Test

Code M

1. Evaluate the following limits.

(a) $\lim\limits_{x\to\infty}\dfrac{8x^5+7x^6+1000x+1}{-23x^4+28x^6+4}$. (b) $\lim\limits_{x\to 0}\dfrac{8x^3}{\sqrt{24x^3+36}-6}$. (c) $\lim\limits_{x\to 0}\dfrac{\sin(x\sin(x)+3x^2\cos(x))}{7x^2}$

(d) $\lim\limits_{x\to\infty}\dfrac{\sqrt{36x^2-49}}{5-x}$ (e) $\lim\limits_{x\to 0}(x+3x^2)\cos\left(\dfrac{1}{x^3}\right)$

Ans.

(a) $\lim\limits_{x\to\infty}\dfrac{8x^5+7x^6+1000x+1}{-23x^4+28x^6+4}=\lim\limits_{x\to\infty}\dfrac{\frac{17}{x^6}+7+\frac{1000}{x^5}\frac{1}{x^6}}{\frac{-23}{x^2}+28+\frac{4}{x^6}}=\dfrac{\lim\limits_{x\to\infty}\frac{17}{x^6}+7+\frac{1000}{x^5}\frac{1}{x^6}}{\lim\limits_{x\to\infty}\frac{-23}{x^2}+28+\frac{4}{x^6}}=\dfrac{7}{28}=\frac{1}{4}$

(b) $\lim\limits_{x\to 0}\dfrac{8x^3}{\sqrt{24x^3+36}-6}=\lim\limits_{x\to 0}\dfrac{8x^3(\sqrt{24x^3+36}+6)}{(\sqrt{24x^3+36}-6)(\sqrt{24x^3+36}+6)}=\lim\limits_{x\to 0}\dfrac{8x^3(\sqrt{24x^3+36}+6)}{24x^3}$

$=\lim\limits_{x\to 0}\dfrac{\sqrt{24x^3+36}+6}{3}=\dfrac{12}{3}=4$

(c)

$\lim\limits_{x\to 0}\dfrac{\sin(x\sin(x)+3x^2\cos(x))}{7x^2}=\lim\limits_{x\to 0}\dfrac{\sin(x\sin(x)+3x^2\cos(x))}{x\sin(x)+3x^2\cos(x)}\cdot\dfrac{(x\sin(x)+3x^2\cos(x))}{7x^2}$

$\lim\limits_{x\to 0}\dfrac{\sin(x\sin(x)+3x^2\cos(x))}{x\sin(x)+3x^2\cos(x)}\cdot\left(\dfrac{1}{7}\cdot\dfrac{\sin(x)}{x}+\dfrac{3}{7}\cos(x)\right)=1\cdot\left(\dfrac{1}{7}\cdot 1+\dfrac{3}{7}\right)=\dfrac{4}{7}$

(d) $\lim\limits_{x\to\infty}\dfrac{\sqrt{36x^2-49}}{5-x}=\lim\limits_{x\to\infty}\dfrac{\sqrt{36x^2-49}/\sqrt{x^2}}{(5-x)/|x|}=\lim\limits_{x\to\infty}\dfrac{\sqrt{36-\frac{49}{x^2}}}{\frac{5}{x}-1}=\dfrac{\lim\limits_{x\to\infty}\sqrt{36-\frac{49}{x^2}}}{\lim\limits_{x\to\infty}(\frac{5}{x}-1)}-6$

(e) $\lim\limits_{x\to 0}(x+3x^2)\cos\left(\dfrac{1}{x^3}\right)=0$ by Squeeze Theorem since

$$-|x||1+3x|\le(x+3x^2)\cos\left(\frac{1}{x^3}\right)\le|x||1+3x| \text{ for } x\ne 0$$

and $\lim\limits_{x\to 0}|x||1+3x|=0$.

2. (a) State Rolle's Theorem.

(b) Let $f(x)=x^3-6x^2+9x-1$

(i) Show that $f(x)=0$ has a solution in the closed interval [0, 1].

(ii) Show that $f(x)=0$ has exactly one solution in [0,1].

(iii) Find the absolute maximum and absolute minimum values of the function f on the interval [0, 4].

(iv) Hence determine the image of [0, 4] under f, i.e., the set $f([0, 4])$.

(a) Rolle's Theorem

Suppose $f : [a, b] \to \mathbf{R}$ is a function defined on $[a, b]$ such that (i) f is continuous on $[a, b]$, (ii) f is differentiable on (a, b) and (iii) $f(a) = f(b)$.

Then there exists a point c in (a, b) such that $f'(c) = 0$.

(b) (i) Note that since f is a polynomial function, f is continuous on [0, 1].

Now $f(1) = 1 - 6 + 9 - 1 = 3$ and $f(0) = -1$. Thus $f(0) < 0 < f(1)$ and so by the Intermediate Value Theorem, there is a point c in [0, 1] such that $f(c) = 0$.

(ii) Now $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$. -------------------- (1)

Therefore, $f'(x) > 0$ for all x in (0, 1). (Why?) Hence f is increasing on [0, 1] and so f is injective on [0, 1]. Since by (i) it has one root in [0, 1], it has exactly one root by injectivity.

Alternatively: Suppose $f(x) = 0$ has two or more solutions in [0, 1]. Let c and d be two such solutions in [0, 1]. Assume without lost of generality that $c < d$. Then $f(c) = f(d) = 0$. Since f is a polynomial function, f is continuous on $[c, d]$ and differentiable on (c, d). Therefore, the condition for Rolle's Theorem is satisfied. Hence there exists a point e in (c, d) such that $f'(e) = 0$. That is, there exists a point e in (0, 1) such that $f'(e) = 0$. But by (1) $f'(x) \neq 0$ for all x in (0, 1) and so $f'(e) \neq 0$ and this contradicts $f'(e) = 0$. Hence $f(x) =0$ cannot have two or more solutions in [0, 1]. Therefore, since it has one solution in [0, 1] by part (i), it has exactly one solution.

(iii) The function f is continuous on [0, 4] and the critical points of f in (0, 4) are 1 and 3.

Now $f(0) = -1$, $f(1) = 3$, $f(3) = -1$, $f(4) = 3$.

Therefore, the absolute minimum value on [0, 4] is -1 and the absolute maximum value on [0, 4] is 3.

(iv) By (iii) since $-1 \leq f(x) \leq 3$ for all x in [0, 4] and so $f([0,4]) \subseteq [-1, 3]$.

By the Intermediate value Theorem, for any value y such that $f(0) = -1 \leq y \leq 3 = f(4)$, there exists k in [0, 4] such that $f(k) = y$. Hence $[-1, 3] \subseteq f([0,4])$. Therefore, $f([0,4]) = [-1, 3]$.

3(a) Let $g(x) = \begin{cases} 2x^3 + 3x^2 - 12x + 3, & 0 \leq x \leq 3 \\ 9x^2 + 6x - 51, & 3 < x \leq 4 \end{cases}$

(i) Show that g is continuous at $x = 3$.

(ii) Consider the function g with [0, 3] as its domain, i.e., $g|_{[0,3]} : [0,3] \to \mathbf{R}$.

Find the intervals on which $g|_{[0,3]} : [0,3] \to \mathbf{R}$ is (1) increasing and (2) decreasing.

(iii) Is g differentiable at $x = 3$? Justify your answer.

(b) Differentiate the following function

(i) $\cos^2(\cos(2x))$ (ii) $\sqrt{x + \sqrt{x^2 + 1 + \sin(x)}}$

Answer:

(a)

(i) First $\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} 2x^3 + 3x^2 - 12x + 3 = 48$ and $\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} 9x^2 + 6x - 51 = 48$, so $\lim_{x \to 3} g(x) = 48$. Thus since g(3) = 48, $\lim_{x \to 3} g(x) = g(3)$. Therefore, g is continuous at $x = 3$.

(ii) $g'(x) = \begin{cases} 6x^2 + 6x - 12, & 0 < x < 3 \\ 18x + 6, & 3 < x < 4 \end{cases} = \begin{cases} 6(x-1)(x+2), & 0 < x < 3 \\ 6(3x+1), & 3 < x < 4 \end{cases}$ ------- (1)

Note that $g|_{[0,3]} : [0,3] \to \mathbf{R}$ is continuous since it is a polynomial function.

g'(x) < 0 for $0 < x < 1$ and g'(x) > 0 for $1 < x < 3$. (Why? Look at the expression in (1))

Therefore, $g|_{[0,3]} : [0,3] \to \mathbf{R}$ is decreasing on [0, 1] and increasing on [1, 3].

(iii) Note that by (i) g is continuous at $x = 3$.

$\lim_{x\to 3^-} g'(x) = \lim_{x\to 3^-} 6x^2 + 6x - 12 = 60$ and $\lim_{x\to 3^+} g'(x) = \lim_{x\to 3^+} 18x + 6 = 60$. Since both limits are finite and are the same g is differentiable at $x = 3$.

Alternatively:

$$\lim_{x\to 0^-} \frac{g(3+h)-g(3)}{h} = \lim_{x\to 0^-} \frac{2(3+h)^3 + 3(3+h)^2 - 12(3+h) + 3 - 48}{h}$$
$$= \lim_{x\to 0^-} \frac{54h + 18h^2 + 2h^3 + 18h + 3h^2 - 12h}{h} = 60$$

and

$$\lim_{x\to 0^+} \frac{g(3+h)-g(3)}{h} = \lim_{x\to 0^-} \frac{9(3+h)^2 + 6(3+h) - 51 - 48}{h} = \lim_{x\to 0^-} \frac{54h + 9h^2 + 6h}{h} = 60$$

Therefore, since the above left and right limits are the same, g is differentiable at $x = 3$.

(b)

(i) $$\frac{d}{dx} cos^2(\cos(2x)) = 2\cos(\cos(2x))(-\sin(\cos(2x)))(-2\sin(2x))$$
$$= 4\cos(\cos(2x))\,\sin(\cos(2x))\,\sin(2x)$$
$$= 2\sin(2\cos(2x))\,\sin(2x)$$

(ii) $$\frac{d}{dx}\sqrt{x + \sqrt{x^2 + 1 + \sin(x)}} = \frac{1}{2\sqrt{x+\sqrt{x^2+1+\sin(x)}}}\left(1 + \frac{1}{2\sqrt{x^2+1+\sin(x)}}(2x + \cos(x))\right)$$