Solution To NUS MA3110 Mathematical Analysis II

SEMESTER 2 EXAMINATION 2011 – 2012

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Question 1

(a) Suppose f is a function differentiable on $[a, b]$ with $a < b$. Suppose $f'(a) = f'(b) = 0$.

By using the function $h(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & a < x \le b \\ f'(a) = 0, & x = a \end{cases}$, or otherwise, prove that there exists a point c in (a, b) such that

$$\frac{f(c) - f(a)}{c - a} = f'(c) .$$

(b) Suppose g is a function twice differentiable on $(0, 1)$ such that for some $K > 0$, $|g''(x)| \le K$ for all x in $(0, 1)$. Prove that g is uniformly continuous on $(0, 1)$.

(Hint: show that g' is bounded on $(0,1)$.)

Solution

Part (a)

Part (a) says that for a continuous function f on the interval $[a, b]$, satisfying the condition in (a), there is always a point c in the interior of $[a, b]$ such that the line joining the points $(a, f(a))$ and $(c, f(c))$ is the tangent line to the graph of f at $(c, f(c))$.

Since f is differentiable on $[a, b]$, f is continuous $[a, b]$. Therefore, h is continuous on $(a, b]$ as $h(x) = \dfrac{f(x) - f(a)}{x - a}$ on $(a, b]$.

Now, $\lim_{x \to a^+} h(x) = \lim_{x \to a^+} \dfrac{f(x) - f(a)}{x - a} = f'(a) = h(a)$. Hence, h is continuous on $[a, b]$.

Since f is differentiable on (a, b) and $h|_{(a,b)}(x) = \dfrac{f(x) - f(a)}{x - a}$, by the Quotient Rule, h is differentiable in (a, b).

By the Extreme Value Theorem, h attains its maximum and minimum in $[a, b]$. If h attains its maximum or minimum at a point c in the interior of $[a, b]$, then since h is differentiable on (a, b), $h'(c) = 0$.

Now, for x in (a, b),

$$h'(x) = \frac{f'(x)}{x-a} - \frac{f(x)-f(a)}{(x-a)^2} = \frac{f'(x)}{x-a} - \frac{h(x)}{x-a} = \frac{f'(x)-h(x)}{x-a} \quad . \text{ ---------------- (1)}$$

Thus, $h'(c) = 0 \Rightarrow f'(c) = h(c) = \dfrac{f(c)-f(a)}{c-a}$.

Now, we look at the case that h does not attain its extremum in the interior of $[a, b]$.

Then its maximum and minimum must occur at the end points of $[a, b]$.

Now $h(a) = f'(a) = 0$.

If $h(b) = 0$, then h must be a constant function. We can thus take any c in (a, b).

(In this case $f(x) = f(a)$ for all x in $[a, b]$.)

Thus, h must have an extremum in the interior of (a, b) giving a contradiction.

It follows that $h(b) \neq 0$.

Suppose now $h(b) < 0$. Then $h(b)$ must be the absolute minimum of h since $h(a) = 0$.

Consequently, $\dfrac{h(x)-h(b)}{x-b} \leq 0$ for $a \leq x < b$. ------------------------ (2)

Note that for x in (a, b),

$$\frac{h(x)-h(b)}{x-b} = \frac{f(x)-f(b)}{x-b} \cdot \frac{1}{b-a} - \frac{f(x)-f(a)}{x-a} \cdot \frac{1}{b-a} .$$

Thus, since f is differentiable at b, $\lim_{x \to b^-} \dfrac{h(x)-h(b)}{x-b}$ exists and

$$\lim_{x \to b^-} \frac{h(x)-h(b)}{x-b} = \lim_{x \to b^-} \frac{f(x)-f(b)}{x-b} \cdot \frac{1}{b-a} - \lim_{x \to b^-} \frac{f(x)-f(a)}{x-a} \cdot \frac{1}{b-a}$$

$$= \frac{f'(b)}{b-a} - \frac{f(b)-f(a)}{(b-a)^2} = \frac{f(a)-f(b)}{(b-a)^2} = -\frac{h(b)}{b-a} > 0 \quad .$$

But as a consequence of (2), $\lim_{x \to b^-} \frac{h(x)-h(b)}{x-b} \le 0$.

Hence, we have a contradiction. So $h(b) > 0$ and $h(b)$ is the absolute maximum of h.

Consequently, $\lim_{x \to b^-} \frac{h(x)-h(b)}{x-b} \ge 0$.

But $\lim_{x \to b^-} \frac{h(x)-h(b)}{x-b} = \lim_{x \to b^-} \frac{f(x)-f(b)}{x-b} \cdot \frac{1}{b-a} - \lim_{x \to b^-} \frac{f(x)-f(a)}{x-a} \cdot \frac{1}{b-a}$

$$= \frac{f'(b)}{b-a} - \frac{f(b)-f(a)}{(b-a)^2} = \frac{f(a)-f(b)}{(b-a)^2} = -\frac{h(b)}{b-a} < 0.$$

So we again arrive at a contradiction.

This means h must have an abolute extremum c in the interior of $[a, b]$ and

$\frac{f(c)-f(a)}{c-a} = f'(c)$.

Part (b)

Follow the hint.

Take a point a in (0, 1).

Since g is twice differentiable in (0,1), by the Mean Value Theorem, for $x \ne a$, $x \in (0,1)$,

$$\frac{g'(x)-g'(a)}{x-a} = g''(y) \quad \text{for some } y \text{ between } x \text{ and } a.$$

Thus, $g'(x) = g'(a) + (x-a)g''(y)$.

Therefore, $|g'(x)| \le |g'(a)| + |x-a||g''(y)| \le |g'(a)| + |g''(y)| \le |g'(a)| + K$.

Hence, $g'(x)$ is bounded by $|g'(a)| + K$ on (0, 1). Let $|g'(a)| + K = M$.

This means g is Lipschitz and so g is uniformly continuous on (0, 1).

We can deduce this as follows:

For $x \ne y$, x, y in (0,1), by the Mean Value Theorem,

$$\frac{g(y)-g(x)}{y-x} = g'(c) \text{ for some } c \text{ between } x \text{ and } y.$$

Therefore,

$$|g(y)-g(x)| = |g'(c)||y-x| \le M|y-x|.$$

This inequality is obviously true for $x = y$.

Given any ε > 0, take $\delta = \dfrac{\varepsilon}{M+1}$. Then for all x and y in (0, 1),

$$|x-y| < \delta \Rightarrow |g(y)-g(x)| \le M \bullet \frac{\varepsilon}{M+1} < \varepsilon \quad .$$

This means that g is uniformly continuous on (0,1).

Question 2

(a) The function $h : [0, \pi] \to \mathbf{R}$ is defined by

$$h(x) = \begin{cases} \sin(x), \text{ if } x \text{ is rational,} \\ \cos(x), \text{ if } x \text{ is irrational.} \end{cases}$$

(i) Show that there exists a sequence of partitions (P_n) of [0, π] and a choice of points C_n in each of the subintervals of P_n with $\|P_n\| \to 0$, such that the Riemann sum $R(h, P_n, C_n) \to 2$ as $n \to \infty$, where

$$R(h, P_n, C_n) = \sum_{k=1}^{L} h(c_k)(x_k - x_{k-1}) \quad ,$$

P_n : $x_0 = 0 < x_1 < \quad x_2 < \ldots < x_L = \pi$ and $c_k \in [x_{k-1}, x_k]$, $k = 1, \ldots, L$.

(ii) Show that there exists a sequence of partitions (Q_n) of [0, π] and a choice of points D_n in each of the subintervals of Q_n with $\|Q_n\| \to 0$, such that the Riemann sum $R(h, Q_n, D_n) \to 0$ as $n \to \infty$.

Hence or otherwise, prove that h is not Riemann integrable.

(b) Suppose the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on an interval I.

(i) Prove that then the sequence of functions $\left(f_n(x) \right)$ converges uniformly on I to the zero constant function.

(ii) Prove that for any $K > 0$, $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2} \right)$ converges uniformly on the closed disk [–*K*, *K*], to a continuous function *f*. Deduce that it converges pointwise to a continuous function on **R** but the convergence is *not* uniform on **R**.

Solution

Part (a)

(i)

Take the partition

$$P_n : \ x_0 = 0 < x_1 < \quad x_2 < \dots < x_n = \pi, \quad \text{with } x_i = \frac{i}{n}\pi \ .$$

Then $\|P_n\| = \frac{\pi}{n}$ and so $\|P_n\| \to 0$.
For each subinterval, [x_{i-1}, x_i] , by the density of the rational numbers, there exists a rational number $c_i \in [x_{i-1}, x_i]$. Let $C_n = (c_1, c_2, \cdots, c_n)$. Then the Riemann sum,

$$R(h, P_n, C_n) = \sum_{i=1}^{n} h(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \sin(c_i)(x_i - x_{i-1}) \ ,$$

is also a Riemann sum for sin(*x*) on [0, π]. Therefore, by the Riemann Sum Convergence Theorem, since sin(*x*) is Riemann integrable on [0, π],

$$R(h, P_n, C_n) \to \int_0^{\pi} \sin(x)dx = \left[-\cos(x) \right]_0^{\pi} = 2 \ .$$

(ii) Let $Q_n = P_n$. By the density of the irrational number, there exists an irrational number $d_i \in [x_{i-1}, x_i]$. Let $D_n = (d_1, d_2, \cdots, d_n)$. The Riemann sum,

$$R(h, Q_n, D_n) = \sum_{i=1}^{n} h(d_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \cos(d_i)(x_i - x_{i-1}) \ ,$$

is also a Riemann sum for cos(*x*) on [0, π]. Therefore, as in part(i), by the Riemann Sum Convergence Theorem, since cos(*x*) is Riemann integrable on [0, π],

$$R(h, Q_n, D_n) \to \int_0^{\pi} \cos(x)dx = \left[\sin(x) \right]_0^{\pi} = 0 \ .$$

Thus, the function *h* cannot be Riemann integrable, because if it were, all Riemann sum must converge to the same limit but we have $R(h, P_n, C_n) \to 2$ and $R(h, Q_n, D_n) \to 0$.

Part (b)

(i)

If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on an interval I, then the series is uniformly Cauchy on I. Hence, given any ε > 0, there exists an integer N such that for all $n \geq N$ and for all $p \geq 1$, $\left|\sum_{k=n+1}^{n+p} f_k(x)\right| < \varepsilon$ for all x in I.

Taking p = 1, we have then that $n \geq N \Rightarrow \left|f_{n+1}(x)\right| < \varepsilon$ for all x in I.

This means $f_n(x) \to 0$ uniformly on I.

(ii)

Observe that $\left|\sin\left(\frac{x}{n^2}\right)\right| \leq \frac{|x|}{n^2} \leq \frac{K}{n^2}$ for all $|x| \leq K$.

Since $\sum_{n=1}^{\infty} \frac{K}{n^2}$ is convergent, by the Weierstrass M-Test, $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ is uniformly convergent on $[-K, K]$.

Since the n-th partial sum of this series is continuous and the convergence is uniform, the series converges to a continuous function f on on $[-K, K]$.

For any x in **R**, we can always take any real number $K > |x|$. (You might like to invoke the Archimedean property of the real numbers.) We have just shown that $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges and converges to a function f continuous on $[-K, K]$. This means that the series $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges at every x in **R** to a function f on **R**. Moreover, the restriction of f to $(-K, K)$ is continuous and so f is continuous at x. It follows that the function f is continuous on **R**.

The convergence of the series $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ cannot be uniform in **R**.

If it were, then by part (i) $\sin\left(\frac{x}{n^2}\right)$ must converge uniformly to the zero constant function on **R**.

For any positive integer n, let $x_n = \frac{\pi}{2} n^2$, then $\sin\left(\frac{x_n}{n^2}\right) = 1$. Take $\varepsilon = ½$. Then for any integer N and for any integer $n \geq N$, $\left|\sin\left(\frac{x_n}{n^2}\right)\right| = 1 > \frac{1}{2} = \varepsilon$. This means $\sin\left(\frac{x}{n^2}\right)$ does not converge uniformly to 0 on **R**.

Question 3

(i) Prove that for $x > 0$, $\sin(x) < x$. By using this inequality and the Cauchy Mean Value Theorem, or otherwise, prove that for $x > 0$,

$$x - \frac{x^3}{3!} < \sin(x) < x - \frac{x^3}{3!} + \frac{x^5}{5!} .$$

(ii) Using the inequality in part (i) or otherwise, prove that the series,

$$\sum_{n=1}^{\infty}\left(\sin\left(\frac{x}{\sqrt{n}}\right) - \frac{x}{\sqrt{n}}\right),$$

converges uniformly on $[0, a]$, for any $a > 0$ to a continuous function $f : [0, a] \to$ **R**. Hence, deduce that $\sum_{n=1}^{\infty}\left(\sin\left(\frac{x}{\sqrt{n}}\right) - \frac{x}{\sqrt{n}}\right)$ converges pointwise to a continuous function f on $[0, \infty)$.

(iii) Prove that the series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}\cos\left(\frac{x}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}}\right)$ converges uniformly to a function g on $[0, a]$, for any $a > 0$. Deduce that $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}\cos\left(\frac{x}{\sqrt{n}}\right) - \frac{1}{\sqrt{n}}\right)$ converges pointwise to a continuous function g on $[0, \infty)$.

(iv) Prove that the function f in part (ii) is differentiable on $(0, \infty)$ and that $f' = g$ on $(0, \infty)$ where g is given in part (iii).

Solution

Part (i)

There are many ways to prove this. An inspection of the graph of $\sin(x)$ or a knowledge of the values of $\sin(x)$ will clearly reveal that for all $x \geq \pi/2$, $x \geq \pi/2 > 1 \geq \sin(x)$. While on the interval $\left(0, \frac{\pi}{2}\right)$, the graph of $\sin(x)$ is concave down. The tangent line at 0 is the function x and so the graph of $\sin(x)$ for $0 < x < \pi/2$ lies below the tangent line at $x = 0$, i.e., $x > \sin(x)$ for $0 < x < \pi/2$. You can prove this statement by using the derivative of $\sin(x)$, i.e., $\cos(x)$, is strictly less than 1 for x in $(0, \pi/2]$. For instance, for x in $(0, \pi/2]$, by the Mean Value Theorem, there exists c between 0 and x such that

$\frac{\sin(x)}{x} = \cos(c) < \cos(0) = 1$ and so $x > \sin(x)$. Thus, $x > \sin(x)$ for all $x > 0$.

Here is another proof.

Let $g(x) = x - \sin(x)$. Then g is differentiable and $g'(x) = 1 - \cos(x)$.

Therefore, $g'(x) > 0$ for all $x \neq 2n\pi$. Therefore, g is strictly increasing on each interval

$[2n\pi, 2(n+1)\pi]$. Hence, g is strictly increasing on **R**. Thus for $x > 0$, $g(x) = x - \sin(x) >$

$g(0) = 0$, i.e., $x > \sin(x)$.

Now, for $x > 0$, consider the quotient $\frac{x - \sin(x)}{x^3/3!}$. By the Cauchy Mean Value Theorem,

there exists c such that $0 < c < x$ and

$$\frac{x - \sin(x)}{x^3/3!} = \frac{x - \sin(x) - 0}{x^3/3! - 0} = \frac{1 - \cos(c)}{c^2/2}.$$

Applying the Cauchy Mean Value Theorem again, there exists b such that $0 < b < c$ and

$$\frac{x - \sin(x)}{x^3/3!} = \frac{x - \sin(x) - 0}{x^3/3! - 0} = \frac{1 - \cos(c)}{c^2/2} = \frac{\sin(b)}{b} < 1. \quad \text{-----------------} \quad (1)$$

Hence, $x - \sin(x) < \frac{x^3}{3!}$ for $x > 0$.

Similarly, applying the Cauchy Mean Value Theorem, there exists c such that $0 < c < x$ and

$$\frac{\sin(x) - x + x^3/3!}{x^5/5!} = \frac{\cos(c) - 1 + c^2/2}{c^4/4!}.$$

Applying again the Cauchy Mean Value Theorem, there exists b such that $0 < b < c$ and

$$\frac{\sin(x) - x + x^3/3!}{x^5/5!} = \frac{\cos(c) - 1 + c^2/2}{c^4/4!} = \frac{-\sin(b) + b}{b^3/3!}.$$

By (1), $\frac{-\sin(b) + b}{b^3/3!} < 1$. And so $\frac{\sin(x) - x + x^3/3!}{x^5/5!} < 1$.

Hence, $\sin(x) - x + x^3/3! < x^5/5!$, i.e., $\sin(x) < x - x^3/3! + x^5/5!$.

So we have for $x > 0$,

$$x - \frac{x^3}{3!} < \sin(x) < x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Part (ii)

By the inequality in part (i), for $x > 0$,

$$-\frac{1}{3!}\cdot\frac{x^3}{n\sqrt{n}} < \sin\left(\frac{x}{\sqrt{n}}\right) - \frac{x}{\sqrt{n}} < -\frac{1}{3!}\cdot\frac{x^3}{n\sqrt{n}} + \frac{1}{5!}\cdot\frac{x^5}{n^2\sqrt{n}} \quad \text{-------------- (2)}$$

Since $\sum_{n=1}^{\infty}\frac{a^3}{n\sqrt{n}}$ and $\sum_{n=1}^{\infty}\frac{a^5}{n^2\sqrt{n}}$ are convergent series as they are a constant times a so-called convergent p-series (you can apply an integral test for the convergence), by the Weierstrass M Test, the series on the left and right of (2), i.e.,

$-\sum_{n=1}^{\infty}\frac{1}{3!}\cdot\frac{x^3}{n\sqrt{n}}$ and $\sum_{n=1}^{\infty}\left(-\frac{1}{3!}\cdot\frac{x^3}{n\sqrt{n}} + \frac{1}{5!}\cdot\frac{x^5}{n^2\sqrt{n}}\right)$ converge uniformly on $[0, a]$, $a > 0$. Hence, both series are uniformly Cauchy on $[0, a]$.

Thus, given $\varepsilon > 0$, there exists M such that for all x in $[0, a]$,

$$n > m \geq M \Rightarrow \left|\sum_{k=m}^{n} -\frac{1}{3!}\cdot\frac{x^3}{k\sqrt{k}}\right| < \varepsilon \quad \text{----------------------- (3)}$$

and there exists L such that for all x in $[0, a]$,

$$n > m \geq L \Rightarrow \left|\sum_{k=m}^{n}\left(-\frac{1}{3!}\cdot\frac{x^3}{k\sqrt{k}} + \frac{1}{5!}\cdot\frac{x^5}{k^2\sqrt{k}}\right)\right| < \varepsilon\,. \quad \text{----------------------- (4)}$$

Let $N = \max(L, M)$. Then from (2), for $n > m \geq N$ and for all x in $[0, a]$,

$$\sum_{k=m}^{n} -\frac{1}{3!}\cdot\frac{x^3}{k\sqrt{k}} < \sum_{k=m}^{n}\left(\sin\left(\frac{x}{\sqrt{k}}\right) - \frac{x}{\sqrt{k}}\right) < \sum_{k=m}^{n}\left(-\frac{1}{3!}\cdot\frac{x^3}{k\sqrt{k}} + \frac{1}{5!}\cdot\frac{x^5}{k^2\sqrt{k}}\right).$$

Hence, it follows from (3) and (4) that for $n > m \geq N$ and for all x in $[0, a]$,

$$\left|\sum_{k=m}^{n}\left(\sin\left(\frac{x}{\sqrt{k}}\right) - \frac{x}{\sqrt{k}}\right)\right| \leq \max\left(\left|\sum_{k=m}^{n}\frac{1}{3!}\cdot\frac{x^3}{k\sqrt{k}}\right|, \left|\sum_{k=m}^{n}\left(-\frac{1}{3!}\cdot\frac{x^3}{k\sqrt{k}} + \frac{1}{5!}\cdot\frac{x^5}{k^2\sqrt{k}}\right)\right|\right) < \varepsilon\,.$$

This means the series $\sum_{n=1}^{\infty}\left(\sin\left(\frac{x}{\sqrt{n}}\right) - \frac{x}{\sqrt{n}}\right)$ is uniformly Cauchy on $[0, a]$ and so the series converges uniformly on $[0, a]$. Since the n-th partial sums of the series are continuous function, the series converges to a continuous function on $[0, a]$.

Take any $x > 0$. Take any $a > x$. By the above argument, not only the series converges at x, it converges to a function f continuous at x. Hence f is continuous on $[0, \infty)$.

(iii)

For any x such that $0 \le x \le a$, $a > 0$,

$$\left|\frac{1}{\sqrt{n}}\cos\left(\frac{x}{\sqrt{n}}\right)-\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}\left|\cos\left(\frac{x}{\sqrt{n}}\right)-1\right| = \frac{2}{\sqrt{n}}\sin^2\left(\frac{x}{2\sqrt{n}}\right) \le \frac{x^2}{2n\sqrt{n}} \le \frac{a^2}{2n\sqrt{n}}.$$

Since $\sum_{n=1}^{\infty}\frac{a^2}{2n\sqrt{n}}$ is convergent, by the Weierstrass M test, the series,

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}\cos\left(\frac{x}{\sqrt{n}}\right)-\frac{1}{\sqrt{n}}\right),$$

converges uniformly on [0, a]. Since the n-th partial sum of this series is continuous, the series converges to a function g continuous on [0, a] for any $a > 0$. Therefore, g is continuous on [0, ∞).

(iv)

Note that the series in part (iii) is obtained from the series in part (ii) by term by term differentiation. That is to say, the series $\sum_{n=1}^{\infty}\left(\sin\left(\frac{x}{\sqrt{n}}\right)-\frac{x}{\sqrt{n}}\right)$ converges to f on [0, a] and its derived series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}\cos\left(\frac{x}{\sqrt{n}}\right)-\frac{1}{\sqrt{n}}\right)$ converges uniformly to g on [0, a].

Hence, f is differentiable on (0, a) and $f'(x) = g(x)$ on (0, a). Since this is true for any $a > 0$, f is differentiable on (0, ∞) and $f' = g$ on (0, ∞).

(For a reference to this fact see Theorem 8 of Chapter 8, Ng Tze Beng, Mathematical Analysis, An Introduction.

https://my-calculus-web.firebaseapp.com/MA3110/Chapter%208%20Uniform%20Convergence%20and%20differentiation.pdf)

Question 4

(a) (i) By considering an appropriate geometric series or otherwise, show that the n-th partial sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{e^{nx}}$ is uniformly bounded on the interval [0, ∞).

(ii) Using part (i) or otherwise, prove that the series,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}e^{-nx}}{\sqrt{n^2+x^2}} \ ,$$

converges uniformly on the interval $[0, \infty)$.

(b) (i) Show that the series $f(x)=\sum_{n=0}^{\infty}\frac{x^{4n}}{(4n)!}$ converges for all x in **R**.

(ii) Show that $f(x)$ satisfies the differential equation

$$f(x) + f'(x) + f''(x) + f'''(x) = e^{x} ,$$

for any x in **R**.

Solution

Part (a)

(i)

The n-th partial sum of the series,

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{e^{kx}} = -\sum_{k=1}^{n}(-e^{-x})^{k} = e^{-x}\sum_{k=0}^{n-1}(-e^{-x})^{k} = e^{-x}\cdot\frac{1-(-e^{-x})^{n}}{1+e^{-x}} = \frac{1-(-1)^{n}\frac{1}{e^{nx}}}{1+e^{x}} .$$

For $x \geq 0$, $e^{x} \geq 1$ and $\frac{1}{e^{x}} \leq 1$ and so

$$\left|\sum_{k=1}^{n}\frac{(-1)^{k+1}}{e^{kx}}\right| = \left|\frac{1-(-1)^{n}\frac{1}{e^{nx}}}{1+e^{x}}\right| \leq \frac{1+\frac{1}{e^{nx}}}{1+e^{x}} \leq \frac{2}{2} = 1 .$$

If we let $f_k(x)=\frac{(-1)^{k+1}}{e^{kx}}$, then the n-th partial sums $s_n(x)=\sum_{k=1}^{n}f_k(x)$ satisfies

$$|s_n(x)| = \left|\sum_{k=1}^{n}f_k(x)\right| = \left|\sum_{k=1}^{n}\frac{(-1)^{k+1}}{e^{kx}}\right| \leq 1 \text{ for all } x \geq 0.$$

Hence, $s_n(x)$ is uniformly bounded by 1 on $[0, \infty)$.

(ii)

Let $g_n(x)=\frac{1}{\sqrt{n^2+x^2}}$. Observe that $|g_n(x)| = \left|\frac{1}{\sqrt{n^2+x^2}}\right| \leq \frac{1}{n}$ for all x in **R** and

$$g_{n+1}(x) = \frac{1}{\sqrt{(n+1)^2 + x^2}} \le \frac{1}{\sqrt{n^2 + x^2}} = g_n(x)$$ for all x in **R** and for all integer $n \ge 1$.

Thus, $(g_n(x))$ is a decreasing sequence of functions on [0, ∞) such that $g_n(x) \to 0$ uniformly on [0, ∞).

Therefore, since we have shown in part (i) that the partial sums $s_n(x)$ is uniformly bounded on [0, ∞), by the Dirichlet's Test, $\sum_{n=1}^{\infty} f_n(x) g_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-nx}}{\sqrt{n^2 + x^2}}$ converges uniformly on [0, ∞).

Part (b)

(i)

For $x \neq 0$, $\left| \frac{x^{4n+4}}{(4n+4)!} \Big/ \frac{x^{4n}}{(4n)!} \right| = \frac{x^4}{(4n+4)(4n+3)(4n+2)(4n+1)} \to 0$ as $n \to \infty$.

Therefore, by the Ratio Test, the series $f(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}$ converges for all $x \neq 0$. It plainly converges at $x = 0$. Hence the series converges for all x in **R**.

(ii)

Similarly, as in part (i), we can deduce that

$$\sum_{n=1}^{\infty} \frac{x^{4n-1}}{(4n-1)!}, \sum_{n=1}^{\infty} \frac{x^{4n-2}}{(4n-2)!} \text{ and } \sum_{n=1}^{\infty} \frac{x^{4n-3}}{(4n-3)!}$$

converge for all x in **R**.

Hence, since the function $f(x)$ has the power series representation $f(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}$, by part (i),

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{4n-1}}{(4n-1)!}, \; f''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-2}}{(4n-2)!} \text{ and } f'''(x) = \sum_{n=1}^{\infty} \frac{x^{4n-3}}{(4n-3)!}.$$

Then we claim that

$$f(x) + f'(x) + f''(x) + f'''(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We prove this as follows:

Let $t_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. Then $t_n(x) \to e^x$ for all x in **R**. Therefore, $t_{4n}(x) \to e^x$ for all x in **R**.

Let $u_n(x) = \sum_{k=0}^{n} \frac{x^{4k}}{(4k)!}$, $v_n(x) = \sum_{k=1}^{n} \frac{x^{4k-1}}{(4k-1)!}$, $w_n(x) = \sum_{k=1}^{n} \frac{x^{4k-2}}{(4k-2)!}$ and $\ell_n(x) = \sum_{k=1}^{n} \frac{x^{4k-3}}{(4k-3)!}$.

Then for all x in **R**,

$$u_n(x) \to f(x),\ v_n(x) \to f'(x),\ w_n(x) \to f''(x) \text{ and } \ell_n(x) \to f'''(x).$$

Note that $t_{4n}(x) = u_n(x) + v_n(x) + w_n(x) + \ell_n(x)$ for $n \geq 1$.

Taking limits,

$$e^x = \lim_{n\to\infty} t_{4n}(x) = \lim_{n\to\infty} u_n(x) + \lim_{n\to\infty} v_n(x) + \lim_{n\to\infty} w_n(x) + \lim_{n\to\infty} \ell_n(x)$$

$$= f(x) + f'(x) + f''(x) + f'''(x).$$