EXAMPLE SESSION 6

Absolute Maximum

Let the function f be defined on **R** by $f(x) = \begin{cases} 2 + x^3, \ x \le 1 \\ 5x - x^2 - 1, \ x > 1 \end{cases}$.

(i) Find the intervals on which is increasing and or decreasing.

(ii) Find the intervals on which the graph of f is concave upward or concave downward.

(iii) Find the relative extrema, absolute extrema of f if any.

(iv) Find the points of inflection of the graph of f.

(v) Sketch the graph of f.

First observe that the function is a piecewise polynomial function.

Then observe that it is continuous. This is deduced below.

f is continuous on **R** because f is a polynomial function on $(-\infty, 1)$ and also on $(1, \infty)$ and that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 3 = f(1).$$

To find the interval on which f is increasing or decreasing we shall have to differentiate as far as we can.

Start by differentiating on the open interval where the function is given by a polynomial (or by a function whose derivative is known -- trigonometric, exponential, log , etc.)

Then $f'(x) = \begin{cases} 3x^2, \ x < 1 \\ 5 - 2x, \ x > 1 \end{cases}$ -------------- (1)

At this point we dont't need to know if the function is differentiable at $x = 1$. Thus for $x < 0$,

$$f'(x) = 3x^2 > 0,$$

and since f is continuous at $x = 0$, f is increasing on the interval $(-\infty, 0]$.

Also for $0 < x < 1$, $f'(x) = 3x^2 > 0$ so that . Hence f is increasing on $[0, 1]$ since f is continuous at $x =$ 0 and at $x = 1$. For $x > 1$, $f'(x) = 5 - 2x$ and $f'(x) = 0$ when $x = 5/2$.

For $1 < x < 5/2$, $(5-2x) > 0$ so that $f'(x) = (5-2x) > 0$. Thus we have that f is increasing on $[1, 5/2]$ since f is continuous at $x = 5/2$ too. Thus f is increasing on $(-\infty, 5/2]$. Finally for $x > 5/2$, $5 - 2x < 0$ and so by (1) $f'(x) < 0$ and we conclude that f is decreasing on $[5/2, \infty)$.

Thus $f(\frac{5}{2})$ is a relative maximum and also an absolute maximum

To investigate the concavity of the graph of f we would use all possible second derivative of f.

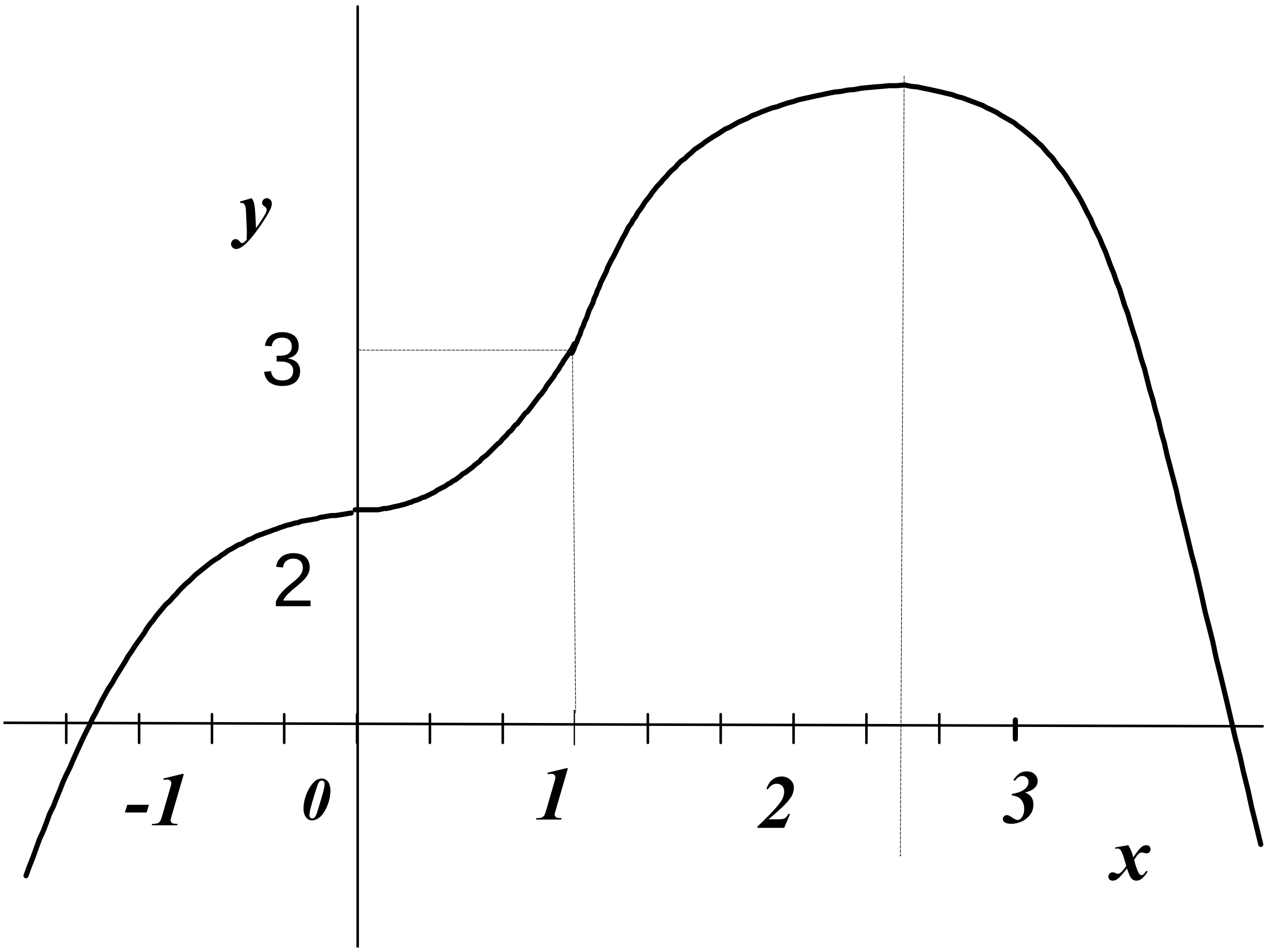
$$f''(x) = \begin{cases} 6x, & x < 1 \\ -2, & x > 1 \end{cases} \qquad \text{----------------} \ (2)$$

From (2) when $x < 0$, $f''(x) = 6x < 0$. Hence the graph of f is concave downward on the interval $(-\infty, 0)$. Also from (2), when $1 > x > 0$, $f''(x) = 6x > 0$.

Thus the graph of f is concave upward on the interval (0, 1). Again from (2), for $x > 1$, and so the graph of f is concave downward on $(1, \infty)$. Thus $(0, f(0))=(0,2)$ and $(1, f(1))=(1,3)$ are the points of inflection of the graph of f. Now $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} -(x-2)^2 + 3 = -\infty$. Thus f has no absolute minimum.

For the purpose of graph sketching note that $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2 + x^3 = -\infty$.

Graph of f.


y
3
2
-1
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