NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2003-2004

MA1102R Calculus

November 2003 — Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **NINE (9)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions in **Section A**. Each question in Section A carries 10 marks.

3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1. [10 marks]

Evaluate each of the following limits.

(a) $\displaystyle\lim_{x\to -1} \frac{x+|x|}{1-x^2}$

(b) $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^x$

Question 2. [10 marks]

Let $f(x) = \ln(\ln(2\sin x))$.

(a) Find the largest subset of $\mathbb{R}$ such that f is defined.

(b) Find $f'(x)$.

Question 3. [10 marks]

Let C be the curve defined by the equation $x^2 + y^2 = 5x^4$.

(a) Find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent line to the curve C at the point $(1, 2)$.

Question 4. [10 marks]

Evaluate each of the following integrals.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx$

(b) $\displaystyle\int \frac{1}{\sqrt[3]{x}+\sqrt{x}}\,dx$

[For (a), You may leave your answer in terms of e and π.]

Question 5. [10 marks]

(a) Evaluate the integral $\int_1^\infty \frac{\ln x}{x^3}\,dx$

(b) Using the Mean Value Theorem, or otherwise, prove that for any $x, y \geq 0$,

$$|e^{-x} - e^{-y}| \leq |x - y|.$$

Question 6. [10 marks]

(a) Find $\frac{d}{dx}\int_{\sin x}^{x^2} \frac{1}{1+t^4}\,dt$.

(b) Let $f(x) = (1+x)^{\frac{5}{2}}$. Find the Maclaurin polynomial of f of degree 2 with remainder.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 7. [20 marks]

(a) Let f be a function such that

$$2x \leq f(x) \leq x^2 + 1, \qquad \text{for all } x \text{ in } (0,2).$$

Show that f is differentiable at $x = 1$ and find $f'(1)$.

(b) Let g be the function defined by

$$g(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ x^2+1 & \text{if } 1 < x \leq 2 \\ x+3 & \text{if } 2 < x \end{cases}.$$

Find an anti-derivative of g.

(c) Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}\sqrt{i^2(n^2-i^2)}$.

Question 8. [20 marks]

Let $f(x) = \dfrac{2-4x}{(2x+1)^2}$.

(a) Find, if any, the x- and y- intercepts of f.

(b) Show that f has a critical point at $x = \frac{3}{2}$.

(c) Find the intervals on which f is (i) increasing, and (ii) decreasing.

(d) Find, if any, the local minima and local maxima of f.

(e) Find the intervals on which the graph of f is (i) concave upward, and (ii) concave downward.

(f) Find, if any, the points of inflection of the graph of f.

(g) Find, if any, the vertical and horizontal asymptotes of the graph f.

(h) Sketch the graph of f.

Question 9. [20 marks]

(a) Let g be a continuous *even* function defined on $\mathbb{R}$.

(i) Prove that

$$\int_0^\pi xg(\cos x)\,dx = \frac{\pi}{2}\int_0^\pi g(\cos x)\,dx.$$

(ii) Using (i), or otherwise, evaluate $\displaystyle\int_0^\pi x\cos^4 x\,dx$.

(b) Let f be a twice differentiable function defined on $\mathbb{R}$ satisfying $f(0) = 0$, $f'(0) = 1$ and $f''(x) > 0$ for all x in $\mathbb{R}$. Define

$$h(x) = \begin{cases} \frac{f(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}.$$

(i) Prove that for any $x > 0$, there exists a number c in $(0, x)$ such that

$$h'(x) = \frac{f'(x) - f'(c)}{x}.$$

(ii) Show that h is increasing on $\mathbb{R}$.

END OF PAPER

Solution

Section A

1. (a) $\displaystyle\lim_{x\to -1}\frac{x+|x|}{1-x^2} = \lim_{x\to -1}\frac{x-x}{1-x^2} = \lim_{x\to -1} 0 = 0.$

(b) Let $y = \left(1+\dfrac{1}{x}+\dfrac{1}{x^2}\right)^x = \left(\dfrac{x^2+x+1}{x^2}\right)^x.$

Thus $\ln y = x\left(\ln(x^2+x+1) - 2\ln x\right) = \dfrac{\ln(x^2+x+1)-2\ln x}{x^{-1}}.$

$$\begin{aligned}
&\lim_{x\to\infty}\ln y\\
=\ &\lim_{x\to\infty}\frac{\ln(x^2+x+1)-2\ln x}{x^{-1}}\\
=\ &\lim_{x\to\infty}\frac{(2x+1)(x^2+x+1)^{-1}-2x^{-1}}{-x^{-2}} \qquad \text{by L'Hôpital's rule,}\\
=\ &\lim_{x\to\infty}\frac{(-x^2)\left(x(2x+1)-2(x^2+x+1)\right)}{x(x^2+x+1)}\\
=\ &\lim_{x\to\infty}\frac{x^2+2x}{x^2+x+1}\\
=\ &\lim_{x\to\infty}\frac{1+2x^{-1}}{1+x^{-1}+x^{-2}} = 1. \qquad \text{Hence, } \lim_{x\to\infty} y = e.
\end{aligned}$$

2. (a)
$$\begin{aligned}
&f(x) = \ln(\ln(2\sin x)) \text{ is defined}\\
\iff\ &\ln(2\sin x) > 0\\
\iff\ &2\sin x > 1\\
\iff\ &\sin x > \tfrac{1}{2}\\
\iff\ &x \in \left(\tfrac{\pi}{6}+2n\pi, \tfrac{5\pi}{6}+2n\pi\right) \quad \text{for some integer } n.
\end{aligned}$$
Thus the domain of f is $\displaystyle\bigcup_{n\in\mathbb{Z}}\left(\frac{\pi}{6}+2n\pi, \frac{5\pi}{6}+2n\pi\right).$

(b) $f'(x) = \dfrac{1}{\ln(2\sin x)}\dfrac{1}{2\sin x}2\cos x = \dfrac{\cot x}{\ln(2\sin x)}.$

3. (a) Differentiating both sides of the equation with respect to x, we have
$$2x + 2yy' = 20x^3.$$
Thus, $y' = \dfrac{10x^3-x}{y}.$

(b) At the point $(1,2)$, $y'(1) = \frac{9}{2}$. Therefore, an equation of the tangent line to the given curve at $(1,2)$ is given by $y-2 = \frac{9}{2}(x-1)$, or equivalently, $9x-2y-5=0$.

4. (a) Using integration by parts twice, we have

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx \\
&= \left[-e^{2x}\cos x\right]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx \\
&= 1 + 2\left[e^{2x}\sin x\right]_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx \\
&= 1 + 2e^{\pi} - 4\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx.
\end{aligned}$$

Therefore, $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx = \frac{1+2e^{\pi}}{5}$.

(b) Let $y = x^{\frac{1}{6}}$. Then $x^{\frac{1}{3}} = y^2$, $x^{\frac{1}{2}} = y^3$, and $dy = \frac{1}{6}x^{-\frac{5}{6}}dx$ so that $dx = 6y^5dy$. Thus,

$$\begin{aligned}
&\int \frac{1}{\sqrt[3]{x}+\sqrt{x}}\,dx \\
&= \int \frac{6y^5}{y^2+y^3}\,dy \\
&= \int \frac{6y^3}{1+y}\,dy \\
&= \int 6y^2 - 6y + 6 - \frac{6}{1+y}\,dy \\
&= 2y^3 - 3y^2 + 6y - 6\ln(1+y) + C \\
&= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln(1+x^{\frac{1}{6}}) + C.
\end{aligned}$$

5. (a) Using integration by parts,

$$\int_1^b \frac{\ln x}{x^3}\,dx = \left[\frac{-\ln x}{2x^2}\right]_1^b + \int_1^b \frac{1}{2x^3}\,dx = -\frac{\ln b}{2b^2} + \left[-\frac{1}{4x^2}\right]_1^b = -\frac{\ln b}{2b^2} - \frac{1}{4b^2} + \frac{1}{4}.$$

Thus, $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\left(-\frac{\ln b}{2b^2} - \frac{1}{4b^2} + \frac{1}{4}\right) = \frac{1}{4}$,

since by L'Hôpital's rule, $\displaystyle\lim_{b\to\infty}\frac{\ln b}{2b^2} = \lim_{b\to\infty}\frac{\frac{1}{b}}{4b} = \lim_{b\to\infty}\frac{1}{4b^2} = 0$.

(b) The inequality clearly holds when $x = y$. Let x and y be distinct nonnegative numbers. Without loss of generality, we may suppose $0 \le x < y$. The function $f(t) = e^{-t}$ is continuous on $[x, y]$ and differentiable in (x, y) with $f'(t) = -e^{-t}$. By Mean Value Theorem, we have $e^{-y} - e^{-x} = -e^{-c}(y - x)$ for some c in (x, y). Since $c > 0$, we have $e^{-c} < 1$. Consequently,

$$|e^{-x} - e^{-y}| = |e^{-c}||x - y| \le |x - y|.$$

6. (a) By Fundamental Theorem of Calculus and the Chain Rule for differentiation, we have

$$\frac{d}{dx}\int_{\sin x}^{x^2} \frac{1}{1+t^4}\,dt = \frac{1}{1+(x^2)^4}\cdot 2x - \frac{1}{1+\sin^4 x}\cdot \cos x.$$

(b) First $f'(x) = \frac{5}{2}(1+x)^{\frac{3}{2}}$, $f''(x) = \frac{15}{4}(1+x)^{\frac{1}{2}}$, and $f'''(x) = \frac{15}{8}(1+x)^{-\frac{1}{2}}$. Thus $f(0) = 1$, $\frac{f'(0)}{1!} = \frac{5}{2}$, and $\frac{f''(0)}{2!} = \frac{15}{8}$. Therefore the Maclaurin polynomial of degree 2 is given by

$$1 + \frac{5}{2}x + \frac{15}{8}x^2,$$

and the remainder is $R_2(x) = \frac{5}{16}(1+c)^{-\frac{1}{2}}x^3$, where c is between 0 and x.

Section B

7. (a) Substituting $x = 1$ into the given inequality, we get $f(1) = 2$. Therefore,

$$2x - 2 \le f(x) - f(1) \le x^2 - 1 \quad \text{for all } x \text{ in } (0,2).$$

For $0 < x < 1$, we have

$$\frac{2x-2}{x-1} \ge \frac{f(x)-f(1)}{x-1} \ge \frac{x^2-1}{x-1},$$

or equivalently,

$$2 \ge \frac{f(x)-f(1)}{x-1} \ge x+1.$$

Thus, by Squeeze Theorem, $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = 2.$

Similarly, for $1 < x < 2$, we have $2 \le \frac{f(x)-f(1)}{x-1} \le x+1$. Again by Squeeze Theorem, $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = 2$. Consequently, $f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = 2.$

(b) Note that g is continuous on $\mathbb{R}$. By Fundamental Theorem of Calculus, $F(x) = \int_0^x f(t)\,dt$ is an anti-derivative of f.

For $x \le 1$, $\int_0^x f(t)\,dt = \int_0^x 2t\,dt = x^2$. In particular $\int_0^1 f(t)\,dt = 1$.

For $1 \le x \le 2$,
$$\begin{aligned}\int_0^x f(t)\,dt &= \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt \\ &= 1 + \int_1^x (t^2+1)\,dt \\ &= 1 + \frac{1}{3}x^3 + x - \frac{4}{3} \\ &= \frac{1}{3}x^3 + x - \frac{1}{3}.\end{aligned}$$

In particular $\int_0^2 f(t)\,dt = \frac{13}{3}$.

For $2 \le x$, $\int_0^x f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^x f(t)\,dt$

$$= \frac{13}{3} + \int_2^x (t+3)\,dt$$
$$= \frac{1}{2}x^2 + 3x - \frac{11}{3}.$$

Therefore,

$$F(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ \frac{1}{3}x^3 + x - \frac{1}{3} & \text{if } 1 < x \le 2 \\ \frac{1}{2}x^2 + 3x - \frac{11}{3} & \text{if } 2 < x \end{cases}.$$

(c) $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}\sqrt{i^2(n^2 - i^2)} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n}\sqrt{1 - \frac{i^2}{n^2}}\,\frac{1}{n}$$
$$= \int_0^1 x\sqrt{1-x^2}\,dx = \left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}.$$

8. Note that the domain of f is $\mathbb{R} \setminus \{-\frac{1}{2}\}$. Since f is a rational function, it is differentiable at each point in its domain. Let's first compute $f'(x)$ and $f''(x)$. We have

$$f'(x) = \frac{4(2x-3)}{(2x+1)^3} \quad \text{and} \quad f''(x) = -\frac{16(2x-5)}{(2x+1)^4}.$$

(a) When $x = 0$, $y = f(0) = 2$, and when $y = 0$, $x = \frac{1}{2}$. Thus, the x-intercept is $\frac{1}{2}$ and the y-intercept is 2.

(b) $f'(x) = \dfrac{4(2x-3)}{(2x+1)^3} = 0$ if and only if $x = \frac{3}{2}$. Therefore, f has a critical point at $x = \frac{3}{2}$.

(c) From the expression of $f'(x)$, we see that $f'(x) > 0$ for x in $(-\infty, -\frac{1}{2}) \cup (\frac{3}{2}, \infty)$ and $f'(x) < 0$ for x in $(-\frac{1}{2}, \frac{3}{2})$. Therefore, f is decreasing on $(-\frac{1}{2}, \frac{3}{2}]$ and is increasing on $(-\infty, -\frac{1}{2}) \cup [\frac{3}{2}, \infty)$.

(d) By the first derivative test, f has a local minimum at $x = \frac{3}{2}$ and $f(\frac{3}{2}) = -\frac{1}{4}$.

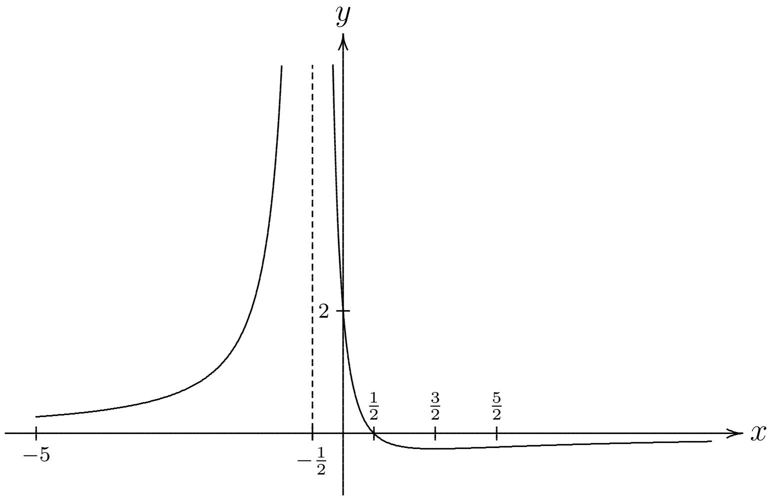
(e) From the expression of $f''(x)$, we see that $f''(x) < 0$ for $x > \frac{5}{2}$ and $f''(x) > 0$ for $x < \frac{5}{2}$ and $x \ne -\frac{1}{2}$.
Thus the graph of f is concave downward in $(\frac{5}{2}, \infty)$ and concave upward in $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \frac{5}{2})$.

(f) Setting $f''(x) = 0$, we see that $f''(x) = 0$ if and only if $x = \frac{5}{2}$. From (e), the graph of f is concave upward in $(-\frac{1}{2}, \frac{5}{2})$ and concave downward in $(\frac{5}{2}, \infty)$. So there is a change of concavity of the graph of f at $x = \frac{5}{2}$. Consequently, there is a point of inflection of the graph of f at $x = \frac{5}{2}$.

(g) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2-4x}{(2x+1)^2} = \lim_{x\to\infty} \frac{\frac{2}{x^2} - \frac{4}{x}}{(2+\frac{1}{x})^2} = 0$. Similarly, $\lim_{x\to-\infty} f(x) = 0$. Therefore, $y = 0$ is a horizontal asymptote of the graph of f.

Also $\lim_{x\to-\frac{1}{2}} f(x) = \lim_{x\to-\frac{1}{2}} \frac{2-4x}{(2x+1)^2} = \infty$. Thus $x = -\frac{1}{2}$ is a vertical asymptote of the graph of f.

(h) The graph of f is shown below.



The graph of $f(x) = (2-4x)(2x+1)^{-2}$

9. (a) (i) Using the substitution $x = \pi - t$, we have

$$\begin{aligned}\int_0^\pi xg(\cos x)\,dx &= \int_\pi^0 (\pi - t)g(\cos(\pi - t))\,(-dt)\\ &= \int_0^\pi (\pi - t)g(-\cos t)\,dt\\ &= \int_0^\pi (\pi - t)g(\cos t)\,dt \qquad \text{since } g \text{ is an even function}\\ &= \pi\int_0^\pi g(\cos t)\,dt - \int_0^\pi tg(\cos t)\,dt\end{aligned}$$

Consequently,

$$\int_0^\pi xg(\cos x)\,dx = \frac{\pi}{2}\int_0^\pi g(\cos x)\,dx.$$

(ii) First, we have $\cos^4(x) = \left[\frac{1}{2}(1+\cos(2x))\right]^2 = \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) = \frac{1}{4}(1+2\cos(2x)+\frac{1}{2}(1+\cos(4x))) = \frac{1}{8}(3+4\cos(2x)+\cos(4x))$.

Thus by (i), $\int_0^\pi x\cos^4 x\,dx = \frac{\pi}{2}\int_0^\pi \cos^4 x\,dx$

$$
\begin{aligned}
&= \frac{\pi}{16}\int_0^{\pi} 3 + 4\cos(2x) + \cos(4x)\,dx \\
&= \frac{\pi}{16}\left[3x + 2\sin(2x) + \frac{1}{4}\sin(4x)\right]_0^{\pi} \\
&= \frac{3\pi^2}{16}.
\end{aligned}
$$

(b) (i) Let $x > 0$. By Mean Value Theorem, there exists c in $(0, x)$ such that $f(x) = f(x) - f(0) = f'(c)x$. Thus

$$h'(x) = \frac{f'(x)x - f(x)}{x^2} = \frac{f'(x)x - f'(c)x}{x^2} = \frac{f'(x) - f'(c)}{x}.$$

(ii) Since $f''(x) > 0$ for all $x > 0$, $f'(x)$ is increasing on $(0, \infty)$. Thus $h'(x) = \dfrac{f'(x) - f'(c)}{x} > 0$ as $x > c$. Therefore, h is increasing on $(0, \infty)$.
Similarly, for $x < 0$, we have

$$h'(x) = \frac{f'(x)x - f(x)}{x^2} = \frac{f'(x)x - f'(c)x}{x^2} = \frac{f'(x) - f'(c)}{x} > 0,$$

where c is in $(x, 0)$. Therefore, h is increasing on $(-\infty, 0)$.
Note that $\lim_{x\to 0} h(x) = \lim_{x\to 0} \dfrac{f(x)}{x} = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = f'(0) = 1 = h(0)$, so that h is continuous at $x = 0$. Thus, h is increasing on $\mathbb{R}$.

Remark An example of such a function f is $f(x) = e^x - 1$.