EXAMPLE ESSION 3

Limits

1. $\lim_{x \to \infty} \frac{3x^3+x^2+1}{5x^3+2x+7}$

Note that the limit of the denominator is + ∞ and that of the numerator is also + ∞.

- First consideration is to rewrite the function so that the denominator will have a finite non-zero limit.
- In our case we divide the denominator as well as the numerator by x^3.

Note the generic $\lim_{x \to \pm\infty} \frac{1}{x} = \lim_{x \to \pm\infty} \frac{1}{x^2} = \lim_{x \to \pm\infty} \frac{1}{x^3}$.

$$\lim_{x \to \infty} \frac{3x^3+x^2+1}{5x^3+2x+7} = \lim_{x \to \infty} \frac{(3x^3+x^2+1)/x^3}{(5x^3+2x+7)/x^3} = \lim_{x \to \infty} \frac{3+1/x+1/x^3}{5+2/x^2+7/x^3} = \frac{3+0+0}{5+0+0} = \frac{3}{5}.$$

2. $\lim_{x \to \infty} \sqrt[3]{\frac{3x^2+x+1}{2x^2+7}} = \sqrt[3]{\lim_{x \to \infty} \frac{3x^2+x+1}{2x^2+7}} = \sqrt[3]{\lim_{x \to \infty} \frac{(3x^2+x+1)/x^2}{(2x^2+7)/x^2}}$

$$= \sqrt[3]{\frac{\lim_{x \to \infty} 3+\frac{1}{x}+\frac{1}{x^2}}{\lim_{x \to \infty} 2+\frac{7}{x^2}}} = \sqrt[3]{\frac{3}{2}}.$$

3. $\lim_{x \to \infty} \frac{\sqrt{x^2+5}}{x-2}$.

- In this case we have square root involved.

To bring x^2 inside the $\sqrt{\quad}$ sign, we must use the identity:

$$\sqrt{x^2} = |x|.$$

$$\lim_{x \to \infty} \frac{\sqrt{x^2+5}}{x-2} = \lim_{x \to \infty} \frac{(\sqrt{x^2+5})/\sqrt{x^2}}{(x-2)/\sqrt{x^2}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1+5/x^2}}{(x-2)/|x|}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1+5/x^2}}{(x-2)/x} \text{ since for x > 0, } \quad |x| = x$$

$$= \lim_{x \to \infty} \frac{\sqrt{1+5/x^2}}{1-2/x} = \frac{\sqrt{1}}{1-0} = 1.$$

4. $$\lim_{x \to \infty} (\sqrt{x^2+1} - x) = \lim_{x \to \infty} \frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{(\sqrt{x^2+1} + x)} = \lim_{x \to \infty} \frac{x^2+1-x^2}{\sqrt{x^2+1} + x}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x} \quad = \lim_{x \to \infty} \frac{1/|x|}{\sqrt{x^2+1}\,/|x| + x/|x|}$$

$$= \lim_{x \to \infty} \frac{1/x}{\sqrt{1+1/x^2} + 1}$$

$$= \lim_{x \to \infty} \frac{0}{\sqrt{1+0} + 1} = \frac{0}{2} = 0.$$

5. Find all values of x at which the following function is continuous.

$$f(x) = \begin{cases} 3x^2 + 1, & x \le 1 \\ 5 - 3x, & 1 < x < 3 \\ x - 7, & x \ge 3 \end{cases} \quad .$$

- Note the following criterion:

If f is continuous on an interval, then it is continuous on any sub-open interval

$E \subseteq I$.

Since $3x^2 + 1$ is continuous on **R,** it is therefore continuous on $(-\infty, 1)$. Now for $x < 1$, $f(x) = 3x^2 + 1$; continuous on $(-\infty, 1)$.

Similarly since $5 - 3x$ is continuous on $(1, 3)$, $f(x)$ is continuous on $(1, 3)$.

Also $x - 7$ is continuous on $(3, \infty)$, that f is continuous on $(3, \infty)$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x^2 + 1 = 4 \text{ and } \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 5 - 3x = 5 - 3 = 2.$$

Since $\lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$, $\lim_{x \to 1} f(x)$ does not exist and so f is not continuous at $x = 1$.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 5-3x = 5-9 = -4 \text{ and } \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x-7 = 3-7 = -4.$$

Therefore $\lim_{x\to 3} f(x) = -4 = f(3)$. Hence f is continuous at $x = 3$.

Thus f is continuous on $\mathbf{R} - \{1\}$.

6. $$\lim_{x\to 0} \frac{\sin(9x)}{2x} = \lim_{x\to 0} \frac{9}{2}\cdot\frac{\sin(9x)}{9x} = \lim_{9x\to 0} \frac{9}{2}\cdot\frac{\sin(9x)}{9x} = \frac{9}{2}\cdot\lim_{t\to 0}\frac{\sin(t)}{t} = \frac{9}{2}\cdot 1 = \frac{9}{2}$$

7. $$\lim_{x\to 0} \frac{\sin(\sin(2x))}{x} = \lim_{x\to 0} \frac{\sin(\sin(2x))}{\sin(2x)}\cdot\frac{\sin(2x)}{2x}\cdot 2$$

$$= \lim_{x\to 0} \frac{\sin(\sin(2x))}{\sin(2x)}\cdot\lim_{x\to 0}\frac{\sin(2x)}{2x}\cdot 2 = 1\cdot 1\cdot 2 = 2.$$

8. $$\lim_{x\to 0} \frac{\sin(x^2)}{x^2+3x} = \lim_{x\to 0} \frac{\sin(x^2)}{x^2}\cdot\frac{1}{1+\frac{3}{x}} = \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot\frac{x}{x+3} = 1\cdot\frac{0}{3} = 0.$$

9. $$\lim_{x\to 0} x\cos(\tfrac{1}{x^2}) = 0.$$

Now $0 \le \left|x\cos(\tfrac{1}{x^2})\right| \le |x|$. Since $\lim_{x\to 0}|x|$, by the Squeeze Theorem

$\lim_{x\to 0}\left|x\cos(\tfrac{1}{x^2})\right| = 0$ and so by a Corollary to the Squeeze Theorem

$$\lim_{x\to 0} x\cos\left(\tfrac{1}{x^2}\right) = 0.$$