^{3t}+\sin^2(t)}dt.$$

Therefore,

$$g'(x) = F'(x^2)\cdot 2x - F'(\ln(x))\ \cdot(\tfrac{1}{x}) \text{ by the } \textit{Chain Rule}$$

$$= \frac{2x}{1+e^{3x^2}+\sin^2(x^2)} - \frac{1}{x(1+x^3+\sin^2(\ln(x)))} \text{ by the FTC.}$$

(b) (i) Since $k(x) = \int_1^x \frac{1}{1+t^3+t^6}dt$, by the FTC,

$k'(x) = \frac{1}{1+x^3+x^6} > 0$ since $1+x^3+x^6 = (x^3+\frac{1}{2})^2 + \frac{3}{4} > 0.$

Therefore, k is increasing on the whole of **R.** Thus k is injective.

(ii) $(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))}$. So we need to know the value of $k^{-1}(0)$. Now

$k^{-1}(0) = x \Leftrightarrow k(x) = 0 \Leftrightarrow \int_1^x \frac{1}{1+t^3+t^6}dt = 0.$ Since

$k(1) = \int_1^1 \frac{1}{1+t^3+t^6}dt = 0$ and k is injective, $x = 1$.

Therefore, $(k^{-1})'(0) = \frac{1}{k'(k^{-1}(0))} = \frac{1}{k'(1)} = \frac{1}{\frac{1}{3}} = 3.$

(c) Try to write the following as a Riemann sum

$$\sum_{i=1}^{n} \frac{i}{n^2}\sin\left(2+4\left(\frac{i}{n}\right)^2\right) = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \cdots < x_n$ is a regular partition and $\Delta x = \Delta x_i = x_i - x_{i-1}$.

Therefore, we can take $x_i = \frac{i}{n}$ so that $\Delta x = \frac{1}{n}, x_0 = 0$ and $x_n = 1$. Thus by comparing $f(x_i)\Delta x$ with $\frac{i}{n^2}\sin\left(2+4\left(\frac{i}{n}\right)^2\right) = \frac{i}{n}\sin\left(2+4\left(\frac{i}{n}\right)^2\right)\cdot\frac{1}{n}$ we would want $f(x_i) = f(\frac{i}{n}) = \frac{i}{n}\sin\left(2+4\left(\frac{i}{n}\right)^2\right)$. Thus $f(x) = x\sin(2+4x^2)$.

Therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{i}{n^2}\sin\left(2+4\left(\frac{i}{n}\right)^2\right) = \int_0^1 x\sin(2+4x^2)dx = \tfrac{1}{8}[-\cos(2+4x^2)]_0^1$$

$$= \tfrac{1}{8}(\cos(2)-\cos(6)).$$