National University of Singapore *Department of Mathematics*

MA1102R Calculus **Semester 2 (2005/2006)**

Test

Code N

1. Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{23x^5 + x\sin(x) + 4x^2 + 1}{-x^4 + 7x^5 + 3}$. (b) $\lim_{x\to 0} \frac{\sqrt{45x^2+25} - 5}{3x^2}$. (c) $\lim_{x\to 0^+} \frac{\sin(x\sin(x) + 2x^3)}{x^3}$

(d) $\lim_{x\to -\infty} \frac{\sqrt{81x^2 - 7}}{7 - 3x}$ (e) $\lim_{x\to 0} \frac{x}{2 + \sin(\frac{1}{x})}$

Ans.

(a)

$$\lim_{x\to\infty} \frac{23x^5 + x\sin(x) + 4x^2 + 1}{-x^4 + 7x^5 + 3} = \lim_{x\to\infty} \frac{23 + \frac{\sin(x)}{x^4} + \frac{4}{x^3} + \frac{1}{x^5}}{-\frac{1}{x} + 7 + \frac{3}{x^5}} = \frac{\lim_{x\to\infty} 23 + \frac{\sin(x)}{x^4} + \frac{4}{x^3} + \frac{1}{x^5}}{\lim_{x\to\infty} -\frac{1}{x} + 7 + \frac{3}{x^5}} = \frac{23}{7}$$

(b)

$$\lim_{x\to 0} \frac{\sqrt{45x^2+25} - 5}{3x^2} = \lim_{x\to 0} \frac{(\sqrt{45x^2+25} - 5)(\sqrt{45x^2+25} + 5)}{3x^2(\sqrt{45x^2+25} + 5)} = \lim_{x\to 0} \frac{45x^2}{3x^2(\sqrt{45x^2+25} + 5)}$$

$$= \lim_{x\to 0} \frac{15}{(\sqrt{45x^2+25} + 5)} = \frac{15}{10} = \frac{3}{2}$$

(c)

$$\lim_{x\to 0^+} \frac{\sin(x\sin(x) + 2x^3)}{x^3} = \lim_{x\to 0^+} \frac{\sin(x\sin(x) + 2x^3)}{x\sin(x) + 2x^3} \cdot \frac{x\sin(x) + 2x^3}{x^3}$$

$$\lim_{x\to 0^+} \frac{\sin(x\sin(x) + 2x^3)}{x\sin(x) + 2x^3} \cdot \left(\frac{1}{x} \cdot \frac{\sin(x)}{x} + 2\right) = +\infty$$

since $\lim_{x\to 0^+} \frac{\sin(x\sin(x) + 2x^3)}{x\sin(x) + 2x^3} = 1$ and $\lim_{x\to 0^+} \left(\frac{1}{x} \cdot \frac{\sin(x)}{x} + 2\right) = +\infty$

(d)

$$\lim_{x\to -\infty} \frac{\sqrt{81x^2 - 7}}{7 - 3x} = \lim_{x\to -\infty} \frac{\sqrt{81x^2 - 7}/\sqrt{x^2}}{(7 - 3x)/|x|} = \lim_{x\to -\infty} \frac{\sqrt{81 - \frac{7}{x^2}}}{-\frac{7}{x} + 3} = \frac{9}{3} = 3$$

(e)

$\lim_{x\to 0} \frac{x}{2 + \sin(\frac{1}{x})} = 0$ by Squeeze Theorem since

$$-|x| \le \frac{x}{2 + \sin(\frac{1}{x})} \le |x| \text{ for } x \ne 0 \text{ and } \lim_{x\to 0} |x| = 0.$$

2. (a) State the Extreme Value Theorem about a continuous function defined on a closed and bounded interval.

(b) Let $f(x) = 2x^3 - 21x^2 + 72x - 61$

(i) Show that $f(x) = 0$ has a solution in the closed interval [0, 2].

(ii) Show that f has exactly one root in [0, 2].

(iii) Find the absolute maximum and absolute minimum values of the function f on the interval [0, 5].

(iv) Hence determine the image of [0, 5] under f, i.e., $f([0, 5])$.

a) Extreme Value Theorem

Suppose $f : [a, b] \to \mathbf{R}$ is continuous and $[a, b]$ is a closed and bounded interval. Then there exists c and d in $[a, b]$ such that that $f(c) \geq f(x) \geq f(d)$ for all x in $[a, b]$, i.e., f has an absolute maximum and an absolute minimum.

(b) (i)

Note that since f is a polynomial function, f is continuous on [0, 2].
Now $f(0) = -61$ and $f(2) = 15$ $f(2) > 0 > f(0)$ and so by the Intermediate Value theorem, there is a c in [0, 2] such that $f(c) = 0$.

(ii)

Now $f'(x) = 6x^2 - 42x + 72 = 6(x^2 - 7x + 12) = 6(x-3)(x-4)$. ----------------- (1)
Therefore, $f'(x) > 0$ for all x in (0, 2). Hence f is increasing on [0, 2] and so f is injective on [0, 2]. Since by (i) it has one root in [0, 2], it has exactly one root by injectivity.

Alternatively:

Suppose $f(x) = 0$ has two or more solutions in [0, 2]. Let c and d be two such solutions in [0, 2]. Assume without lost of generality that $c < d$. Then $f(c) = f(d) = 0$. Since f is a polynomial function, f is continuous on $[c, d]$ and differentiable on (c, d). Therefore, the condition for Rolle's Theorem is satisfied. Hence there exists a point e in (c, d) such that $f'(e) = 0$. That is, there exists a point e in (0, 1) such that $f'(e) = 0$. But by (1) $f'(x) \neq 0$ for all x in (0, 2) and so $f'(e) \neq 0$ and this contradicts $f'(e) = 0$. Hence $f(x) = 0$ cannot have two or more solutions in [0, 1]. Therefore, since it has one solution in [0, 1] by part (i), it has exactly one solution.

(iii) The function f is continuous on [0, 5].
From (ii), there are two critical points of f in (0, 5), namely 3 and 4.
Now $f(0) = -61$, $f(3) = 20$, $f(4) = 19$, $f(5) = 24$.
Therefore, the absolute minimum value on [0, 5] is -61 and the absolute maximum value on [0, 5] is 24.

(iv) By (iii) since $-61 \leq f(x) \leq 24$ for all x in [0, 5], $f([0, 5]) \subseteq [-61, 24]$.
By the Intermediate value Theorem, for any value y such that $f(0) = -61 \leq y \leq 24 = f(5)$, there exists k in [0, 5] such that $f(k) = y$. Hence $[-61, 24] \subseteq f([0, 5])$. Therefore, $f([0, 5]) = [-61, 24]$.

3. (a) Let $g(x) = \begin{cases} 2x^3 + 3x^2 - 36x + 1, & 1 \leq x \leq 4 \\ x^2 - 6x + 41, & 4 < x \leq 5 \end{cases}$

(i) Show that g is continuous at $x = 4$.

(ii) Consider the function g with [1, 4] as its domain, i.e., $g|_{[1,4]} : [1,4] \to \mathbf{R}$.
Find the intervals on which $g|_{[1,4]} : [1,4] \to \mathbf{R}$ is (1) increasing and (2) decreasing.

(iii) Is g differentiable at $x = 4$? Justify your answer.

(b) Differentiate the following function

(i) $\cos^5(\cos(2x^2 + x) + 2x)$ (ii) $\tan(\sin(x) + \cos^2(x))$

Answer:

(a)

(i) First $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} 2x^3+3x^2-36x+1 = 33$ and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} x^2-6x+41 = 33$ so $\lim_{x\to 4} g(x) = 33$. Thus since g(4) = 33, $\lim_{x\to 4} g(x) = g(4)$. therefore, g is continuous at x = 4.

(ii)

$$g'(x) = \begin{cases} 6x^2+6x-36, & 1<x<4 \\ 2x-6, & 4<x<5 \end{cases} = \begin{cases} 6(x-2)(x+3), & 1<x<4 \\ 2(x-3), & 4<x<5 \end{cases}$$

Note that $g|_{[1,4]}\ :\ [1,4]\to\mathbf{R}$ is continuous since it is a polynomial function.

g'(x) < 0 for 1 < x < 2 and g'(x) > 0 for 2 < x < 4.

Therefore, $g|_{[1,4]}\ :\ [1,4]\to\mathbf{R}$ is decreasing on [1, 2] and increasing on [2, 4].

(iii) $\lim_{x\to 4^-} g'(x) = \lim_{x\to 4^-} 6x^2+6x-36 = 84$ and $\lim_{x\to 4^+} g'(x) = \lim_{x\to 4^+} 2x-6 = 2$. Since both limits are finite and are not the same g is not differentiable at x = 4.

Alternatively:

$$\lim_{x\to 0^-} \frac{g(4+h)-g(4)}{h} = \lim_{x\to 0^-} \frac{2(4+h)^3+3(4+h)^2-36(4+h)+1-33}{h}$$

$$\lim_{x\to 0^-} \frac{84h+27h^2+2h^3}{h} = 84$$

and

$$\lim_{x\to 0^+} \frac{g(4+h)-g(4)}{h} = \lim_{x\to 0^-} \frac{(4+h)^2-6(4+h)+41-33}{h} = \lim_{x\to 0^-} \frac{2h+h^2}{h} = 2$$

Therefore, since the above left and right limits are not the same, g is not differentiable at x = 4.

(b)

(i) $\frac{d}{dx} cos^5(\cos(2x^2+x)+2x)$

$= 5\cos^4(\cos(2x^2+x)+2x)(-\sin(\cos(2x^2+x)+2x))(2-(1+4x)\sin(2x^2+x))$

$= -5\cos^4(\cos(2x^2+x)+2x)\sin(\cos(2x^2+x)+2x))(2-(1+4x)\sin(2x^2+x))$

(ii) $\frac{d}{dx} tan(\sin(x)+\cos^2(x))$ $= \sec^2(\sin(x)+\cos^2(x))(\cos(x)-\sin(2x))$