Answer and Guide To MA1102 Calculus Mock Test 1998-99 Semester 1

1. This question tests the concept of the range of a function, continuity, differentiability and integration over a piece-wise polynomial.

The function $f$ is defined by $f(x)=\begin{cases}(x+3)^2, & x\le -3\\ 2x, & -3<x<1\\ x^2+1, & x\ge 1\end{cases}$.

Recall range $f = \{f(x); x\in \mathbf{R}\}$. Examine this definition carefully. The intuitive geometrical preconception about the range depends on quite a number of concepts among them one which assume, without proof, what the graph of a function would look like in its entirety. Indeed this would involve the methods of calculus.
Very often the range does not call for this kind of analysis.

$y$ is in the range of $f$ if and only if we can find an element $x$ in the domain of $f$ such that

$$f(x) = y$$

So we need to know when we can solve this equation for $x$ in the domain of $f$. Usually this would take the form of a condition on $y$ which would allow us to specify the range of $f$. But our function is defined in a piecewise manner. So we consider our function as three functions with each of the following intervals $(-\infty,-3]$, $(-3,1)$, $[1,\infty)$ as their respective domains. So the range of $f$ is the union of the range of these three functions. More precisely, the range of $f$ is the union of the images of these three intervals under $f$.

(a) For $x\le -3$, $f(x)=(x+3)^2\ge 0$. Thus if $y$ is in the image of $(-\infty,-3]$ $y\ge 0$.
Now for any $y\ge 0$ $f(x) = y$ can be solved for $x\le -3$. This is done as follows. For $y\ge 0$ and $f(x)=(x+3)^2=y$, we can take $x+3=-\sqrt{y}$ so that $x=-3-\sqrt{y}\le -3$. Therefore, *the image of* $(-\infty,-3]$ under $f$ is $[0,+\infty)$. Also, $-3<x<1$ if and only if $-6<2x<2$. Thus, since for $-3<x<1$, $f(x)=2x$, $f$ maps $(-3,1)$ onto $(-6,2)$. Therefore, the image of $(-3,1)$ under $f$ is $(-6,2)$. Finally for $x\ge 1$, $f(x)=x^2+1\ge 2$. And for any $y\ge 2$, we can solve $f(x)=x^2+1=y$ for $x\ge 1$ by taking $x=\sqrt{y-1}\ge 1$. Therefore, the image of $[1,\infty)$ under $f$ is $[2,\infty)$.

Hence the range of $f$ is $[0,\infty)\cup(-6,2)\cup[2,\infty)=(-6,\infty)$.

(b) (i) From part (a) $-5$ is in the image of $(-3,1)$ under $f$. Thus, to find the *preimage* we need to solve the equation $2x=-5$ for $x<-4$. Solving this gives $x=\frac{-5}{2}$.

(ii) From part (a) $-7$ is not in the range of $f$. Thus, there is no value of $x$ for which $f(x)=-7$.

(iii) 0 is in the images of (-3,1) and $(-\infty,-3]$. Solving $f(x)=0$ for $x$ in $(-\infty,-3]$ means solving $(x+3)^2=0$ which gives $x=-3$. Solving $f(x)=0$ for $x$ in $(-3,1)$ means solving $2x=0$ which gives $x=0$.

(c) When $x<-3$, $f(x)=(x+3)^2$, which is a polynomial function, therefore $f$ is continuous on $(-\infty,-3)$, since any polynomial function is continuous on the reals and so is continuous on any interval. Similarly, when $-3<x<1$, $f(x)$ is a polynomial function and so $f$ is continuous on this interval. Finally when $x>1$, $f(x)$ is also a polynomial function and so it is continuous for $x>1$. Thus it remains to check if $f$ is continuous at $x=-3$ or 1. Consider the left limit at $x=-3$,

$$\lim_{x\to(-3)^-} f(x)=\lim_{x\to(-3)^-}(x+3)^2=0 \quad \text{and the right limit at } x=-3$$

$$\lim_{x\to(-3)^+} f(x)=\lim_{x\to(-3)^+} 2x=-6.$$

Therefor, the left and the right limits are not the same. Thus the limit at $x=-3$ does not exist. Therefore $f$ is not continuous at $x=-3$. Now consider the left limit of $f$ at $x=1$,

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-} 2x=2 \text{ and the right limit at } x=1,$$

$$\lim_{x\to1^+} f(x)=\lim_{x\to1^+} x^2+1=1^2+1=2=f(1).$$

Therefore, the left and the right limits of $f$ at $x=1$ are the same and is equal to the value of the function $f$ at $x=1$ and so $f$ is continuous at $x=1$. Thus $f$ is continuous at $x$ for all $x\neq-3$.

(d) $f$ is differentiable at $x=1$. This is seen as follows.

$$\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^-}\frac{2(1+h)-2}{h}=2 \quad \text{and}$$

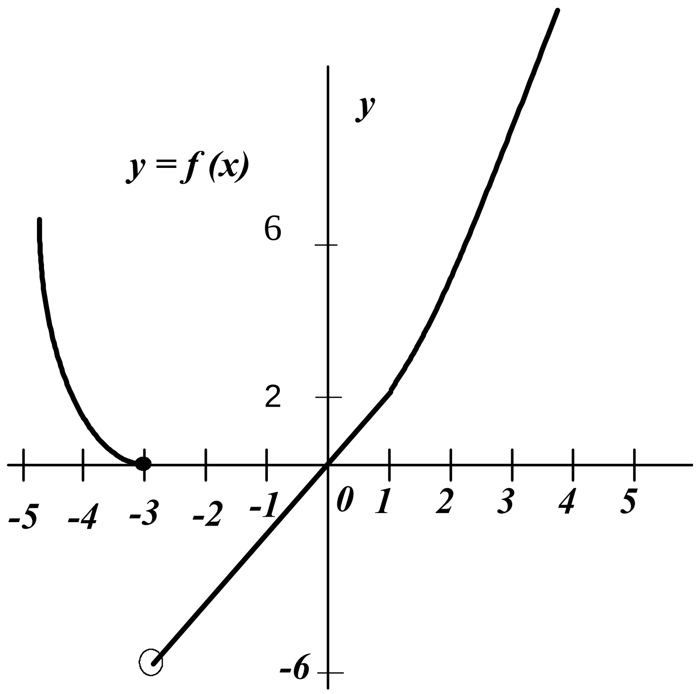
$$\lim_{h\to0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to0^+}\frac{(1+h)^2+1-2}{h}=\lim_{h\to0^+}\frac{h^2+2h}{h}=2=\lim_{h\to0^-}\frac{f(1+h)-f(1)}{h}.$$

Therefore, $\lim_{h\to0}\frac{f(1+h)-f(1)}{h}=2$ and so $f$ is differentiable at $x=1$ and $f'(1)=2$.

(e) $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 2x dx + \int_1^2 (x^2+1)dx$

$$= [x^2]_0^1 + \left[\frac{x^3}{3} + x\right]_1^2 = 1 + \tfrac{1}{3}(8-1) + 1 = 4\tfrac{1}{3}.$$

(f)




2. You can use L'Hôpital's Rule here.

(a) $\lim\limits_{x\to\infty} \dfrac{10+9x^3-x^2}{3x^3-7+5x} = \lim\limits_{x\to\infty} \dfrac{\frac{10}{x^3}+9-\frac{1}{x}}{3-\frac{7}{x^3}+\frac{5}{x^2}} = \dfrac{0+9+0}{3-0+0} = \dfrac{9}{3} = 3.$

(b) $\lim\limits_{x\to0} \dfrac{\sqrt{5x^2+4}-2}{x^2} = \lim\limits_{x\to0} \dfrac{(\sqrt{5x^2+4}-2)(\sqrt{5x^2+4}+2)}{x^2(\sqrt{5x^2+4}+2)} = \lim\limits_{x\to0} \dfrac{5x^2+4-4}{x^2(\sqrt{5x^2+4}+2)}$

$$= \lim_{x\to0} \frac{5}{(\sqrt{5x^2+4}+2)} = \frac{5}{4}.$$

*Or you can use* L'Hôpital's Rule:

$$\lim_{x\to0} \frac{\sqrt{5x^2+4}-2}{x^2} = \lim_{x\to0} \frac{\frac{1}{2}(5x^2+4)^{-1/2}\cdot 10x}{2x} = \lim_{x\to0} \tfrac{5}{2}(5x^2+4)^{-1/2} = \frac{5}{4}.$$

(c) $\lim\limits_{x\to0} \dfrac{\sin(x)}{7x-x^2} = \lim\limits_{x\to0} \dfrac{\sin(x)}{x} \cdot \dfrac{1}{(7-x)} = 1 \cdot \dfrac{1}{7-0} = \dfrac{1}{7}$ since $\lim\limits_{x\to0} \dfrac{\sin(x)}{x} = 1$.

*Or* $\lim\limits_{x\to0} \dfrac{\sin(x)}{7x-x^2} = \lim\limits_{x\to0} \dfrac{\cos(x)}{7-2x}$ by L'Hôpital's Rule

$$= \frac{\cos(0)}{7-0} = \frac{1}{7}.$$

(d) $\lim\limits_{x\to\infty}\left(\sqrt{16x^2+3}-4x\right)=\lim\limits_{x\to\infty}\left(\sqrt{16x^2+3}-4x\right)\cdot\dfrac{\sqrt{16x^2+3}+4x}{\sqrt{16x^2+3}+4x}$

$$=\lim_{x\to\infty}\frac{16x^2+3-16x^2}{\sqrt{16x^2+3}+4x}=\lim_{x\to\infty}\frac{3/x}{\sqrt{16+\frac{3}{x^2}}+4}=\frac{0}{8}=0.$$

Notice here we make use of the fact that for $x > 0$, $\sqrt{x^2}=x$.

(e) $\lim\limits_{x\to1}\dfrac{\sin(x-1)}{\sqrt{x}-1}=\lim\limits_{x\to1}\dfrac{\sin(x-1)}{\sqrt{x}-1}\cdot\dfrac{\sqrt{x}+1}{\sqrt{x}+1}=\lim\limits_{x\to1}\dfrac{\sin(x-1)}{x-1}\cdot(\sqrt{x}+1)$

$$=\lim_{x\to1}\frac{\sin(x-1)}{x-1}\cdot\lim_{x\to1}(\sqrt{x}+1)=1\cdot(1+1)=2\ .$$

Or $\lim\limits_{x\to1}\dfrac{\sin(x-1)}{\sqrt{x}-1}=\lim\limits_{x\to1}\dfrac{\cos(x-1)}{\frac{1}{2\sqrt{x}}}$ by L'Hôpital's Rule

$$=\frac{\cos(0)}{1/2}=2.$$

(f) $\lim\limits_{x\to0}(1+7x^2)^{\frac{1}{x^2}}$ .

Let $y=(1+7x^2)^{\frac{1}{x^2}}$ . Then $\ln(y)=\frac{1}{x^2}\ln(1+7x^2)$.

$\lim\limits_{x\to0}\ln(y)=\lim\limits_{x\to0}\dfrac{\ln(1+7x^2)}{x^2}=\lim\limits_{x\to0}\dfrac{\frac{14x}{1+7x^2}}{2x}$ by L'Hôpital's Rule

$$=\lim_{x\to0}\frac{7}{1+7x^2}=7.$$

Therefore, $\lim\limits_{x\to0}(1+7x^2)^{\frac{1}{x^2}}=e^{\lim\limits_{x\to0}\ln(y)}=e^7$ .

3 (a) (i) For $g(x)=x^5+x+5$ , $g(0) = 5 > 0$ and $g(-2) = -32-2 +5 = -29 < 0$ . Since $g$ is a polynomial function on $[-2, 0]$, g is continuous on $[-2, 0]$. Therefore, by the *Intermediate Value Theorem*, there is a point $c$ in $(-2, 0)$ such that $g(c) = 0$.

(ii) For $x$ in **R**, $g'(x)=5x^4+1$ .

Suppose $g$ has two distinct roots say $c$ and $c$ in **R**. Without loss of generality we may assume that $c < c$ . Then since $g$ is differentiable on the whole of **R,** $g$ is continuous on $[c, c\ ]$, differentiable on $(c, c\ )$. Obviously, $g(c) = g(c\ ) = 0$. Therefore, by Rolle's theorem, there is a point $d$ in $(c, c\ )$ with $g'(d) = 0$. But $g'(d)=5d^4+1>0$. This contradiction shows that g can have only one root. Thus by part (i) $g$ has exactly one such root $c$.

*alternatively*, since $g'(x)=5x^4+1>0$ $g$ is increasing on **R** and so $g$ is injective on **R**. Therefore by part (i) there is exactly one root $c$ in **R.**

(b) $h(x) = x^4 - 4x^2 + 16$ on $[-2, 4]$. Therefore, since $h$ is a polynomial function $h$ is continuous on [-2, 4] and the derivative $h'(x) = 4x^3 - 8x = 4x(x - \sqrt{2})(x + \sqrt{2})$ on the open interval $(-2, 4)$. Therefore, in the interval (-2, 4)

$$h'(x) = 0 \Leftrightarrow 4x(x - \sqrt{2})(x + \sqrt{2})x = 0 \Leftrightarrow x = 0, \sqrt{2} \text{ or } -\sqrt{2} .$$

Thus, there are only three critical points in $(-2, 4)$ occurring at $x = 0$ , $\sqrt{2}$ and $-\sqrt{2}$ .

$h(0) = 16$, $h(\sqrt{2}) = h(-\sqrt{2}) = 4 - 8 + 16 = 12$, $h(-2) = 16 - 4 \cdot 4 + 16 = 16$,

$h(4) = 16 \cdot 4^2 - 4 \cdot 4^2 + 16 = 13 \times 16 = 208$ .

Therefore, the absolute minimum value of $h$ on $[-2, 4]$ is 12 and the absolute maximum value of $h$ on $[-2,4]$ is 208.

(c) $y^2 - \sin(y) = 2x$.

Differentiating implicitly we get $2y\dfrac{dy}{dx} - \cos(y)\dfrac{dy}{dx} = 2$ --------------------------- (1)

Differentiating (1) implicitly again we get

$$2\frac{dy}{dx}\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} + \sin(y)\frac{dy}{dx}\frac{dy}{dx} - \cos(y)\frac{d^2y}{dx^2} = 0 .$$

Thus, $(2y - \cos(y))\dfrac{d^2y}{dx^2} = -\left(\dfrac{dy}{dx}\right)^2 (\sin(y) + 2)$ -------------------------------------(2)

From (1) we know that $(2y - \cos(y)) \neq 0$ and $\dfrac{dy}{dx} = -\dfrac{2}{2y - \cos(y)}$. Thus,

$$\frac{d^2y}{dx^2} = -\frac{4(\sin(y) + 2)}{(2y - \cos(y))^3} .$$

4. Since $f(x) = \dfrac{2 + x - x^2}{(x-1)^2}$ , we note that $f$ is continuous on $\mathbf{R} - \{1\}$ because $f$ is a rational function. Then we can rewrite the function in a simpler form as follows.

$$f'(x) = \frac{2 - x(x-1)}{(x-1)^2} = \frac{2}{(x-1)^2} - \frac{x}{(x-1)} = \frac{2}{(x-1)^2} - \frac{1}{(x-1)} - 1 .$$

Then $f'(x) = -\dfrac{4}{(x-1)^3} + \dfrac{1}{(x-1)^2} = \dfrac{-4 + (x-1)}{(x-1)^3} = \dfrac{x-5}{(x-1)^3}$ ------------------ (1)

$$f''(x) = \frac{12}{(x-1)^4} - \frac{2}{(x-1)^3} = \frac{12 - 2(x-1)}{(x-1)^4} = 2\frac{7-x}{(x-1)^4} \quad \text{------------------ (2)}$$

(a) When $x < 1$, $(x-1)^3 < 0$ and $x - 5 < 0$ so that by (1), $f'(x) > 0$. Thus $f$ is increasing on the interval $(-\infty, 1)$.

For $1 < x < 5$, $(x-1)^3 > 0$ and $x - 5 < 0$ so that by (1), $f'(x) < 0$ . Hence $f$ is decreasing on $(1, 5]$, since $f$ is continuous at $x = 5$ .

Finally for $x > 5$, $(x-1)^3 > 0$ and $x - 5 > 0$ and so by (1) $f'(x) > 0$ and we conclude that $f$ is increasing on $[5, \infty)$ since $f$ is continuous at $x = 5$.

(b) Since $f$ is differentiable on its domain, by (1) it has only one critical point, namely $x = 5$.

(c) From part (b) since $f$ is differentiable on its domain the critical point is also a stationary point. Therefore, it can have only one relative extremum and

$f(5) = \dfrac{2+5-25}{16} = -\dfrac{9}{8}$ is a relative minimum since $f$ is decreasing on (1, 5] and increasing on $[5, \infty)$. There are no relative maxima.

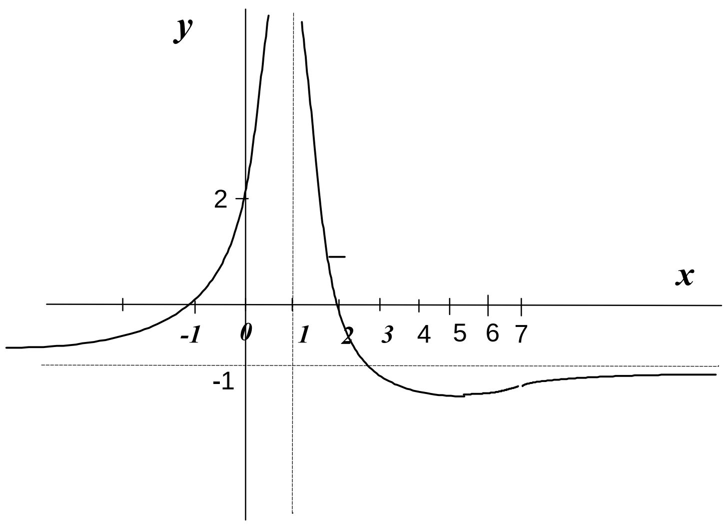
(d) When $x < 7$ and $x \neq 1$, $7 - x > 0$ and so by (2) $f''(x) > 0$. Hence the graph of $f$ is concave upward on the intervals $(-\infty, 1)$ and $(1, 7)$. When $x > 7$, i.e., $7 - x < 0$, by (2), $f''(x) < 0$. Thus the graph of $f$ is concave downward on the interval $(7, \infty)$.

(e) $(7, f(7)) = (7, \dfrac{2+7-49}{36}) = (7, -\dfrac{40}{36}) = (7, -\dfrac{10}{9})$ is a point of inflection since before and after the point $x = 7$ there is a change of concavity.

(f) Now $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{2+x-x^2}{(x-1)^2} = \lim\limits_{x\to 1} \dfrac{1}{(x-1)^2} \cdot (2+x-x^2) = \infty$. This is because $\lim\limits_{x\to 1} \dfrac{1}{(x-1)^2} = \infty$ and $\lim\limits_{x\to 1} (2+x-x^2) = 2 > 0$. Therefore, the line $x = 1$ is a vertical asymptote of the graph of $f$. Also $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{2+x-x^2}{(x-1)^2} = \lim\limits_{x\to\pm\infty} \dfrac{-1+\frac{1}{x}+\frac{2}{x^2}}{(1-\frac{1}{x})^2} = -1$.

Thus $y = -1$ is a horizontal asymptote of the graph of $f$.

(g)



The graph of $f$ (not drawn to scale)

5. (a) $\int \frac{x^2dx}{(1+x^2)^2} = \int(\frac{1+x^2-1}{(1+x^2)^2}dx = \int(\frac{1}{(1+x^2)} - \frac{1}{(1+x^2)^2})dx$

$= \tan^{-1}(x) - \int \frac{1}{(1+x^2)^2}dx$.

Now $\int \frac{1}{(1+x^2)^2}dx = \int \frac{1}{(1+\tan^2(\theta))^2}\sec^2(\theta)d\theta = \int \cos^2(\theta)d\theta$

where $x = \tan(\theta)$ so that $dx = \sec^2(\theta)d\theta$

$= \int \frac{1}{2}(1+\cos(2\theta))d\theta = \frac{1}{2}(\theta + \frac{1}{2}\sin(2\theta)) + C$

$= \frac{1}{2}\tan^{-1}(x) + \frac{1}{2}\frac{\tan(\theta)}{\sec^2(\theta)} + C = \frac{1}{2}\tan^{-1}(x) + \frac{1}{2}\frac{\tan(\theta)}{1+\tan^2(\theta)} + C$

$= \frac{1}{2}\tan^{-1}(x) + \frac{1}{2}\frac{x}{1+x^2} + C$.

Therefore, $\int \frac{x^2dx}{(1+x^2)^2} = \frac{1}{2}(\tan^{-1}(x) - \frac{x}{1+x^2}) + C$. Thus

$\int_0^1 \frac{1}{(1+x^2)^2}dx = \frac{1}{2}\left[\tan^{-1}(x) - \frac{x}{1+x^2}\right]_0^1 = \frac{1}{2}(\tan^{-1}(1) - \frac{1}{2}) = \frac{\pi}{8} - \frac{1}{4}$.

(b) $\int(\ln(3x))^2dx = x(\ln(3x))^2 - \int x \cdot 2\ln(3x) \cdot \frac{1}{x}dx$ by integration by parts

$= x(\ln(3x))^2 - 2\int \ln(3x)dx = x(\ln(3x))^2 - 2(x\ln(3x) - \int x \cdot \frac{1}{x}dx)$

by integration by parts

$= x(\ln(3x))^2 - 2x\ln(3x) + 2x + C$.

(c) $\int \cos(\sin(y))\cos(y)dy = \int \cos(u)du$, where $u = \sin(y)$ so that $du = \cos(y)dy$

$= \sin(u) + C = \sin(\sin(y)) + C$.

(d) $\int \frac{e^x}{e^{2x}+3e^x+2}dx = \int \frac{1}{u^2+3u+2}du$, where $u = e^x$ so that $du = e^x dx$

$= \int \frac{1}{(u+2)(u+1)}du = \int(\frac{1}{u+1} - \frac{1}{u+2})du = \ln|u+1| - \ln|u+2| + C$

$= \ln\left|\frac{u+1}{u+2}\right| + C = \ln(\frac{e^x+1}{e^x+2}) + C$..

(e) $\int \frac{1}{\sqrt{2\sqrt{x}+5}}dx$

Use the substitution $y = \sqrt{2\sqrt{x}+5}$ so that $y^2 = 2\sqrt{x}+5$ and $2ydy = \frac{1}{\sqrt{x}}dx$.

Note that $dx = 2y(y^2-5)/2dy = y(y^2-5)dy$. Therefore,

$$\int \frac{1}{\sqrt{2\sqrt{x}+5}}dx = \int \frac{y(y^2-5)}{y}dy = \int (y^2-5)dy = \frac{y^3}{3} - 5y + C$$

$$= \frac{1}{3}(2\sqrt{x}+5)^{\frac{3}{2}} - 5(2\sqrt{x}+5)^{\frac{1}{2}} + C..$$

6. (a) $\int_2^5(|x-3|+|x-4|)dx = \int_2^3(|x-3|+|x-4|)dx$

$$+\int_3^4(|x-3|+|x-4|)dx + \int_4^5(|x-3|+|x-4|)dx$$

$$= -\int_2^3((x-3)+(x-4))dx + \int_3^4((x-3)-(x-4))dx + \int_4^5((x-3)+(x-4))dx$$

$$= -\int_2^3(2x-7)dx + \int_3^4 1dx + \int_4^5(2x-7)dx$$

$$= -[x^2-7x]_2^3 + 1 + [x^2-7x]_4^5 = -5+9+1 = 5.$$

(b) Write the following as a Riemann sum

$$\sum_{i=1}^{n} \frac{\pi}{2n}\sin(\frac{\pi}{2}\cdot\frac{i}{n}) = \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $x_0 < x_1 < \cdots < x_n$ is a regular partition and $\Delta x = \Delta x_i = x_i - x_{i-1}$. Therefore, we can take $x_i = \frac{\pi}{2}\cdot\frac{i}{n}$ so that $\Delta x = \frac{\pi}{2n}, x_0 = 0$ and $x_n = \frac{\pi}{2}$. Thus by comparing

$$f(x_i)\Delta x \text{ with } \frac{\pi}{2n}\sin(\frac{\pi}{2}\cdot\frac{i}{n}),$$

we would want $f(x_i) = \sin(\frac{\pi}{2}\cdot\frac{i}{n}) = \sin(x_i)$ Thus $f(x) = \sin(x)$. Therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{2n}\sin(\frac{\pi}{2}\cdot\frac{i}{n}) = \int_0^{\frac{\pi}{2}}\sin(x)dx = [-\cos(x)]_0^{\frac{\pi}{2}} = \cos(0) = 1.$$

(c) Since $g(x) = x\int_0^x f(x)dx$, by the Product Rule and the Fundamental Theorem of calculus,

$$g'(x) = \int_0^x f(x) + xf(x).$$

Since it is given that for all $x$ in $\mathbf{R}$, $f(x) > 0$, for any $x > 0$, $xf(x) > 0$ and the integral $\int_0^x f(x) \geq 0$. Therefore, $g'(x) = \int_0^x f(x) + xf(x) > 0$ for $x > 0$. Hence $g$ is increasing on $(0,\infty)$.

7. (a) (i) $g(x) = \int_{x^3}^{x^4}\frac{1}{1+t^2}dt = \int_0^{x^4}\frac{1}{1+t^2}dt + \int_{x^3}^{0}\frac{1}{1+t^2}dt$

$= \int_0^{x^4}\frac{1}{1+t^2}dt - \int_0^{x^3}\frac{1}{1+t^2}dt = F(x^4) - F(x^3)$, where $F(x) = \int_0^x \frac{1}{1+t^2}dt$.

Therefore, $g'(x) = F'(x^4)\cdot 4x^3 - F'(x^3)\cdot 3x^2$ by the *Chain Rule*

$= \frac{4x^3}{1+x^8} - \frac{3x^2}{1+x^6}$ by the Fundamental Theorem of Calculus.

(ii) Since $h(x) = 3^{(x^2)}$ , $\ln(h(x)) = x^2 \ln(3)$. Thus, differentiating the above on both sides gives $\dfrac{h'(x)}{h(x)} = \ln(3)(2x)$. Therefore, $h'(x) = 3^{(x^2)} \ln(3)(2x)$ .

(iii) Since $k(x) = (1+x^2)^{\ln(x)}$ , $\ln(k(x)) = \ln(x)\ln(1+x^2)$

Differentiating this equation on both sides gives

$$\frac{k'(x)}{k(x)} = \frac{1}{x}\ln(1+x^2) + \ln(x)\cdot\frac{2x}{1+x^2}.$$

Therefore, $k'(x) = (1+x^2)^{\ln(x)}\left(\dfrac{\ln(1+x^2)}{x} + \dfrac{2x\ln(x)}{1+x^2}\right)$.

(b) (i) Since $f(x) = \int_1^x \sqrt{8+t^2}\,dt$ , by the Fundamental Theorem of Calculus,

$$f'(x) = \sqrt{8+x^2} \geq \sqrt{8} > 0 .$$

Therefore, $f$ is increasing on the whole of **R.** Thus $f$ is injective.

(ii) Now $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))}$. So we need to know the value of $f^{-1}(0)$ . Now

$f^{-1}(0) = x \Leftrightarrow f(x) = 0 \Leftrightarrow \int_1^x \sqrt{8+t^2}\,dt = 0$ . Since $f(1) = \int_1^1 \sqrt{8+t^2}\,dt = 0$

and $f$ is injective, $x = 1$. Therefore, $f^{-1}(0) = 1$.

Thus, $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))} = \dfrac{1}{f'(1)} = \dfrac{1}{\sqrt{8+1}} = \dfrac{1}{3}$.

8 (a) Since $f(x) = \begin{cases} \dfrac{\sin(3x)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ , $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin(3x)}{x} = \lim\limits_{x\to 0} \dfrac{\sin(3x)}{3x}\cdot 3 = 1\cdot 3 = 3$

because $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{3x} = 1$. (You can use L'Hôpital's Rule here.)

Now recall the definition of continuity of a function at a point. $f$ is continuous at $x = 0$ if and only if the limit $\lim\limits_{x\to 0} f(x)$ exists and is equal to $f(0)$. This means

$$\lim_{x\to 0} f(x) = f(0) = k.$$

Hence $k = 3$.

(b) This is a very good question. You will have to refer to the definition of differentiability and work with it. Condition (3) is a statement about differentiability of $f$ at $x = 0$. It says $f$ (0) = 1. I,e,

$$\lim_{h\to 0} \frac{f(h+0) - f(0)}{h} = 1 \text{ ---------------------} \quad (*)$$

he function $f$ is differentiable at the point $x$ if and only if the limit

$$\lim_{h\to 0} \frac{f(h+x) - f(x)}{h}$$

exists. So we shall start with this limit

$$\lim_{h\to 0}\frac{f(h+x)-f(x)}{h}=\lim_{h\to 0}\frac{f(h)f(x)-f(x)}{h}$$ by condition (1)

( $f(h+x)=f(h)f(x)$)

$$=\lim_{h\to 0} f(x)\frac{f(h)-1}{h}$$

$$=\lim_{h\to 0} f(x)\frac{f(h)-f(0)}{h}$$ since $f(0) = 1$ by Condition (2)

$$= f(x)\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$= f(x)f'(0)$ by (*) (i.e., by condition (3) which says that the limit exists)

$= f(x)\cdot 1$ since $f'(0) = 1$ by Condition (3)

$= f(x)$.

Therefore, the function $f$ is differentiable at $x$ for any $x$ in **R** and for any $x$ in **R**, $f'(x) = f(x)$.