Example Session 2

Limits

1. Find $\lim_{x \to 2} \frac{x^3+2x^2-6x-4}{x-2}$

- **Check if the limit of the denominator is 0.**

In this case it is 0.

- **Next we check if the limit of the numerator is 0.**

(+) If it is not 0, then we have to deal with it differently. But we can say for sure that the limit does not exist.

There is the possibility that the limit can be either $+\infty$ or $-\infty$ **a phrase that describes the "behaviour" of the function, which we shall meet later.**

If it is 0, then since the function is a rational function, we can try to eliminate the common factor until the denominator gives non-zero limit.

If we can't do this, then we are back in situation (+) above.

$$\lim_{x \to 2} \frac{x^3+2x^2-6x-4}{x-2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2+4x+2)}{x-2}$$

$$= \lim_{x \to 2} (x^2+4x+2) = \mathbf{4 + 8 + 2 = 14.}$$

The next question will involve the use of the identity

$$(a^2-b^2) = (a-b)(a+b)$$

2. Find $\lim_{x \to 2} \frac{\sqrt{x^2+3}-\sqrt{7}}{x-2}$

- **Note that both the denominator and the numerator are zero when $x = 2$.**

This "suggests" that if we can somehow remove the $\sqrt{}$ sign from the numerator we should be able to cancel the factor $(x - 2)$.

This suggests the use of the above identity.

For other rational power, e.g. cube root, the following generalization of the above identity is useful.

$$(a^n - b^n) = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$

$$(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$$

$$\lim_{x\to 2} \frac{\sqrt{x^2+3} - \sqrt{7}}{x-2}$$

$$= \lim_{x\to 2} \frac{(\sqrt{x^2+3} - \sqrt{7})(\sqrt{x^2+3} + \sqrt{7})}{(x-2)(\sqrt{x^2+3} + \sqrt{7})}$$

$$= \lim_{x\to 2} \frac{x^2+3-7}{(x-2)(\sqrt{x^2+3} + \sqrt{7})}$$

$$= \lim_{x\to 2} \frac{x^2-4}{(x-2)(\sqrt{x^2+3} + \sqrt{7})}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+3} + \sqrt{7})}$$

$$= \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+3} + \sqrt{7}}$$

$$= \frac{2+2}{\sqrt{4+3} + \sqrt{7}} = \frac{2}{\sqrt{7}}.$$

Similarly if we have rational powers involved in the denominator or both the numerator and the denominator, the same technique should apply.

3. $\lim_{x\to 4} \frac{x^2-3x-4}{2-\sqrt{x}}$

$$= \lim_{x\to 4} \frac{(x^2-3x-4)(2+\sqrt{x})}{(2-\sqrt{x})(2+\sqrt{x})}$$

$$= \lim_{x\to 4} \frac{(x^2-3x-4)(2+\sqrt{x}\,)}{4-x}$$

$$= \lim_{x\to 4} \frac{(x-4)(x+1)(2+\sqrt{x}\,)}{4-x}$$

$$= \lim_{x\to 4} \; -(x+1)(2+\sqrt{x}\,)$$

$$= 5 \cdot (2+2) = -20.$$

The next example will involve what we call a piecewise polynomial function.

4. $f(x) = \begin{cases} 7x-5, & x<1 \\ x+1, & x \geq 1 \end{cases}$.

$\lim_{x\to 1} f(x) = 2$**.**

- **To show this we look at both the left and the right limits at $x = 1$.**

We can determine the left or the right limits by using the computation for the limit for a polynomial.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (7x-5)$$

$$\textbf{(} = \lim_{x\to 1} (7x-5)\textbf{)}$$

$$= 7-5 = 2.$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = \lim_{x\to 1} (x+1)$$

$$= 1+1 = 2.$$

Since the left and the right limits at $x = 1$ are the same as 2, the limit of f at $x = 1$ is equal to 2.

- **The next trick is a familiar one. If we can rewrite the not so obvious function as a piecewise polynomial around the point where the limit is to be taken, then the evaluation of the limit of this not so obvious function can be pursued as in the above example.**

5. Find $\lim_{x\to 1^+} ([x+1] - |x-1|)$**and**

$\lim_{x\to 1^+} ([x+1] - |x-1|)$**.**

Let $f(x) = [x+1] - |x-1|$.

$1 < x < 2 \Rightarrow 2 < x+1 < 3$

$\Rightarrow [x+1] = 2$. **---------- (1)**

Also $1 < x < 2 \Rightarrow x - 1 > 0$

$\Rightarrow |x-1| = x - 1$. **---- (2)**

Thus $1 < x < 2 \Rightarrow f(x) = [x+1] - |x-1|$

$= 2 - (x-1)$ **by (1) and (2)**

$= 3 - x$.

Therefore, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3 - x)$

$= 3 - 1 = 2.$

Now $0 < x < 1 \Rightarrow 1 < x+1 < 2 \Rightarrow [x+1] = 1$, **and**

$0 < x < 1 \Rightarrow x - 1 < 0 \Rightarrow |x-1| = -(x-1)$.

Thus $0 < x < 1 \Rightarrow f(x) = [x+1] - |x-1|$

$= 1 + (x-1) = x$.

Therefore, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$. **Notice that** $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$,

therefore $\lim_{x \to 1} f(x)$**does mot exist.**

6. Find condition for the constant *a* and *b* such that the function *f* defined below has a limit at *x* = 1.

$$f(x) = \begin{cases} x + a, \ x < 1 \\ b - x^2, \ x > 1 \\ 1, \ x = 1 \end{cases} .$$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x + a = 1 + a$ **and** $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} b - x^2 = b - 1$.

Thus $\lim_{x \to 1} f(x)$**exists if and only if** $1 + a = b - 1$. **I.e.,** $b = a + 2$.